CHAPTER 5  COORDINATE SYSTEMS CONTINUED

SECTION 5.0  REVIEW

2 x 2 determinants

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

computing an n x n determinant by expansion by minors

First you have to know what minors and cofactors are.
Every entry in a square array has a minor and a cofactor, defined as follows.
The minor of an entry is the determinant you get by deleting the row and col of that entry.
The cofactor of an entry is the minor with a sign attached according to the entry's location in the checkerboard pattern in Fig 1.

\[
\begin{array}{ccc}
+ & - & + \\
- & + & - \\
+ & - & + \\
- & + & - \\
\end{array}
\]

FIG 1

For example if you start with

\[
\begin{array}{cccc}
1 & 2 & 7 & 8 \\
4 & 3 & 1 & 9 \\
5 & 2 & 9 & 6 \\
8 & 1 & 3 & 8 \\
\end{array}
\]

then to get the minor of the 7 in row 1, col 3, form a determinant by deleting row 1 and col 3:

\[
\begin{vmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 2 \\ 8 & 1 \end{vmatrix} = 439 - 526 - 818 = +439
\]

To get the cofactor, choose the sign in row 1 col 3 of the checkerboard in Fig 1.

\[
\begin{vmatrix} 4 & 3 & 9 \\ 5 & 2 & 6 \\ 8 & 1 & 8 \end{vmatrix} = +439
\]

(in this case the minor and the cofactor are the same)

Similarly,

\[
\begin{vmatrix} 1 & 2 & 8 \\ 5 & 2 & 6 \\ 8 & 1 & 8 \end{vmatrix} = -128
\]

Here's how to compute say a 3 x 3 det (same idea for an n x n).
Pick any row.
\[ \text{det} = 1\text{st entry in the row} \times \text{its cofactor} \]
\[ + \text{2nd entry in the row} \times \text{its cofactor} \]
\[ + \text{3rd entry in the row} \times \text{its cofactor} \]

You can also find the det by picking any col. Then

\[ \text{det} = 1\text{st entry in the col} \times \text{its cofactor} \]
\[ + \text{2nd entry in the col} \times \text{its cofactor} \]
\[ + \text{3rd entry in the col} \times \text{its cofactor} \]

Here's a determinant expanded down column 2:

\[
\begin{vmatrix}
10 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix}
= -2 \begin{vmatrix}
4 & 6 \\
7 & 9
\end{vmatrix}
+ 5 \begin{vmatrix}
10 & 3 \\
7 & 9
\end{vmatrix}
- 8 \begin{vmatrix}
10 & 3 \\
4 & 6
\end{vmatrix}
= -27
\]

**example 1**

Here's an expansion down column 4, a good column to use because it has some zero entries:

\[
\begin{vmatrix}
1 & 2 & 3 & 0 \\
1 & -1 & 3 & 2 \\
0 & 1 & 2 & 3 \\
2 & 1 & 4 & 0
\end{vmatrix}
= 2 \begin{vmatrix}
1 & 2 & 3 \\
1 & 4 \\
2 & 1
\end{vmatrix}
+ 2 \begin{vmatrix}
2 & 3 \\
1 & 2 \\
1 & 4
\end{vmatrix}
- 3 \begin{vmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
2 & 1 & 4
\end{vmatrix}
= 2(2 + 2) - 3(-7 + 4 + 9)
= -10
\]

**some properties of determinants**

1. Interchanging two rows (or two cols) changes the sign of the determinant.
   For example,
   \[
   \begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 5 \\
7 & 7 & 7
\end{vmatrix}
= - \begin{vmatrix}
7 & 7 & 7 \\
4 & 5 & 5 \\
1 & 2 & 3
\end{vmatrix}
\]
   (rows 1 and 3 were switched)

2. Multiplying a row (or col) by a number will multiply the entire det by that number.
   If say row 2 is doubled then new det = 2 \times old det:
   \[
   \begin{vmatrix}
a & b & c \\
2d & 2e & 2f \\
g & h & i
\end{vmatrix}
= 2 \begin{vmatrix}
a & b & c \\
d & e & f \\
h & i
\end{vmatrix}
\]
   In other words, a common factor can be pulled out of a row or a col.

   For example,
   \[
   \begin{vmatrix}
6 & 3 & 9 \\
2 & 1 & 5 \\
2 & 3 & 1
\end{vmatrix}
= 2 \begin{vmatrix}
3 & 3 & 9 \\
1 & 1 & 5 \\
1 & 3 & 1
\end{vmatrix}
= 6 \begin{vmatrix}
1 & 1 & 3 \\
1 & 1 & 5 \\
1 & 3 & 1
\end{vmatrix}
\]
   pull 2 out of col 1  pull 3 out of row 1
If $M$ is $7 \times 7$ and every entry of $M$ is quadrupled, then you can pull a 4 out of each of the 7 rows (or equivalently out of each of the 7 columns) so the determinant of $M$ is multiplied by $4^7$.

3. If you take the transpose, i.e., turn the rows into cols and the cols into rows, the determinant doesn't change.

For example

\[
\begin{vmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3 \\
\end{vmatrix}
= \begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3 \\
\end{vmatrix}
\]

the scalar triple product

If $u,v,w$ are 3-dim vectors then $u \cdot v \times w$ is called a scalar triple product. If

\[
\begin{align*}
    u &= (u_1, u_2, u_3) \\
    v &= (v_1, v_2, v_3) \\
    w &= (w_1, w_2, w_3)
\end{align*}
\]

then

\[
(1) \quad u \cdot v \times w = \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3 \\
\end{vmatrix}
\]

For example, if

\[
\begin{align*}
    u &= (1,2,3), \quad v = (4,6,-1), \quad w = (0,3,2)
\end{align*}
\]

then

\[
\begin{vmatrix}
    1 & 2 & 3 \\
    4 & 6 & -1 \\
    0 & 3 & 2 \\
\end{vmatrix} = \ldots = 35
\]

or equivalently

\[
\begin{align*}
    v \times w &= (15,-8,12) \\
    u \cdot v \times w &= (1,2,3) \cdot (15,-8,12) = 35
\end{align*}
\]

The absolute value of $u \cdot v \times w$ is the volume of the parallelepiped determined by the vectors $u,v,w$ (Fig 2).

For non-zero vectors $u,v,w$, $u \cdot v \times w = 0$ iff the arrows $u,v,w$ are coplanar when attached to a common tail.
cyclic permutations

Start with say

\[ a\ b\ c\ d\ e \]

and rearrange (permute) in the following special way: Picture the letters as beads on a bracelet and slide one of the end letters around (Fig 3) (I'm sliding clockwise but you can also slide counterclockwise).

The new permutation is

\[ e\ a\ b\ c\ d \]

It's called a cyclic permutation of \( a\ b\ c\ d\ e \).

You can do it again three more times getting

\[
\begin{align*}
 & a\ b\ c\ d\ e \\
\rightarrow & d\ e\ a\ b\ c \\
\rightarrow & c\ d\ e\ a\ b \\
\rightarrow & b\ c\ d\ e\ a
\end{align*}
\]

Once more and you're back to the original \( a\ b\ c\ d\ e \).

**cyclic permutation and scalar triple products**

There are six scalar triple products involving factors \( p, q, r \).

- Three of the products have the same value. The other three have the opposite value (e.g., if one of the scalar triple products is 7 then two more are 7 and the other three are -7).

Here's the rule for telling which ones go together.

If you permute the letters cyclically (leaving the dot and cross alone) then the value of the scalar triple product doesn't change. If you permute non-cyclically then the sign changes.

\[
\begin{align*}
 & p\cdot q\times r \quad \text{same value} \\
 & r\cdot p\times q \\
 & q\cdot r\times p \\
 & q\cdot p\times r \quad \text{same value} \\
 & r\cdot q\times p \\
 & p\cdot r\times q \quad \text{opposite value}
\end{align*}
\]
SECTION 5.1 dA and dV

area elements

Start with a \( u,v \) coordinate system in the plane. Sweep out a patch, called an area element, by starting at a point \( P \) and changing \( u \) by \( du \) and \( v \) by \( dv \). Its area is called \( dA \).

Fig 1 shows the patch in polar coords and Fig 2 shows it in parabolic coords.

Similarly, in a \( u,v,w \) coordinate system in 3-space, a volume element is the little region swept out by changing \( u \) by \( du \), \( v \) by \( dv \), \( w \) by \( dw \). Its volume is called \( dV \).

finding \( dA \) and \( dV \) in orthogonal coordinate systems

If the \( u,v \) coordinate system is orthogonal then the area element (Fig 3) is (almost) a rectangle with sides \( ds_u = h_u \ du \) and \( ds_v = h_v \ dv \).

The area of the rectangle is the product of the sides so

\[
dA = h_u h_v \ du \ dv
\]

Similarly, if a \( u,v,w \) coordinate system is orthogonal then

\[
dV = h_u h_v h_w \ du \ dv \ dw
\]
example 1 (dA in polar coords)
In polar coords (Fig 4), \( h_r = 1, h_\theta = r \) so \( dA = h_r h_\theta \, dr \, d\theta = r \, dr \, d\theta \).

![Figure 4](image)

warning
The area and vol mag factors are products of the scale factors only in an orthog coord system. Otherwise, use the more general formulas coming up in (5) and (7).

dV = h_\rho h_\phi h_\theta \, d\rho \, d\phi \, d\theta = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta

example 2 (dV in spherical coordinates)
Fig 5 shows a volume element in spherical coordinates swept out starting at point A with coordinates \( \rho, \phi, \theta \).
The faces of the box are coordinate surfaces:
The face ABFE lies on a cone with cone angle \( \phi \).
The opposite face DCGH lies on a cone with cone angle \( \phi + d\phi \).
The face ADCB lies on a sphere with radius \( \rho \).
The opposite face EHGCF lies on a sphere with radius \( \rho + d\rho \).
The face ADHE lies on a half-plane, hinged along the z-axis at angle \( \theta \).
The opposite face BCGF lies on a half-plane with angle \( \theta + d\theta \).
The edges of the box are \( \rho \)-curves, \( \phi \)-curves and \( \theta \)-curves. For instance, the curve AD is on a \( \phi \)-curve; it was traced out by changing \( \phi \) by \( d\phi \) while \( \rho \) and \( \theta \) are fixed.

In spherical coordinates, \( h_\rho = 1, h_\phi = \rho, h_\theta = \rho \sin \phi \).
Length AE = \( ds_\rho = h_\rho \, d\rho = d\rho \)
Length AD = \( ds_\phi = h_\phi \, d\phi = \rho \, d\phi \)
Length AB = \( ds_\theta = h_\theta \, d\theta = \rho \sin \phi \, d\theta \)

\[ dV = h_\rho h_\phi h_\theta \, d\rho \, d\phi \, d\theta = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]
### definition of the Jacobian determinant

Given equations \( x = x(u,v) \), \( y = y(u,v) \). The Jacobian determinant is defined like this:

\[
\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]

You can also put the partials w.r.t. \( u \) down the first column instead of across the row, and the partials w.r.t. \( v \) down the second column instead of across the first row and still get the same determinant. In other words, we also have

\[
\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]

Similarly, given equations \( x = x(u,v,w), y = y(u,v,w), z = z(u,v,w) \) then

\[
\frac{\partial (x,y,z)}{\partial (u,v,w)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix}
\]

And so on.

### example 3

If

\[
\begin{align*}
x &= 3v^2 + u + 5w \\
y &= 2uv \\
z &= 3vw
\end{align*}
\]

then

\[
\frac{\partial (x,y,z)}{\partial (u,v,w)} = \begin{vmatrix}
1 & 2v & 0 \\
6v & 2u & 3w \\
5 & 0 & 3v
\end{vmatrix}
\]

**Note** It doesn't matter whether you start by putting \( 1 \ 2v \ 0 \) across the first row, as I did, or down the first column. You'll get the same answer either way.

I'll expand across row 1 (to take advantage of one of the zero entries):

\[
\frac{\partial (x,y,z)}{\partial (u,v,w)} = 1 \begin{vmatrix} 2u & 3w & -2v \\ 0 & 3v & 6v \\ 6v & 3w & 5 & 3v \end{vmatrix}
\]

\[
= 6uv - 2v(18v^2 - 15vw)
\]

\[
= 6uv - 36v^3 + 30vw
\]

The Jacobian is a scalar field. There is a Jacobian value at each point.
dA in a not necessarily orthogonal $u,v$ coordinate system

Fig 6 shows an area element in an arbitrary $u,v$ coordinate system.

![Fig 6](image)

\[ dA = \frac{\partial (x,y)}{\partial (u,v)} \left| \begin{array}{l} du \\ dv \end{array} \right| \] (Fig 6)

The vertical bars in (5) mean the absolute value of the determinant inside. The formula in (5) is more general than the formula in (1) because (5) holds whether the coordinate system is orthogonal or not.

**why (5) works**

Look at the left edge of the area element in Fig 6. It has parametric equations

\[
x = x(u,v_0) \\
y = y(u,v_0) \\
u_0 \leq u \leq u_0 + du \quad \text{(the parameter is $u$)}
\]

By (2) in Section 2.0, the left edge is approximated by arrow

\[ du \left( \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} \right) \] (Fig 7)

Similarly the upper edge of the patch is approximated by the arrow

\[ dv \left( \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j} \right) \]
Now you can think of the patch as almost the parallelogram in Fig 7 determined by the two arrows in which case

\[
dA = ||\text{cross product of arrows}|| \quad \text{(the double bars mean norm)}
\]

where

\[
\text{cross product} = du \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right) \times dv \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right)
\]

\[
= du \ dv \begin{vmatrix} 0 & 0 & \frac{\partial x}{\partial u} \\ 0 & 0 & \frac{\partial y}{\partial u} \\ \partial x/\partial v & \partial y/\partial v & 0 \end{vmatrix} \quad \text{(vertical bars mean determinant)}
\]

\[
= du \ dv \left( 0, 0, \frac{\partial (x,y)}{\partial (u,v)} \right)
\]

The norm of a vector of the form \((0,0,k)\) is \(|k|\) (vert bars mean abs value) so

\[
dA = |\frac{\partial (x,y)}{\partial (u,v)}| \ du \ dv \quad \text{(vertical bars mean absolute value)} \quad \text{QED}
\]

example 4 (dA in polar coords again)
If \(x = r \cos \theta, y = r \sin \theta\) then

\[
\frac{\partial (x,y)}{\partial (r,\theta)} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r
\]

\[
dA = |\frac{\partial (x,y)}{\partial (r,\theta)}| \ dr \ d\theta
\]

\[
= |r| \ dr \ d\theta
\]

\[
= r \ dr \ d\theta \quad \text{(since the polar coord r is always \geq 0)}
\]

example 5
If a \(u,v\) coordinate system is defined by
\[x = u^2 + v, \ y = 3u + 2v\]

then

\[
dA = \left| \frac{\partial (x,y)}{\partial (u,v)} \right| \ du \ dv
\]

\[
= \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} \ du \ dv \quad \text{(outer bars are abs value, inner bars are det)}
\]

\[
= |4u - 3| \ du \ dv \quad \text{(bars are abs value signs)}
\]

warning
Get the absolute value signs right. And be consistent.
Here are some wrong ways to write in example 5.

(a) \[
\left| \frac{\partial (x,y)}{\partial (u,v)} \right| = \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} \quad \text{wrong}
\]

It's wrong because the left side is the absolute value of the Jacobian determinant but the right side is the determinant written without absolute values. For a correct version of (a), either write
\[ \frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} \quad \text{(absolute value signs on neither side)} \]

or write

\[ \left| \frac{\partial (x,y)}{\partial (u,v)} \right| = \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} \quad \text{(absolute value signs on both sides)} \]

(b) \( \text{dA} = \frac{\partial (x,y)}{\partial (u,v)} \text{ du dv} \quad \text{wrong} \)

It's wrong because it leaves out the absolute value signs around the Jacobian. The correct version is in (5):

\[ \text{dA} = \left| \frac{\partial (x,y)}{\partial (u,v)} \right| \text{ du dv} \]

(c) \[ \begin{vmatrix} 2u & 3 \\ 1 & 2 \end{vmatrix} = |4u - 3| \quad \text{wrong} \]

It's wrong because the left side is a determinant without absolute value signs and the right side suddenly includes absolute value signs.

taking absolute values correctly

If \( \text{Jac} = -6u \) then it is not necessarily true that \( |\text{Jac}| = 6u \); i.e., the absolute value of \(-6u\) is not necessarily \(6u\). If \( u \) is negative then \( |6u| \) happens to be \(-6u\). So unless you know whether \( u \) is positive or negative the best you can do is write \( |\text{Jac}| = |-6u| \) (same as \( |6u| \)). On the other hand if \( \text{Jac} = -6u^2 \) then \( |\text{Jac}| = 6u^2 \) because \( 6u^2 \) is always \( \geq 0 \).

dV in a not necessarily orthogonal \( u,v,w \) coordinate system

\[ \text{dV} = \left| \frac{\partial (x,y,z)}{\partial (u,v,w)} \right| \text{ du dv dw (Fig 7)} \quad \text{(6)} \]

The vertical bars in (6) mean the absolute value of the determinant inside.
The formula in (6) is more general than the formula in (2) because (6) holds whether the coord system is orthogonal or not.
why (6) works
The volume element in Fig 8 is (almost) a parallelepiped determined by arrows

\[
\begin{align*}
\text{vel}_u \ du &= \left( \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \right) du \\
\text{vel}_v \ dv &= \left( \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k} \right) dv \\
\text{vel}_w \ dw &= \left( \frac{\partial x}{\partial w} \mathbf{i} + \frac{\partial y}{\partial w} \mathbf{j} + \frac{\partial z}{\partial w} \mathbf{k} \right) dw
\end{align*}
\]

By the rule in Fig 2 in §5.0,

\[
dV = | \text{vel}_u \ du \cdot \text{vel}_v \ dv \times \text{vel}_w \ dw| \quad \text{(vert bars are abs value)}
\]

\[
= |\text{vel}_u \cdot \text{vel}_v \times \text{vel}_w| \ du \ dv \ dw \quad \text{(pull out the positive scalars)}
\]

\[
= \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\
\frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w}
\end{vmatrix} \ du \ dv \ dw \quad \text{by (1) in §5.0}
\]

(outer bars are still abs value, inner bars are det signs)

\[
= \left| \frac{\partial (x,y,z)}{\partial (u,v,w)} \right| \ du \ dv \ dw \quad \text{(vert bars are abs value)} \quad \text{QED}
\]

sign of the Jacobian in 3-space
Let

\[
\begin{align*}
x &= x(u,v,w) \\
y &= y(u,v,w) \\
z &= z(u,v,w)
\end{align*}
\]

define a u,v,w coordinate system superimposed on the usual (righthanded) x,y,z coord system in 3-space.

Look at the Jacobian \( \frac{\partial (x,y,z)}{\partial (u,v,w)} \).

If the Jacobian is positive at a point then at that point the vectors \( e_u, e_v, e_w \) in that order are a righthanded triple (Fig 9) meaning that if you curl the fingers of your right hand like \( e_u \) turning into \( e_v \), your thumb points more or less like \( e_w \) (i.e., makes an acute angle with \( e_w \)).

And if the Jac is negative then \( e_u, e_v, e_w \) are lefthanded (Fig 10) meaning \( e_w \) makes an obtuse angle with your thumb.
Here's why this works.

\[
\begin{vmatrix}
\frac{\partial (x,y,z)}{\partial (u,v,w)} = \\
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\
\frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w}
\end{vmatrix}
\]

(7) \(v^e_u \cdot v^e_v \times v^e_w\)

Fig 11 shows a typical \(v^e_v\) and \(v^e_w\) and their cross product.

If the Jac is positive then the dot product in (7) is positive and \(v^e_u\) makes an acute angle with the cross product \(v^e_v \times v^e_w\). I drew a typical such \(v^e_u\) in the picture.

Now just look at Fig 11 (and curl your fingers etc) to see that the following are all righthanded triples.

\[\begin{align*}
&v^e_v, v^e_w, v^e_u \\
&v^e_u, v^e_v, v^e_w \quad \text{(that's the one I was after)} \\
&v^e_w, v^e_u, v^e_v
\end{align*}\]

footnote

You can look at pictures to see that cyclically permuting three vectors doesn't change "handedness". For instance, if \(p, q, r\) are a lefthanded triple then so are \(r, p, q\) and \(q, r, p\).

So the vectors \(e^e_u, e^e_v, e^e_w\) (and cyclic permutations of them) are righthanded because they point like \(v^e_u, v^e_v, v^e_w\) (the only difference is that the \(e\)'s are unit vectors).

sign of the Jacobian in 2-space

Let

\[
\begin{align*}
x &= x(u,v) \\
y &= y(u,v)
\end{align*}
\]

define a \((u,v)\) coord system superimposed on the usual (righthanded) \((x,y)\) coord system in 2-space.

Look at the Jacobian \(\frac{\partial (x,y)}{\partial (u,v)}\).

If the Jacobian is positive at a point (Fig 12) then as the fingers of your right hand curl like \(e^e_u\) turning into \(e^e_v\), your thumb points out of the page.

If the Jac is negative (Fig 13) as your fingers curl like \(e^e_u\) turning into \(e^e_v\), your thumb points into the page.
Here's why this works. Think of \( e_u \) and \( e_v \) in Figs 12 and 13 as if they were in the \( x,y \) plane in a righthanded \( x,y,z \) coord system \textit{in 3-space}, where \( k \) comes out of the page at you (Figs 14, 15).

Algebraically this amounts to using the \( u,v,z \) cylindrical coord system where

\[
\begin{align*}
x &= x(u,v) \\
y &= y(u,v) \\
z &= z 
\end{align*}
\]

In Fig 14, the vectors \( e_u, e_v, k \) are righthanded so \( \frac{\partial(x,y,z)}{\partial(u,v,z)} \) is positive.

In Fig 15, the vectors \( e_u, e_v, k \) are lefthanded so \( \frac{\partial(x,y,z)}{\partial(u,v,z)} \) is negative.

But you will see in problem #7 that \( \frac{\partial(x,y,z)}{\partial(u,v,z)} = \frac{\partial(x,y)}{\partial(u,v)} \).

So back in Fig 12, \( \frac{\partial(x,y)}{\partial(u,v)} \) is positive and back in Fig 13, \( \frac{\partial(x,y)}{\partial(u,v)} \) is negative.

\textbf{Warning}

\texttt{Order counts!}

\[
\frac{\partial(x,y,z)}{\partial(a,q,m)} \text{ means } \begin{vmatrix}
x/\partial a & y/\partial a & z/\partial a \\
x/\partial q & y/\partial q & z/\partial q \\
x/\partial m & y/\partial m & z/\partial m
\end{vmatrix}.
\]

The derivatives w.r.t. \( a \) are in the first row (or col), derivatives w.r.t. \( q \) are in the second row (or col) and derivatives w.r.t. \( m \) are in the third row (or col).

If this Jacobian is positive then arrows \( e_a, e_q, e_m \) \textit{in that order} will be righthanded.

Jacobians that use different orders, such as

\[
\begin{align*}
\frac{\partial(x,y,z)}{\partial(q,a,m)} & \quad \frac{\partial(x,y,z)}{\partial(m,q,a)} & \quad \frac{\partial(x,y,z)}{\partial(m,a,q)}
\end{align*}
\]

all have the same absolute value, and no matter what order you use it is still true that

\[
dV = |\text{Jacobian}| \ da \ dq \ dm
\]

but the signs of the Jacobians may differ, depending on how you scramble the rows or cols of the Jacobian determinant.
the Jacobian in an orthogonal coord system

Suppose the \( u,v \) coord system is orthogonal. Then the area mag factors in (1) and (5) must agree. So

\[
\frac{\partial(x,y)}{\partial(u,v)} = h_u \ h_v
\]

Furthermore, the Jacobian is positive if the coord system is righthanded and negative if the coord system is lefthanded so all in all

\[
\frac{\partial(x,y)}{\partial(u,v)} = \begin{cases} 
  h_u \ h_v & \text{if the system is righthanded} \\
  -h_u \ h_v & \text{if the system is lefthanded}
\end{cases}
\]

Similarly, suppose the \( u,v,w \) coord system is orthogonal. Then the vol mag factors in (2) and (6) must agree. So

\[
\frac{\partial(x,y,z)}{\partial(u,v,w)} = h_u \ h_v \ h_w
\]

In particular,

\[
\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{cases} 
  h_u \ h_v \ h_w & \text{if the system is righthanded} \\
  -h_u \ h_v \ h_w & \text{if the system is lefthanded}
\end{cases}
\]

PROBLEMS FOR SECTION 5.1

1. Find the Jacobian \( \frac{\partial(x,y)}{\partial(u,v)} \) for the parabolic coordinate system two ways.

2. Let \( x = u + v, \ y = u - v \).
   (a) Sketch a piece of \( u,v \) coord paper.
   (b) Find \( e_u \) and \( e_v \).
   (c) Is the \( u,v \) system orthogonal?
   (d) Find \( \frac{\partial(x,y)}{\partial(u,v)} \).
   (e) Sketch an area element and find its area.
   (f) Is the \( u,v \) system lefthanded or righthanded.

3. Let \( x = 3u^2 + 7v, \ y = 2v \).
   (a) Is the \( u,v \) coord system orthogonal.
   (b) Find \( dA \) in the \( u,v \) system.
   (c) Is the \( u,v \) coord system lefthanded or righthanded.

4. Start at a point \( P \) with cylindrical coordinates \( r, \theta, z \).
   (a) Sketch the volume element swept out by changing \( r \) by \( dr \), \( \theta \) by \( d\theta \) and \( z \) by \( dz \).
   Label \( P, \ r, \ \theta, \ z, \ dr, \ d\theta, \ dz \) in the picture and find \( dV \).
   (b) Suppose \( r \) changes by \( dr \) and \( z \) change by \( dz \) while \( \theta \) stays fixed. What is traced out (draw a picture) and what is its length/area/volume.

5. (boring) Find \( \frac{\partial(x,y,z)}{\partial(u,v,w)} \) if \( x = v^3 w, \ y = 2u + 3v + 4w, \ z = uvw \).
6. Define a $u,v$ coordinate system with the equations
   \[ x = x(u,v) \]
   \[ y = y(u,v) \]

   Call it the blue coord system.

   Then the blue \textit{cylindrical} coord system is defined by
   \[ x = x(u,v) \]
   \[ y = y(u,v) \]
   \[ z = z \]

   Start computing $\frac{\partial (x,y)}{\partial (u,v)}$ and $\frac{\partial (x,y,z)}{\partial (u,v,z)}$ and just go far enough to show that they are equal.
**SECTION 5.2 MAPPINGS**

**mappings from a u,v plane to an x,y plane**

I've been thinking of equations of the form

\[
x = x(u,v), \quad y = y(u,v)
\]
as relating the usual Cartesian x,y coordinate system with a new u,v coordinate system all in the same plane.

They can also be thought of as a mapping from a plane with a u,v Cartesian coordinate system to a plane with an x,y Cartesian coordinate system.

For example, from one point of view the equations

\[
x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad v \geq 0,
\]
define parabolic coordinates. Now think mapping! Point u=2, v=1 (Fig 1a) maps to the image point \(x = \frac{3}{2}, y = 2\) (Fig 1b). The half-line \(u = 2, v \geq 0\) (Fig 2a) maps to the curve in the x,y plane with parametric equations

\[
x = \frac{1}{2}(4 - v^2), \quad y = 2v, \quad v \geq 0,
\]
This curve has plain equation \(x = \frac{1}{2}(4 - \frac{1}{4}y^2)\) (Fig 2b). It's the top half of a parabola. The half-parabola is called the image of the half-line; the half-line is called the pre-image of the half-parabola.

**area and volume magnification factors**

This is just a restatement of ideas in the preceding section from a mapping point of view.

Think of

\[
x = x(u,v) \\
y = y(u,v)
\]
as a mapping from a u,v plane to an x,y plane.

Create a rectangle in the u,v plane by changing u by du and v by dv (Fig 3a)
The rectangle has area \(du \, dv\).
In the image world (Fig 3b), the image area \(dA\) is given by

\[
dA = \left| \frac{\partial (x,y)}{\partial (u,v)} \right| \, du \, dv
\]
The value of the Jacobian changes from point to point so its absolute value is a "local" mag factor only. Two \(du \, dv\) patches in different parts of the old plane may be magnified differently when they are mapped to the new plane.
Similarly think of
\[ x = x(u,v,w) \]
\[ y = y(u,v,w) \]
\[ z = z(u,v,w) \]
as mapping from a \( u,v,w \) space to an \( x,y,z \) space.

Make a little box in a \( u,v,w \) world by making changes \( du, dv, dw \). The box has volume \( du \cdot dv \cdot dw \).

Its image in \( x,y,z \) space has volume \( dV \) where
\[
    dV = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du \cdot dv \cdot dw
\]

**example 1**

The equations \( x = r \cos \theta, y = r \sin \theta \) can be thought of as mapping from an \( r,\theta \) plane to an \( x,y \) plane.

\[
    \frac{\partial(x,y)}{\partial(r,\theta)} = r
\]

The area mag factor is \( |r| = r \). The little rectangle with area \( dr \cdot d\theta \) in the old plane (Fig 4a) maps to a patch with area \( r \cdot dr \cdot d\theta \) in the new plane (Fig 4b).

**example 2**

Here's a clever way to find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = R^2 \). Let

\[
    u = \frac{x}{a}, \quad v = \frac{y}{b}
\]

where \( a \) and \( b \) are the positive square roots of \( a^2 \) and \( b^2 \) respectively.

These equations map from an \( x,y \) plane to a \( u,v \) plane. The ellipse in the \( x,y \) plane maps to the circle \( u^2 + v^2 = R^2 \) in the \( u,v \) plane.

Rewrite the equations as

\[
    x = au, \quad y = bv
\]

which map from a \( u,v \) plane to an \( x,y \) plane. Then
\[
    \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab
\]

\[
    \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = |ab| = ab
\]

The area mag factor is \( ab \) and it's constant (same magnification at every point) so any region in the \( u,v \) plane should be magnified by \( ab \) when it is mapped to the \( x,y \) plane.

Circle area in the \( u,v \) plane \( \times ab \) = ellipse area in the \( x,y \) plane.

Ellipse area \( = ab \times \pi R^2 = \pi ab R^2 \)
You can stick with the original mapping equations $u = x/a$, $v = y/b$. Then

$$\frac{\partial (u,v)}{\partial (x,y)} = \left| \begin{array}{cc} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{array} \right| = \frac{1}{ab}$$

$$| \frac{\partial (u,v)}{\partial (x,y)} | = | \frac{1}{ab} | = \frac{1}{ab} \quad \text{(because } a \text{ and } b \text{ are positive)}$$

The Jacobian is constant so regions in the $x,y$ plane are magnified by $1/ab$ when they are mapped to the $u,v$ plane.

Ellipse area in $x,y$ plane $\approx \frac{1}{ab} = \text{circle area in } u,v$ plane

Ellipse area $= ab \times \text{circle area} = ab \times \pi R^2 = \pi ab R^2$

**sign of the Jacobian of a mapping from a $u,v$ plane to an $x,y$ plane**

Let

\begin{equation}
(2) \quad x = x(u,v), \quad y = y(u,v)
\end{equation}

define a mapping from a plane with a Cartesian $u,v$ coord system to a plane with a Cartesian $x,y$ coordinate system.

There are two ways to walk around a point in any plane — clockwise and ccl.

Look at the Jacobian $\frac{\partial (x,y)}{\partial (u,v)}$.

Let $P$ be a point in the $u,v$ plane and let $P'$ be its image in the $x,y$ plane.

Walk on a small loop around $P$.

If the Jacobian is positive at $P$ then the image walker goes the same way around $P'$ (i.e., either both walkers go clockwise or both go ccl). We say orientation is preserved.

If the Jac is negative at $P$ then the image walker goes the opposite way around $P'$ (if one goes clockwise then the other goes ccl). We say orientation is reversed.

Here's why.

Suppose $P$ has coordinates $u_o, v_o$ in the $u,v$ plane.

Look at the closed counterclockwise path $PABCP$ in Fig 5.

I'll find the image path in the $x,y$ plane. Think of the equations in (2) as determining a $u, v$ coordinate system in the $x, y$ plane (Fig 6). The image point $P'$ is the intersection of the $v$-curve $u = u_o$ and the $u$-curve $v = v_o$. I happened to draw $P'$ in quadrant II — doesn't matter where it is. I also drew the hypothetical $v$-curve $u = u_o + du$. 

\begin{center}
\textbf{FIG 5}
\end{center}
The image walker goes from P' to A' in Fig 6, in the direction of increasing \( u \) corresponding to the PA path in Fig 5, and then turns in the direction of increasing \( v \) corresponding to path AB in Fig 5. But does she turn left or right from A'.

Consider the Jacobian \( \frac{\partial(x,y)}{\partial(u,v)} \) at point \( u=u_0 \), \( v=v_0 \), i.e., at point P'. If it's positive then \( e_v \) must be ahead of \( e_u \) (see the preceding section), Fig 7 fleshes out to Fig 6 and the image walker turns left at A' to go in the increasing \( v \) direction. If the Jac is negative at P' then \( e_u \) must be ahead of \( e_v \), Fig 6 becomes Fig 8 and the image walker turns right at A' to go in the increasing \( v \) direction.

---

**example 3**

Let \( x = u^2 + v^2 \), \( y = u^3 v \). This is a mapping from a \( u,v \) plane to an \( x,y \) plane.

\[
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 3u^2 v \\ 3u^2 v & u^3 \end{vmatrix} = 2u^4 - 6u^2 v^2.
\]

If \( u=1, v=2 \) the Jacobian is -22. A small region around point \( u=1, v=2 \) in the \( u,v \) plane maps to a region around point \( x=5, y=2 \) with about 22 times as much area. And if a particle moves clockwise around the point \( u=1, v=2 \), its image moves ccl around the image point \( x=5, y=2 \) in the \( x,y \) plane.

**inverse Jacobians**

Suppose the equations

\[
(3) \quad x = x(u,v), \quad y = y(u,v)
\]

can be solved for \( u \) and \( v \) to get equations

\[
(4) \quad u = u(x,y), \quad v = v(x,y).
\]

Then
why the inverse rule in (5) works

The equations in (3) define a mapping from a u,v plane to an x,y plane. The equations in (4) define the inverse mapping from the x,y plane back to the u,v plane.

Suppose \( \frac{\partial(x,y)}{\partial(u,v)} \) is -7 at point \((u_0, v_0)\). Then a region in the u,v plane around point \((u_0, v_0)\) gets magnified sevenfold when it is mapped to a region around the image point \((x_0, y_0)\) in the x,y plane, and orientations are reversed (e.g., a clockwise walk maps to a ccl walk). So the inverse mapping which takes a region around \((x_0, y_0)\) back to a region around \((u_0, v_0)\) should also reverse orientation (it maps a ccl walk back to a clockwise walk) and should shrink area to 1/7-th its original size. So at point \((x_0, y_0)\) you should have \( \frac{\partial(u,v)}{\partial(x,y)} = -\frac{1}{7} \).

example 4

Find \( \frac{\partial(x,y)}{\partial(u,v)} \) if \( u = x^3 + 2y, \ v = x^3 \).

method 1

First solve for x and y:

\[
\begin{align*}
x &= \frac{3}{\sqrt{v}}, \quad y = \frac{1}{2}(u - v)
\end{align*}
\]

Then find the Jacobian directly.

\[
\begin{align*}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix}
0 & 1 \\
\frac{1}{3}v^{-2/3} & -\frac{1}{2}
\end{vmatrix} = -\frac{1}{6}v^{-2/3}
\end{align*}
\]

method 2

\[
\begin{align*}
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix}
3x^2 & 3x^2 \\
2 & 0
\end{vmatrix} = -6x^2
\end{align*}
\]

(7) \( \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-6x^2} \)

The answers look different but if you replace the x in (7) by \( \frac{3}{\sqrt{v}} \) from (6) you’ll see that they agree.

PROBLEMS FOR SECTION 5.2

1. Let \( u = 2x + y, \ v = 3x - 4y \). Find \( \frac{\partial(x,y)}{\partial(u,v)} \) twice.

(a) Without solving for x and y explicitly
(b) By solving for x and y explicitly
2. Let \( u = x^2 - y^2, \quad v = x^2 + y^2, \quad x \geq 0, \quad y \geq 0 \).

(a) Find \( \frac{\partial (u,v)}{\partial (x,y)} \).

(b) Solve the equations for \( x \) and \( y \) to get the equations for the inverse mapping. Then use the inverse equations to find \( \frac{\partial (x,y)}{\partial (u,v)} \) directly to see if it really is the reciprocal of the Jacobian in part (a).

3. Let \( x = u + v - 2w^3, \quad y = u + 2v, \quad z = v \).

(a) Find \( \frac{\partial (x,y,z)}{\partial (u,v,w)} \).

(b) Use part (a) to find \( \frac{\partial (u,v,w)}{\partial (x,y,z)} \) at point \( x=61, \quad y=12, \quad z=5 \).

4. The equations \( u = 2x - y, \quad v = x - 2y \) define a mapping from the \( x,y \) plane to a \( u,v \) plane.

   The diagram shows a triangular region \( ABC \) in the \( x,y \) plane.

   (a) Find the area of the region \( ABC \)

   (b) Use Jacobians to find the area of the image region in the \( u,v \) plane.

   (c) Actually find the image of the region and then find its area directly to see if the answer agrees with part (b).

5. Find the volume inside the surface \( (x + y + z)^2 + (2y + 5z)^2 + 9z^2 = R^2 \) by choosing a mapping that turns the surface into a sphere.
6. For each of these mappings, predict the Jacobian \( \frac{\partial (x,y)}{\partial (u,v)} \) by thinking about magnification and orientation and then compute the Jacobian to confirm your prediction.

(a) \( x = u + 2, \ y = v - 1 \)
(b) \( x = -u, \ y = v \)

7. When you compute \( \frac{\partial (p,q)}{\partial (r,s)} \) directly what letters do you expect to find in your answer; i.e., is the Jacobian a function of \( p \) and \( q \) or is it a function of \( r \) and \( s \) or all four?
SECTION 5.3 DOUBLE AND TRIPLE INTEGRALS IN A NEW COORDINATE SYSTEM

double integration in a (not necessarily orthogonal) u,v coordinate system

Here's how to find \( \int_{\text{region}} f(x,y) \, dA \) using a new u,v coordinate system.

(I) Use the equations relating x and y with u and v to replace all x's and y's in the integrand with u's and v's.

(II) Instead of using \( dA = dx \, dy \), use the area element created by changing u by du and v by dv:

\[
\frac{\partial (x,y)}{\partial (u,v)} \, du \, dv
\]

In the special orthog case, \( dA = h_u \, h_v \, du \, dv \).

(III) Put in u,v limits that sweep out the region; i.e.,

\[
\int_{\text{largest } v}^{\text{smallest } v} \int_{\text{exiting } u \, \text{boundary}}^{\text{entering } u \, \text{boundary}} du \, dv \quad \text{or} \quad \int_{\text{largest } u}^{\text{smallest } u} \int_{\text{exiting } v \, \text{boundary}}^{\text{entering } v \, \text{boundary}} du \, dv
\]

Integrating in u,v coordinates is also referred to as making a change of variables or substitution.

A similar idea works for triple integrals.

example 1

I'll find \( \int_{\text{region in Fig 1}} x^2 y^2 \, dA \).

The region of integration is messy in x,y coordinates because the lower and upper boundaries each consist of two curves, and the left and right boundaries each consist of two curves. Another method is to use a u,v coordinate system defined by

\[
u = \frac{x^3}{y}, \quad v = \frac{y^3}{x}
\]

because in this system the region of integration is the "rectangle" where

\[4 \leq u \leq 9, \quad 16 \leq v \leq 25 \quad (\text{Fig 2}).\]

In fact, instead of thinking of a u,v coordinate system you can think of the equations in (1) as mapping from an x,y plane to a u,v plane. The image of the region in Fig 1 is literally the rectangle in Fig 3.

Then
\[ \frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} 3x^2/y & -y^3/x^2 \\ -x^3/y^2 & 3y^2/x \end{vmatrix} = 8xy \]

\[ \frac{\partial (x,y)}{\partial (u,v)} = \frac{1}{8xy} \text{ by the inverse rule} \]

\[ \left| \frac{\partial (x,y)}{\partial (u,v)} \right| = \left| \frac{1}{8xy} \right| = \frac{1}{8xy} \text{ since } 8xy \text{ is positive in the region} \]

\[ \int_{\text{region}} x^2 y^2 \, dA = \int_{u=4}^{9} \int_{v=16}^{25} x^2 y^2 \frac{1}{8xy} \, dv \, du \]
\[ = \int_{u=4}^{9} \int_{v=16}^{25} \frac{1}{8} xy \, dv \, du \]

From (1), \(uv = x^2 y^2\) so \(xy = \sqrt{uv}\) (take the positive square root since \(xy \geq 0\) in the region). So

\[ \int_{\text{region}} x^2 y^2 \, dA = \frac{1}{8} \int_{u=4}^{9} \int_{v=16}^{25} \sqrt{uv} \, dv \, du \]

inner integral = \[ \sqrt{u} \frac{2}{3} \frac{v^{3/2}}{v} \bigg|_{u=4}^{25} \int_{v=16}^{25} = \frac{2}{3} 61 \sqrt{u} \]

outer integral = \[ \frac{1}{8} 3 \frac{2}{3} 61 \frac{2}{3} u^{3/2} \bigg|_{u=4}^{9} = \frac{1159}{18} \]

warning

1. When you integrate in a u,v coord system, dA is \[ \frac{\partial (x,y)}{\partial (u,v)} \, du \, dv, \not \]
   \[ \frac{\partial (u,v)}{\partial (x,y)} \, du \, dv \]. One way to remember the correct version is to picture it
   "canceling" like this:
   \[ \frac{\partial (x,y)}{\partial (u,v)} \, du \, dv \]

2. And don't forget the absolute value signs around the Jacobian.

3. When you put in u,v limits of integration, a lower limit of integration should always be smaller than the corresponding upper limit.

   For instance, you should not have \[ \int_{v=6}^{4} \].

4. In example 1, the substitution \(u=x^3\) would not be useful.

   The substitution \(u=x^3/y\) was useful because two of the bounding curves of the region have equations of the form \(x^3/y = \text{constant}\).

   One way to spot a potentially useful substitution is to rewrite the equations of the bounding curves so that all the letters are on one side.

5. \[ \int_{\text{region in Fig 1}} x^2 y^2 \, dA \] is not the area of the region.

   Only \[ \int_{\text{region in Fig 1}} 1 \, dA \] is the area of the region.

   So when you compute a double integral, don't refer to it as an "area".
So what good is the integral in example 1. If you insist on a geometric interpretation the integral is the volume in 3-space under the graph of \( z = x^2y^2 \) and over the region in Fig 1. (Well, it's actually the volume above the \( xy \) plane minus the volume below the \( xy \) plane but in this case the graph happens to lie entirely above the \( xy \) plane so the integral really is a plain volume.)

The integral is also for instance the total mass of the region if its density is \( x^2y^2 \) grams per square cm.

PROBLEMS FOR SECTION 5.3

1. Switch to a good \( u,v \) coord system and find \( \int (x^2 - y^2) \, dA \) over the region in quadrant III bounded by \( y = x + 3, \ y = x - 3, \ xy = 16, \ xy = 4 \).

2. Switch to a useful \( u,v \) coord system and set up \( \int x^2 \, dA \) over the region bounded by lines \( y = 2x + 3, \ y = 2x - 5, \ y = -x + 1, \ y = -x + 6 \). (If you do the integral directly, you have to divide the region of integration into three parts.)

3. Make a substitution (switch to \( u 's \) and \( v 's \)) and find \( \int \sin xy \, dA \) over the region in quadrant I bounded by the curves \( xy = 2, \ xy = 1, \ y = x \) and \( y = 4x \).

4. Make a substitution to set up \( \int e^{(y-x)/(y+x)} \, dA \) over the region ABC in the diagram.

footnote: Mathematica can do the original integral directly so a substitution is not necessary. We're just practicing.

```
In[24]:= Integrate[Exp[(y - x)/(y + x)], {x, 0, 1}, {y, 0, 1 - x}]

Out[24]=
\[\frac{-1 + e}{4 e}\]
```

5. Set up \( \int xy \, dA \) over the region in the diagram between the ellipses \( \frac{x^2}{4} + \frac{y^2}{25} = 1 \) and \( \frac{x^2}{4} + \frac{y^2}{25} = 2 \).

(a) using the coordinate system defined by \( x = 2r \cos \theta, \ y = 5r \sin \theta, \ r \geq 0, \ 0 \leq \theta \leq 2\pi \)

(b) using the substitution \( u = x/2, \ v = y/5 \)
6. Look at \( \int \frac{1}{x + 2y} \, dA \) over the region bounded by the axes and lines \( x + 2y = 4 \), \( x + 2y = 1 \).
(a) Set it up in the \( x,y \) system.
(b) Set it up in some nice \( u,v \) coordinate system.

7. The diagram shows the parallelepiped determined by the arrows
\[ \vec{v}_1 = (-1,0,4), \quad \vec{v}_2 = (1,4,-2), \quad \vec{v}_3 = (-2,1,3) \]
attached to the origin. Switch to a convenient coord system and set up \( \int y^2 \, dv \) over the solid parallelepiped.
Remember (Section 1.0) that if a plane has normal vector \((a,b,c)\) and goes through point \((x_0,y_0,z_0)\) then the plane has equation \( a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \).

8. Let \( N \) be a positive integer.
Look at \( \int (x^2 - y^2)^N \, dA \) over the region in the diagram. It can be set up in \( x,y \) coordinates if you divide up the region of integration, say into a top and bottom half:
\[
\int (x^2 - y^2)^N \, dA = \int_{y=-1/2}^{y=1/2} \, \int_{x=-y}^{x=y} (x^2 - y^2)^N \, dx \, dy + \int_{y=0}^{y=-1} \, \int_{x=y}^{x=-y} (x^2 - y^2)^N \, dx \, dy
\]
But when I tried to do these integrals with Mathematica I found that it could do the integration for any specific value of \( N \), but not for the abstract \( N \). So find some new \( u,v \) coord system in which you can actually compute the integral.
REVIEW PROBLEMS FOR CHAPTER 5

1. The region in the diagram is bounded by circles $x^2 + y^2 = 10$, $x^2 + y^2 = 11$ and hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 2$. Switch to a convenient coordinate system to set up $\int x \, dA$ over the region so that you can avoid having to split up the region of integration.

2. Let $n$ and $m$ be positive integers. Then

$$\int (x + y)^n (3x - y)^m \, dA \text{ over the region in the diagram}$$

$$= \int_0^0 \int_{y=-3-x}^{x=-3} (x + y)^n (3x - y)^m \, dy \, dx$$

But Mathematica couldn't do this integration.
Set it up with a substitution.

3. Find the area enclosed by the curve $(x+y)^2 + (x+6y)^2 = 100$

4. Look at the region $R$ in the diagram, bounded by parabolas $y=5x^2$, $y=x^2$, the circle with radius 2 and center $(2,0)$ and the circle with radius 1 and center $(1,0)$.
(a) Find a $u,v$ coordinate system that turns the region into a "rectangle".

(b) Make up an integrand so that your class (of dopes) can do $\int_R \ldots \, dA$ over the region easily with the substitution in part (a).

(c) Find some more rigged integrands so that $\int_R \ldots \, dA$ can be done easily.
5. Let \( x = u^2 - v \), \( y = uv \) define a new \( u,v \) coordinate system. If \( u \) starts at 0 and changes by \( du \), \( v \) starts at -10 and changes by \( dv \), how much area is swept out in the \( x, y \) plane.

6. Since the spherical coord system is righthanded, \( \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \) should come out to be \( h_\rho h_\phi h_\theta \). Compute the Jacobian to check.

7. Fill in the blanks.
   (a) I have an \( x,y \) Cartesian coordinate system as usual. If \( x = x(u,v) \), \( y = y(u,v) \) and \( u \) changes by \( du \) and \( v \) changes by \( dv \) then the _____ traced out is ______.

   (b) If \( x = x(u,v,w) \), \( y = y(u,v,w) \), \( z = z(u,v,w) \) and \( u \) changes by \( du \) and \( v \) changes by \( dv \) and \( w \) changes by \( dw \) then the _____ traced out is ______.
SUMMARY OF MAG FACTORS FROM CHAPTERS 2,3,5 (REVIEW THIS FOR THE FINAL EXAM)

- Let
  \[ x = x(u,v) \]
  \[ y = y(u,v) \]

  These equations can be thought of as mapping from a plane with a \( u,v \) Cartesian coordinate system to a plane with an \( x,y \) Cartesian coordinate system.

  They can also be thought of as defining a \( u,v \) coordinate system on top of the usual Cartesian coordinate system in an \( x,y \) plane.

  If \( u \) changes by \( du \) and \( v \) changes by \( dv \) then an area element is swept out in the \( x,y \) plane. Its area is given by
  \[
  dA = \frac{\partial(x,y)}{\partial(u,v)} \, du \, dv
  \]

  Special case: If the \( u,v \) coordinate system is orthogonal then \( dA = h_u h_v \, du \, dv \)

- Let
  \[ x = x(u,v,w) \]
  \[ y = y(u,v,w) \]
  \[ z = z(u,v,w) \]

  These equations can be thought of as mapping from a space with a \( u,v,w \) Cartesian coordinate system to a space with an \( x,y,z \) Cartesian coordinate system.

  They can also be thought of as defining a \( u,v,w \) coordinate system on top of the usual Cartesian coordinate system in an \( x,y,z \) space.

  If \( u \) changes by \( du \), \( v \) changes by \( dv \) and \( w \) changes by \( dw \) then a volume element is swept out in the \( x,y,z \) space. Its volume is given by
  \[
  dV = \frac{\partial(x,y,z)}{\partial(u,v,w)} \, du \, dv \, dw
  \]

  Special case: If the \( u,v,w \) coordinate system is orthogonal then \( dV = h_u h_v h_w \, du \, dv \, dw \)

- Let
  \[ x = x(t) \]
  \[ y = y(t) \]

  These are parametric equations of a curve in 2-space.

  If \( t \) changes by \( dt \) then a little curve you could call an arclength element is swept out. Its arclength is given by
  \[
  ds = \|v\| \, dt \quad \text{where} \quad v = (dx/dt, dy/dt)
  \]

- Let
  \[ x = x(t) \]
  \[ y = y(t) \]
  \[ z = z(t) \]

  These are parametric equations of a curve in 3-space.

  If \( t \) changes by \( dt \) then an arclength element is swept out. Its arclength is given by
  \[
  ds = \|v\| \, dt \quad \text{where} \quad v = (dx/dt, dy/dt, dz/dt)\]
Let
\[ x = x(u,v) \]
\[ y = y(u,v) \]
\[ z = z(u,v) \]

These are parametric equations of a surface in 3-space (there are three variables, \( x, y, z \) and two parameters \( u,v \)). If \( u \) changes by \( du \) and \( v \) changes by \( dv \) then a surface area element is traced out on the surface. Its surface area is given by

\[ dS = \|n\| \, du \, dv \]

where \( n = \text{vel}_u \times \text{vel}_v \)

Special cases: See Sections 4.3 and 4.4

PROBLEMS

1. Let
   \[ x = x(u,v) \]
   \[ y = y(u,v) \]
   \[ z = z(u,v) \]
   
   If \( u \) changes by \( du \) while \( v \) is fixed what geometric thing is traced out? Where? And what is the size of the geometric thing?

2. Let
   \[ x = x(u,v) \]
   \[ y = y(u,v) \]
   
   If \( u \) changes by \( du \) while \( v \) is fixed then what geometric thing is traced out? Where? And what is the size of the geometric thing?

3. Let
   \[ x = \rho \sin \phi \cos \theta \]
   \[ y = \rho \sin \phi \sin \theta \]
   \[ z = \rho \cos \phi \]
   
   If \( \rho \) changes by \( d\rho \), \( \theta \) changes by \( d\theta \), \( \phi \) changes by \( d\phi \) then what geometric thing is traced out. Draw a picture of it and find its size.

4. Let
   \[ x = \rho \sin \phi \cos \theta \]
   \[ y = \rho \sin \phi \sin \theta \]
   \[ z = \rho \cos \phi \]
   
   If \( \rho \) change by \( d\rho \) and \( \theta \) changes by \( d\theta \), what geometric thing is traced out. Where? Draw a picture of it and find its size.

5. Let
   \[ x = \rho \sin \phi \cos \theta \]
   \[ y = \rho \sin \phi \sin \theta \]
   \[ z = \rho \cos \phi \]
   
   If \( \rho \) changes by \( d\rho \) and \( \phi \) changes by \( d\phi \) what geometric thing is traced out. Where? Draw a picture of it and find its size.

6. Let
   \[ x = r \cos \theta \]
   \[ y = r \sin \theta \]
   \[ z = z \]
   
   If \( \theta \) changes by \( d\theta \) and \( z \) changes by \( dz \) then what geometric thing is traced out. Where? Draw a picture of it and find its size.