YE OLDE COMPUTATIONS

1. Introduction

This write-up is going to be my attempt to understand the computations done by Hirosi Toda in the olden days – the content of the book [To]. The results are compiled on pages 186-191, with \( \pi_{n+k}(S^n) \) being computed for \( k < 20 \). This work is rather remarkable, since these groups are notoriously difficult to compute. I guess, that is why computing them can present interest to some people. I may wander off the book’s course and talk of something that (hopefully) fits into the context, but I will, at least, make an attempt to stick to the book.

The first part of the book tries to take somewhat of a pure approach of using the bracketing operations and the EHP spectral sequence. After some point it does become necessary to introduce new (cohomological) techniques, as with the increase of dimension the computations become harder. Anyway, this seems like a good thing to get into at leisure time.

2. Basics and Three Elements

The most trivial computation of homotopy groups, that one can think of, is the computation of \( \pi_1(S^1) \). Now usually people show that this group is isomorphic to \( \mathbb{Z} \) via direct methods, since van Kampen theorem (at least the classical one) does not work for the circle and waiting to develop the covering space theory would take too much time. Here is a silly argument: \( S^1 \) is a topological group, so \( \pi_1(S^1) \) is abelian, i.e. isomorphic to \( H_1(S^1) \), which can be easily computed to be \( \mathbb{Z} \). Check a standard text. The generator of the group is the class of the identity map \( \mathit{v}_1 \).

We can get slightly fancy, by asking what is \( \pi_m(S^1) \) for the rest of the values of \( m \), i.e. \( m \neq 1 \). We know that \( \mathbb{R} \) is the universal cover of \( S^1 \). With an easy use of covering space theory we can instantly determine that the map \( \pi_m(\mathbb{R}) \to \pi_m(S^1) \) induced from the projection is an isomorphism, and since \( \mathbb{R} \) is contractible, then \( \pi_m(S^1) \) is trivial for \( m \geq 2 \).

Nextly, let me address something that was written in the introduction. One may ask why to write \( \pi_{n+k}(S^n) \), and what is the significance of \( k \)? There are two reasons. The first one is that \( k \) is only interesting when it is non-negative. Indeed, if \( k \) is negative then the group is trivial. If you are not an opponent of differential geometry, then you can argue as follow: given a map \( S^{n+k} \to S^n \) (with \( k \) negative, of course) homotope it to something smooth; then the entire image consists of critical points, which have to have measure 0 by Sard’s Theorem, which means that the map is not surjective, and there is a factorization \( S^{n+k} \to \mathbb{R}^n \to S^n \). The other reason stems from Fruedenthal Suspension Theorem, which states that once \( n > k + 1 \), then \( \pi_{n+k}(S^n) \) stabilizes via the suspension map. Let us remind the reader if given a pointed map \( f : X \to Y \), then we can form its suspension my smashing the left on the with the identity map on \( S^1 \). We obtain a map \( \Sigma f : \Sigma X \to \Sigma Y \). This procedure respects homotopies, so we get suspensions of homotopy classes. If we fix \( k \), we only need to determine the homotopy groups for finitely many \( n \). The stabilized group \( \text{collim}_n \pi_{n+k}(S^n) \) is called the stable \( k \)-stem and it is usually written as \( \pi_k^S \). Fruedenthal Suspension Theorem also tells us that the suspension map \( \pi_{2k+1}(S^{k+1}) \to \pi_{2k+2}(S^{k+2}) \) is surjective. This means that \( \pi_1(S^1) \to \pi_n(S^n) \) is surjective. This map is also injective, since \( m \in \pi_1(S^1) \) maps to a degree \( m \) map of \( S^n \), which cannot be null. Thus, \( \pi_n(S^n) \simeq \mathbb{Z} \). This group is also generated by the identity map \( \mathit{v}_n \), since \( \Sigma^0 \mathit{v}_1 = \mathit{v}_n \). This shows that \( \pi_0^S \simeq \mathbb{Z} \) with the generator \( \mathit{v} \), the image of \( \mathit{v}_1 \).

I would like to pause here and share a slick proof of Fruedenthal Suspension Theorem using the Serre Spectral Sequence. The theorem states that if \( X \) is \( n \)-connected and \( n \geq 1 \), i.e. if \( \pi_k(X) = 0 \) for \( 1 \leq k < n \), then the suspension map \( \Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X) \) is an isomorphism for \( k < 2n - 1 \) and is a surjection for \( 2n - 1 \). The important thing is to understand what realizes the suspension map. Recall the \((\Sigma, \Omega)\) adjunction on topological spaces. Because of this adjunction there is the unit map \( X \to \Omega \Sigma X \). If one plays around with the adjunction a little, one can realize that if we hit this map with \( \pi_k \) functor we obtain the suspension map.

We would like to show that the homotopy fiber \( F \) of this map is \((2n - 1)\)-connective. If we show that the homology of the fiber vanishes in this range, the connectivity would follow from Hurewicz’ theorem. This could be demonstrated by using the Serre Spectral Sequence, once we understand the homology of \( \Omega \Sigma X \) in terms of homology \( \Sigma X \) and the effect of the unit map on homology. To do this we set up the path-loop fibration \( \Omega \Sigma X \to \Omega \Sigma X \to \Sigma X \). We set up the cohomology Serre Spectral Sequence, that has the \( E^2 \)-page \( H_k(\Sigma X, H_n((\Omega \Sigma X; A))) \), where \( A \) is an abelian group, converging to the homology of a point with coefficients in \( A \). We know that if \( 0 < k < n + 1 \), then \( \pi_k(\Sigma X) = 0 \) by Hurewicz’ theorem. Similarly, \( \pi_k((\Omega \Sigma X) = 0 \) for \( 0 < k < n \). Based on this observation we can draw the spectral sequence (we omit writing the coefficients due to space considerations)
The area in the spectral sequence marked blue consists of zeros only. As a consequence of that, all the transgressions are of the form $H^k(\Omega\Sigma X; A) \to H^k(X; A) \simeq H^{k+1}(\Sigma X; A)$ and are isomorphisms in the range $n \leq k \leq 2n - 1$ for all coefficient groups $A$. This looks fairly promising. There is a problem, however: we do not have an interpretation of these transgressions. We would like to say that these transgressions are induced by the unit map $X \to \Omega\Sigma X$. Usually, tracking down these differentials geometrically is pretty hard. We have another tool in our hands: the naturality. This allows us to reduce to the universal case $X = H(A, k)$, an Eilenberg-MacLane space. Checking in this case is easy.

The next step is to consider the following fibration $F \to X \to \Omega\Sigma X$. The Serre Spectral Sequence with the fact that the edge homomorphism realizes the map $H^k(\Omega\Sigma X) \to H^k(X)$, shows that $H^k(F; A) = 0$ for $1 \leq k < 2n - 1$. The group $H^{2n-1}(F; A)$ may be non-zero, since it can support a differential. This actually implies that $H_k(F) = 0$ for $1 \leq k < 2n - 1$, which implies that $\pi_k(F) = 0$ for $1 \leq k < 2n - 1$, which is precisely what we wanted.

Now let’s look at the unstable 1-stem, i.e. when $k = 1$ and $n = 2$. Heinz Hopf noticed that $\mathbb{C}P^1 \simeq S^2$ and that $S^3$ is really the set of unit vectors in $\mathbb{C}^2$, and we tautologically get a map, $S^3 \to S^2$ by sending $(z_1, z_2)$ to the line $[z_1 : z_2]$ in $\mathbb{C}^2$. This map is, in fact, a fiber bundle with generic fiber being $S^1$. Thus, we have a fiber sequence $S^1 \to S^3 \to S^2$, which produces a long exact sequence in homotopy

$$\cdots \to 0 \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to 0 \to \cdots$$

which instantly implies that $\pi_3(S^2) \simeq \mathbb{Z}$ and it is generated by the class of the Hopf map, which we will call $\eta_2$. There is another way of seeing why $\eta_2$ is non-trivial, and, in fact, it will give us a way of showing that the stable image of the Hopf map, $\eta$, is non-zero (essential). We simply note that the cofiber of $\eta_2$ is $\mathbb{C}P^2$, which has a cohomology ring $\mathbb{Z}[x]/(x^3)$, where $|x| = 2$. This shows already that $\eta$ is non-trivial, since otherwise the cohomology ring would’ve had a trivial multiplication. This implies that $H^*(\mathbb{C}P^2; F_2) = F_2[x]/(x^3)$, so we can see that $Sq^2x = x^2$, so $Sq^2(\Sigma^nx) = \Sigma^nx(x^2)$, which could not have happened if $\Sigma^{n-2}\eta_2 = \eta_4$ were 0. Thus, it survives in $\pi_3^h$. One can show that $\pi_3^h$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, but we will get to it later.

There are two other fibre bundles of similar type. The next can be constructed using quaternions, $\mathbb{H}$. Consider the space $\mathbb{H}^2$ on which $\mathbb{H}$ acts on the right. Then we take the quotient space of $\mathbb{H}^2 - \{0\}$ under the left action of $\mathbb{H}^\times$ on this space. We will call this space $\mathbb{H}P^1$ – the space of left 1-dimensional $\mathbb{H}$-submodules of $\mathbb{H}^2$. This space can be shown to be homeomorphic to $S^3 \simeq \mathbb{H} \cup \{\infty\}$ by sending $(z_1, z_2)$ to $z_1\bar{z}_2^{-1}$. We can consider the composite $S^7 \to \mathbb{H}^2 - \{0\} \to S^4$. We will call the homotopy class of this map $\nu_4$. The cofiber of this map can be identified with $\mathbb{H}P^2$ – the space of left 1-dimensional $\mathbb{H}$-submodules of $\mathbb{H}^3$. Then one can easily see $H^*(\mathbb{H}P^2; \mathbb{Z}) = \mathbb{Z}[x]/(x^3)$ and $|x| = 4$, say by means of Poincaré duality. Just like in the previous case, this shows that $\nu_4$ is not trivial. Now let $f : S^7 \to S^4$ be any continuous map. Then if $Cf$ is the cofiber of $f$, then we have a cofiber sequence of form $S^3 \to Cf \to S^8$. One can show that as a $\mathbb{Z}$-module $H^*(Cf)$ is generated by $x$ and $y$ of degrees 4 and 8, respectively. We can fix a generator of $H^8(S^8; \mathbb{Z})$ and let $y$ be the pullback of this generator. The generator of $H^4(S^4; \mathbb{Z})$ is determined up to a sign. We see that $x^2 = h(f)y$ for some integer $h(f)$. The integer $h(f)$ depends on the homotopy class of $f$ and not $f$ itself. This defines a map $h : \pi_7(S^4) \to \mathbb{Z}$, called the Hopf invariant map. This can be checked to be a group homomorphism, and since $h(\nu_4) = 1$, we see that $h$ is surjective. This implies that $\nu_4$ generates a free $\mathbb{Z}$ summand of $\pi_7(S^4)$.
We can define the Hopf invariant more generally, $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$ if $n > 1$. The procedure is the same, take a map $f : S^{2n-1} \to S^n$ and define $h$ using cup product structure on the cohomology of its cofiber. It is known that for $n$ even, the image of $h$ contains $2\mathbb{Z}$; we'll show this later. In particular, this means that $\pi_{4n-1}(S^{2n})$'s contain free $\mathbb{Z}$ summands, and along with $\pi_n(S^n)$'s these are the only infinite homotopy groups of spheres. The elements $\eta_2$, $\nu_4$ are elements that have Hopf invariant 1. There is another element obtained via similar construction. This time the stars of the show are the Cayley octonions, $\mathbb{O}$. The trouble with this algebra is that it is not associative, so we don't quite have a notion of an $\mathbb{O}$-module, $\mathbb{O}^2$, as in the previous cases. However, we do have a unit and left, right inverses, i.e. it is a division algebra. We can identify $S^8$ as the set of elements in $\mathbb{O}^2$, $(z_1, z_2)$, such that $|z_1|^2 + |z_2|^2 = 1$. Then we can write the map $S^8 \to \mathbb{O} \cup \{\infty\} \approx S^8$ by sending $(z_1, z_2)$ to $z_1^{-1}z_2$, where we consider the left inverse. The homotopy class of this map, $\sigma_8$, has Hopf invariant 1. Here is a strategy of showing this. Notice that $\mathbb{O}$ can be considered a complex vector space. We can define a complex vector bundle over $S^8 \approx \mathbb{O} \cup \{\infty\}$ by taking $S^8 \times \mathbb{O}^2$ considering the subspace of point $\{(z, z_1, z_2)\mid z = z_1^{-1}z_2\}$. Call this space $\zeta$. Now the projection $\zeta \to S^8$ defines a fiber bundle, which can be reduced to complex vector bundle. Then one notes that the cofiber of the map representing $\sigma_8$ is homeomorphic to $T_\zeta$, the Thom space of this vector bundle. Complex vector bundles are automatically orientable, so $H^*(T_\zeta) \approx \mathbb{Z}[x]/(x^3)$ for $|x| = 8$. Thus, we have a free $\mathbb{Z}$ summand in $\pi_{15}(S^8)$.

We will talk more about these Hopf invariant 1 elements, especially since the ones mentioned above are the only examples of such maps. As a consequence, one can demonstrate that there are no division algebras over $\mathbb{R}$, except for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

3. The James Construction

My first goal here would be to construct the EHP sequence. Along the way we will need something called the James construction. This construction offers a model for an otherwise unruly space $\Omega \Sigma X$, provided that $X$ is sufficiently nice. Here by model we, of course, mean something weakly equivalent.

There are a couple of reasons why we may suspect that $\Omega \Sigma X$ may have a nice model. First reason is that its cohomology is strongly linked to the cohomology of $X$. The Serre spectral sequence has an $E_2$-page, $H^{*+1}(X; H^*(\Omega \Sigma X))$, and we would like to make it converge to the cohomology a point. The analysis of this spectral sometimes gives complete answer for $H^*(\Omega \Sigma X)$. For instance, a very simple computation will reveal that $H^*(\Omega S^{2n+1}) \approx \mathbb{Z}[x]/(x^n)$ with $|x| = 2n$, and $H^*(\Omega S^{2n}) \approx \mathbb{Z}[x]/(x^n)$ with $|x| = 2n - 1$ and $|x| = 4n - 2$. Here $\Gamma[x]$ is the divided polynomial algebra on $x$, i.e. an algebra generated by $\{x_n\}_{n=1}^\infty$ subject to relation $nx_n = x^n$. We can even determine $H^*(\Omega(V, S^{2n}))$, which is simply $\bigotimes H^*(\Omega S^{2n})$. Anyway, some work with the Serre spectral sequence will demonstrate the nice the resulting cohomology rings for $\Omega \Sigma X$.

The second reason for the suspicion is more substantial and, in fact, lies at the heart of the James construction. Notice that $\Omega \Sigma X$ is a topological monoid because it is a loop space. Besides that we also get a natural map $X \leftarrow \Omega \Sigma X$ that maps the basepoint to the identity as consequence of the adjunction $(\Sigma, \Omega)$ in $\mathcal{Top}_*$. Now the reasonable thing at this point would be to look at a space that is a “free” monoid over $X$. It is not difficult to guess what it ought to be: we first take $\coprod_{n=0}^{\infty} X^n$, which is basically the space of all the words with letters in $X$, and then we identify two words if one can be obtained from the other by adding or remove the basepoint of $X$ to the sequence. Actually, this is it − the James construction. We will write $JX$ for this space. Now how far is exactly $JX$ from $\Omega \Sigma X$? As a matter of fact, they are the same for all practical purposes, i.e. they are weakly equivalent if $X$ is a connected CW-complex.

So we need to understand how exactly they are weakly equivalent. One way to show this, would be by constructing a map $JX \rightarrow \Omega \Sigma X$ and then show that it is a weak equivalence. However, the naive guesses for the maps end up not being continuous, so in order to make it continuous one needs to make choices, such as a metric on the space $X$. We will not pursue this path. There is a relatively cleaner way of constructing an equivalence, but we would no longer have a map between the spaces; instead, we will have a zigzag. There is a more general framework of operads, algebras over which model all sort of loopspaces, including the infinite ones. We will not talk about that in here, since it’ll take us too far afield, but one should know where to look for a more general approach. Essentially, what we’ll be mimicking some of the constructions in [May1].

The zigzag will be the simplest possible one: $JX \leftarrow LX \rightarrow \Omega \Sigma X$, except I have to tell you what LX is and what the maps are. One ought to think of LX as some sort of fattening of JX that helps us to map into $\Omega \Sigma X$ more or less canonically. Let $L(n)$ be the space of unordered $n$-tuples of closed subintervals of I that have non-intersecting interia. We will write the elements of $L(n)$ as $\{I_1, \ldots, I_n\}$ in correspondence with their order on the unit interval. Consider the following coproduct $\bigoplus_{n=0}^{\infty} L(n) \times X^n$. The space $L(n)$ will be a quotient of this space. The equivalence relation is almost the same as that for JX, except we
can add or remove any appropriate interval times a point into the sequence, e.g. \( \{I_1, I_2, I_3\} \times (x_1, *, x_3) \simeq \{I_1, I_2\} \times \{x_1, x_3\} \simeq \{I_1, I_3, I_4\} \times (x_1, x_3, *) \), where \( I_2 < I_3 < I_4 \). Now we can construct a natural map \( LX \to JX \). There exist projections \( L(n) \times X^n \to X^n \). These projections descend to a map from \( LX \) to \( JX \). We would like to show that this map is a weak equivalence.

We need to say a couple of things about \( L \). One thing is that \( LX \) is an \( H \)-space, which is good, since we would it to be homotopy equivalent to a topological monoid. There is a map \( L(2) \times L(n) \times L(m) \to L(n + m) \), which sends \( \{J_1, J_2\} \times \{I_1^{(1)}, \ldots, I_n^{(1)}\} \times \{I_1^{(2)}, \ldots, I_m^{(2)}\} \) to \( \{J_1^{(1)}, \ldots, J_n^{(1)}, J_1^{(2)}, \ldots, J_m^{(2)}\} \). \( I_1^{(k)} \) are linearly scaled versions of \( I_1^{(k)} \) so that they fit into \( J_k \). Choosing a point in \( L(2) \) gives us a set of maps \( \mu_{m,n} : L(n) \times L(m) \to L(n + m) \). This gives us, in an obvious fashion, a multiplication on \( \prod_{n=0}^\infty L(n) \times X^n \), and it descends to \( LX \). This map is clearly unital with the basepoint being the unit the class of the sole point of \( L(0) \times X^0 \). The multiplication has another (among many others) virtue: it is homotopy associative. We will sketch why it ought to be true. There is a triple product \( L(3) \times L(n) \times L(m) \times L(k) \to L(n + m + k) \), defined just as above. If we fix a point in \( L(2) \) to define the multiplications, then there are two points of \( L(3) \) defined the two options for triple multiplication. However, all of \( L(n)'s \) are contractible, and in particular, \( L(3) \) is contractible. This means that the two multiplications that we have can be deformed to one another.

Let’s finally get to the promised equivalence. We filter \( LX \) and \( JX \) by \( F_n \) and \( P_n \), respectively, which are the images of \( L(n) \times X^n \) and \( X^n \), respectively. The above map clearly respects this filtration. The situation is better, since \( F_n/P_{n-1} \simeq L(n) \times P_n/P_{n-1} \) and the induced map to \( P_n/P_{n-1} \) is the projection. Thus, we have two filtration homology spectral sequences with \( E^1 \)-pages, \( \tilde{H}_*(F_n/F_{n-1}) = \tilde{H}_*(L(n) \times P_n/P_{n-1}) \) and \( \tilde{H}_*(P_n/P_{n-1}) \), and the induced map is by projection. Thus, the map between the spectral sequences is an isomorphism on the \( E^1 \)-page, since \( L(n)'s \) are contractible. This implies that we have an isomorphism of spectral sequences, so \( \tilde{H}_*(LX) \to \tilde{H}_*(JX) \) is an isomorphism. This implies that the map is a weak equivalence, since both \( LX \) and \( JX \) are \( H \)-spaces.

Now we need to construct the map \( LX \to \Omega \Sigma X \), and then show that it is a weak equivalence. We can define map \( L(n) \times X^n \to \Omega \Sigma X \) as follows. Note that the preimage under projection of a point in \( x \in X \) in \( X \times I \) can be thought of as a path. After passing to \( \Sigma X \) by quotienting \( X \times I \) this path turns into a loop based at the basepoint call it \( l_x \). Now say we are given \( \{I_1, \ldots, I_n\} \times (x_1, \ldots, x_n) \); we can form a loop in \( \Sigma X \) by sending everything outside of the given intervals in \( I \) to the basepoint and by tracing \( l_x \), as it goes over \( I_n \). Hopefully, it makes some sense. Some careful checking will verify that these maps define correctly a map from \( LX \) to \( \Omega \Sigma X \).

To show that this map is a weak equivalence, we will need to construct the following diagram

\[
\begin{array}{ccc}
LX & \longrightarrow & EX \\
\downarrow & & \downarrow \\
\Omega \Sigma X & \longrightarrow & \Sigma X \\
\end{array}
\]

with the sequence on top a quasifibration and \( EX \) is weakly contractible. The weak equivalence instantly follows from 5-lemma. We first consider the coproduct \( \coprod_{n=1}^\infty L(n) \times X^{n-1} \times CX \), where \( CX \) denotes the reduced cone on \( X \). The we impose the equivalence relation on the subspace \( \coprod_{n=1}^\infty L(n) \times X^{n-1} \times X \), just as we did previously in order to obtain the space \( LX \). The resulting space will be denoted by \( EX \). If we consider the image of \( L(n) \times X^{n-1} \times CX \) in \( EX \), which we will call \( G_n \), then this sets up a filtration of \( EX \).

We claim that \( \tilde{H}_*(G_n/G_{n-1}) = 0 \) if \( n \geq 1 \). We have to notice that

\[
G_n/G_{n-1} \simeq L(n) \times X^{n-1} \times CX \left/ \left( \bigcup_{i=0}^{n-2} L(n) \times X^i \times X^{n-i-2} \times CX \right) \right. \cup (L(n) \times X^{n-1}).
\]

However, the inclusion of the latter subspace is a homotopy equivalence, so the homology of the quotient is trivial. The \( E^1 \)-page of the filtration spectral sequence is \( \tilde{H}_*(G_s/G_{s-1}) \), so everything is 0 except the \((0,0)\)-entry, where we have \( \tilde{H}_0(CX/\emptyset) \simeq \tilde{H}_0(CX) \simeq \mathbb{Z} \). Thus, \( EX \) has a single connected component, and trivial homology in positive dimensions. To show that it is contractible we need to demonstrate that \( EX \) is an \( H \)-space. Recall that if we fix an element in \( L(2) \) we obtain products \( L(n) \times L(m) \to L(n + m) \). To define a product completely we have to define a map of form \( CX \times CX \to X \times CX \). It is kind of a drag that we are using the reduced cone, since we have to use local metric on \( X \). Let \( d : X \to \mathbb{R} \) be a continuous map such that \( d(x) \geq 0 \) for all \( x \in X \), and \( d(x) = 0 \) if and only if \( x = * \). If \( X \) is a CW-complex then we
definitely have such a map (actually normality is enough to have such a map). The product is defined by sending \((x_1, t_1) \times (x_2, t_2)\) to \(x_1 \times x_2, d(x_1)t_1t_2\). We can now define a product by combining these maps in the obvious way that makes sense. We need to verify that this product is homotopy unital. The unit for the multiplication will be the image of \(I \times * \in L(1) \times CX\). It is straightforward to check that this is true. This concludes that \(EX\) is weakly contractible.

We need to define two maps from \(EX\) – one to \(\Sigma X\) and the other to \(\Sigma \Sigma X\), and one from \(LX\) to \(EX\). It is clear that, in fact, \(LX\) includes into \(EX\). Also, note that we have maps \(L(n) \times X^{n-1} \times CX \to CX \to \Sigma X\). These maps descend to a map \(EX \to \Sigma X\). In fact the preimage of the basepoint of \(\Sigma X\) is precisely \(LX\). The map \(EX \to \Sigma \Sigma X\) is almost like the map \(LX \to \Sigma \Sigma X\), except we don’t complete to a loop – the last factor \(CX\) makes it possible. The diagram above commutes.

To finish the proof we need to show that the map \(EX \to \Sigma X\) is a quasifibration. We will need to talk about quasifibrations in general. Recall that a map \(p : E \to B\) is a quasifibration if it is surjective and the maps \(\pi_n(E, p^{-1}(b), e) \to \pi_n(B, b)\) are isomorphisms for all \(b \in B\), \(e \in p^{-1}(b)\) and \(n\), or equivalently, if the set theoretic fibers are weakly equivalent to homotopy fibers for all choices of basepoints. We need a couple of facts about quasifibrations. The first fact goes as follows: if \(U, V\) be subsets of \(B\), such that their interia cover \(B\), and \(p^{-1}(U) \to U\), \(p^{-1}(V) \to V\) and \(p^{-1}(U \cap V) \to U \cap V\) are quasifibrations, then so is \(p\). This statement is rather technical to prove, so I am going to skip it. The second fact is that if \(E : E \times I \to E\) is a weak deformation retraction to a subspace \(E'\) that covers a weak deformation retraction \(F : B \times I \to B\) to a subspace \(B'\), and the map \(E' \to B'\) is a quasifibration and the maps \(p^{-1}(b) \to p^{-1}(F_1(b))\) is a weak equivalence for all \(b \in B\), then \(p\) is a quasifibration. This one is fairly straightforward, so I am going to skip it.

We put these two facts together to show that we have a quasifibration. We split \(CX\) into two subspaces: one of them \(U = CX - \{\ast\}\) and the other a subspace \(V\) that weakly deformation retracts to \(X\). One can easily check that \(p^{-1}(U) \to U\) is actually a trivial fiber bundle with fiber an appropriate quotient of \(\prod_{n=1}^{\infty} L(n) \times X^{n-1}\), where \(p\) is the map \(EX \to \Sigma X\). We will write \(LX\) for this quotient. The same statement applies to \(U \cap V\). The existence of \(V\) is basically guaranteed by the fact that CW-complexes are locally contractible. Now let \(H : V \times I \to V\) be a weak deformation retraction to \(X\). Then it induces a homotopy \(F\) on \(\prod_{n=1}^{\infty} L(n) \times X^{n-1} \times V\) and on the image of this subspace in \(EX\). This homotopy in \(EX\) covers a deformation retraction of an open subset to a point in \(\Sigma X\), so the time 1 projection is a quasifibration. Now if \(b \in \Sigma X\), then the map \(\tilde{LX} \simeq p^{-1}(b) \to p^{-1}(F_1(b))\) is the one obtained by quotienting the map \(\prod_{n=1}^{\infty} L(n) \times X^{n-1} \to \prod_{n=1}^{\infty} L(n) \times X^n\) that is formed via \(H_1(b)\) section \(X^{n-1} \to X^n\). If \(X\) is connected, we can consider without the loss of generality that \(H_1(b) = \ast\). Note that \(\tilde{LX}\) maps to \(JX\) and the following diagram commutes

\[
\begin{array}{ccc}
\tilde{LX} & \longrightarrow & LX \\
\downarrow & & \downarrow \\
JX & \longrightarrow & \ast
\end{array}
\]

The slanted arrows are weak equivalences; therefore, so is the horizontal one. This concludes the proof.

4. The EHP Spectral Sequence

Before looking at the exactness of any sequences let us first try to interpret some of the maps that we are going to deal with. First off, there is the \(m\)-skeleton inclusion \(S^m \to JS^m\), and so there is a corresponding map \(S^m \to \Omega S^{m+1}\). The question is, what element in \(\pi_m(\Omega S^{m+1}) \simeq \pi_{m+1}(\Omega S^{m+1})\) does it represent? Consider the induced map \(\pi_m(S^m) \to \pi_m(\Omega S^{m+1})\), the image of the class of the identity map will correspond to the element that we are looking for. By Hurewicz isomorphism we can look at the induced map on homology: \(H_m(S^m) \to H_m(\Omega S^{m+1})\), which is an isomorphism, since we had an \(m\)-skeleton inclusion. So the class of the map \(S^m \to \Omega S^{m+1}\) is the adjoint of the identity. This implies that the induced map \(\pi_m(S^m) \to \pi_m(\Omega S^{m+1}) \simeq \pi_{m+1}(\Omega S^{m+1})\) is the suspension map. We denote this map by \(\Sigma\), but in the olden days it was written as \(E\), which stood for “Einüngung”, the German word for suspension. The two letters look similar anyway.

The next map is \(H : JS^m \to JS^{2m}\) defined by the formula \(H(x_1, \ldots, x_n) = (x_1 \wedge x_2, x_1 \wedge x_3, \ldots, x_{n-1} \wedge x_n)\), where the ordering of the entries in the image is done lexicographically. One can verify that this map is well defined. So this gives a map \(H : \Omega S^{m+1} \to \Omega S^{2m+1}\). It is trickier to say the effect of this map on the homotopy. However, the induced map on \(2m\)-the homotopy \(\pi_{2m+1}(S^{m+1}) \to \pi_{2m+1}(S^{2m+1}) \simeq \mathbb{Z}\) coincides up to sign with Hopf invariant map defined previously. This justifies the use of letter \(H\) for the map. The
proof of this statement is not incredible difficult, but it did not strike as being intuitive. One can find it in [Ha1], pp. 47-48 of chapter 1.

Now assume that m is odd. Then we claim that \( S^m \xrightarrow{\varepsilon} \Omega S^{m+1} \xrightarrow{\varepsilon} \Omega S^{2m+1} \) is actually a fiber sequence, and its associated long exact sequence is called EHP sequence. There will more exact sequences that will be called EHP. Let \( FH \) denote the fiber of the Hopf map. Then we get a map \( S^m \rightarrow FH \), which is an isomorphism on \( \pi_n \). Thus, \( H_*(S^m) \rightarrow H_*(FH) \) is also an isomorphism. The same works for cohomology: \( H^n(FH) \rightarrow H^n(S^m) \) is an isomorphism. Now the Serre spectral sequence for the fibration has an \( E_2 \)-page \( \pi_* \Omega S^{2m+1}; H^*(FH) \), which looks like this up to the horizontal line at \( m \)

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & Z & Z \times Z & Z \times \frac{Z}{2} & \cdots & \\
0 & Z & Z \times Z & Z \times \frac{Z}{2} & \cdots & \\
2m & 4m & \cdots & & & \\
\end{array}
$$

since \( H^n(FH) \simeq Z \), by Hurewicz and universal coefficients theorem. Clearly, \( z \) is a non-vanishing permanent cycle, so it is the image of the generator of \( H^n(\Omega S^{m+1}) \). The element \( x_1 \) is also a non-vanishing permanent cycle, since \( x_1 \) pulls back to the generator of \( H^2(\Omega S^{m+1}) \). However, then the rest of depicted elements are also permanent cycle. None of them can be trivial, since otherwise the cup products won’t match. What makes this work is that the cup product \( H^n(\Omega S^{m+1}) \times H^2(\Omega S^{m+1}) \rightarrow H^{n+2}(\Omega S^{m+1}) \) is an isomorphism. Basically this tells us this is all to the spectral sequence. Thus, we see that \( S^m \rightarrow FH \) is a homology, and whence, a homology equivalence.

Let \( m = 2n - 1 \), so our fiber sequence is \( S^{2n-1} \rightarrow \Omega S^{2n} \rightarrow \Omega S^{4n-1} \). We are interested in understanding the connecting map \( \Omega^2 S^{4n-1} \rightarrow S^{2n-1} \); more specifically, we are interested in its effect on homotopy \( Z \simeq \pi_{4n-1}(\Omega S^{4n-1}) \rightarrow \pi_{4n-3}(S^{2n-1}) \). Notice that we have the isomorphism \( \pi_{4n-3}(\Omega^2 S^{4n-1}) \simeq \pi_{4n-1}(JS^{2n-1}, S^{2n-1}) \simeq \pi_{4n-1}(JS^{2n-1}/S^{2n-1}) \), the last one being a consequence of the homotopy excision theorem ([May2], p. 81-82). Thus, \( \pi_{4n-1}(JS^{2n-1}, S^{2n-1}) \) is generated by the inclusion of the \( 4n-1 \)-skeleton, \( (D^{4n-2}, S^{4n-3}) \rightarrow (JS^{2n-1}, S^{2n-1}) \). The map \( S^{4n-3} \rightarrow S^{2n-1} \) in the pair is what we are looking for, and it is clear from the James construction that this is nothing but the Whitehead product, \( [\tau_{2n-1}, \tau_{2n-1}] \). This fact is the last piece of the puzzle behind the name EHP: “P” stands “product”.

Let us play around with it a little bit before moving forward. So we plug \( m = 1 \) and get \( S^1 \rightarrow \Omega^2 S^2 \rightarrow \Omega S^3 \). Then we have a long exact sequence in homotopy

$$
\cdots \rightarrow \pi_n(S^1) \rightarrow \pi_{n+1}(S^3) \rightarrow \pi_{n+1}(S^3) \rightarrow \pi_{n-1}(S^1) \rightarrow \cdots
$$

Thus, \( \pi_n(S^2) \simeq \pi_n(S^3) \) if \( n \geq 3 \), but we knew this from the Hopf fibration \( S^1 \rightarrow S^3 \rightarrow S^2 \). Note that if we continue the fibration we obtain a map \( \Omega S^2 \rightarrow S^3 \), which is an isomorphism on \( \pi_1 \). Then we consider the product map \( \Omega S^2 \rightarrow S^3 \times \Omega S^3 \) and it is a weak equivalence. Notice that the cohomologies of both of the spaces are the same, so that’s nice. Now let’s plug \( m = 3 \) and get \( S^3 \rightarrow \Omega S^4 \rightarrow \Omega S^7 \). Recall that the construction of the map \( \nu_4 : S^7 \rightarrow S^4 \). If we inspect it we will notice that it is a fiber bundle with generic fiber \( S^3 \). So we get a map \( \Omega S^4 \rightarrow S^3 \). This gives another weak equivalence \( \Omega S^4 \rightarrow S^3 \times \Omega S^7 \). We get another splitting like this if we set \( m = 7 \), \( \Omega S^8 \simeq S^7 \times \Omega S^{15} \). Actually, another way to see evidence of these splitting to be true is the fact that \( S^1 \), \( S^3 \) and \( S^7 \) are H-spaces. This implies that \( [\tau_{2n-1}, \tau_{2n-1}] = 0 \) for \( n = 1, 2, 4 \). The maps \( \Omega S^{2n} \rightarrow S^{2n-1} \) for \( n = 1 \) and \( 2 \) can be constructed using the monoidal structure on \( S^{2n-1} \); indeed, being a monoid implies that there is map \( JS^{2n-1} \rightarrow S^{2n-1} \). Unfortunately, there are no more splittings like these.

However, this splitting occurs once we invert 2. Let us keep the integral setting for just a moment. Note that we have a map \( \Omega [\tau_{2n}, \tau_{2n}] : \Omega S^{4n-1} \rightarrow \Omega S^{2n} \). Let us denote \( [\tau_{2n}, \tau_{2n}] \) by \( \tau \) for future convenience. We are interested in the composition \( H(\Omega \tau) : \Omega S^{4n-1} \rightarrow \Omega S^{4n} \). Let us consider the effect of this map on \( \pi_{4n-2} \). The image of \( \tau_{4n-1} \in \pi_{4n-1}(S^{4n-1}) \) in \( \pi_{4n-1}(S^{2n}) \) is \( \tau \), obviously. As we’ve discussed earlier \( \tau \in \pi_{4n-1}(S^{2n}) \) in its turn gets sent to \( \pm h(\tau) \in \pi_{4n-1}(S^{4n}) \) by the map \( H \). The following diagram will help us:

$$
\begin{array}{cccccc}
S^{4n-1} & \rightarrow & S^{2n} & \vee & S^{2n} & \leftarrow & S^{2n} \times S^{2n} & \xrightarrow{j} & S^{4n} \\
S^{4n-1} & \xrightarrow{\tau} & S^{2n} & \xrightarrow{f} & C_\tau & \xrightarrow{g} & S^{4n} \\
\end{array}
$$
The horizontal sequences are cofiber sequences. Let \( x \) be the generator of \( H^{2n}(C/\tau) \) and \( y \) the generator of \( H^{4n}(C/\tau) \). Let \( y \) be the image of \( z \in H^{4n}(S^{4n}) \) and let \( w \) be equal to \( f^*(\chi) \), i.e. a generator of \( H^{2n}(S^{2n}) \). Then we can view \( H^{*}(S^{2n} \times S^{2n}) \) as \( \mathbb{Z}[w_1, w_2]/(w_1^2, w_2^2) \) and \( H^{*}(S^{2n} \vee S^{2n}) \) as \( \text{(the ring)} \mathbb{Z}[w_1] \oplus \mathbb{Z}[w_2] \). By the definition of the Hopf invariant, \( x^2 = h(\tau) \) (disregard the sign issue). Now, \( i^* k^*(x) = \nabla^* f^*(x) = \nabla^*(w) = w_1 + w_2 \) and since in degree \( 2n \), \( i^* \) is an isorphism, then \( k^*(x) = w_1 + w_2 \). Thus, \( h(\tau) k^*(y) = (w_1 + w_2)^2 = 2w_1 w_2 \); on the other hand, \( k^*(y) = k^*(z) = h(\tau) = 2 \).

Now we invert \( 2 \). Then what we see is that \( H(\Omega \tau) \) induces an isomorphism on \( H^{n-2} \). Since \( H^*(\Omega S^{4n-1}) \) is a divided polynomial ring, then we get an isomorphism in all degrees. Thus, \( H(\Omega \tau) \) is a cohomology isomorphism of \( H \)-spaces, so it can be easily shown that it is a homotopy equivalence. Then we can define a map \( \Omega \tau \cdot i : \Omega S^{4n-1} \times S^{2n-1} \to \Omega S^{2n} \), and it is a weak equivalence, since \( \pi_n(\Omega \tau) \) provides splitting for the EHP spectral sequence. As a consequence we get the following nice isomorphism, \( \pi_{k+1}(S^{4n-1}) \times \pi_k(S^{2n-1}) \cong \pi_{k+1}(S^{2n}) \).

The idea of localizing produces some good results. One of them is that \( S^m \to \Omega S^{m+1} \to \Omega S^{m+1} \) is fiber sequence 2-locally for all \( m \). Actually this is true for the same reason why it is true integrally for odd \( m \): the cup product \( H^m(\Omega S^{m+1}) \times H^{2m}(\Omega S^{m+1}) \to H^{3m}(\Omega S^{m+1}) \) is an isomorphism 2-locally. Using these fibrations we can set up a new spectral sequence via the following exact couple: This the 2-local EHP spectral sequence. Let’s see what it does. The \( E_1 \)-page of this spectral sequence is the bottom term of the exact couple. We provide it with the following filtration, \( E_{k,m}^1 = \pi_{m+k}(S^{2m-1}) \). The grading of the differentials is the following: \( d_r : E_{k,m}^r \to E_{k-r,m-r}^r \). We claim that this spectral sequence converges to \( \pi_* \otimes \mathbb{Z}(2) \). That’s, at least, what [Rav] tells us. The filtration of \( \pi_*^2 \) is the following \( 0 \subset \operatorname{im} \pi_{k+1}(S^1) \subset \operatorname{im} \pi_{k+2}(S^2) \subset \cdots \subset \operatorname{im} \pi_{2k+1}(S^{k+1}) = \pi_*^2 \) and \( \operatorname{im} \pi_k^2 \) is \( \operatorname{im} \pi_{m+k}(S^m)/\operatorname{im} \pi_{m+k-1}(S^{m-1}) \otimes \mathbb{Z}(2) \). The proof should be some sort of meditation on long exact sequence: so I’ll let you meditate. How does this spectral sequence look like. Here is a picture

Here we’ve written out the groups that we already know. The differential on \( E_1 \)-page can be explained as follows. So we are looking at the \( d_1 \) differential \( \pi_{n+1}(S^{4n+1}) \to \pi_{n-1}(S^{4n-1}) \). This differential is obtained by composing the EHP boundary map \( \Delta : \pi_{4n+1}(S^{4n+1}) \to \pi_{4n-1}(S^{4n-1}) \) with the Hopf map \( H_* : \pi_{4n-1}(S^{2n}) \to \pi_{4n-1}(S^{2n-1}) \). Recall that \( \Delta(\lambda_{4n+1}) = [\tau_{2n}, \epsilon_{2n}] \); therefore, \( H_*\Delta(\lambda_{4n+1}) = H_*([\lambda_{2n}, \epsilon_{2n}]) = 2\lambda_{4n-1} \) according to calculation on the previous page. Furthermore, since \( H_*([\lambda_{2n-1}, \epsilon_{2n-1}]) = 0 \), we conclude that no \( d_1 \) differential starts at \( \pi_{4n-1}(S^{4n-1}) \). Thus, we see that for in stance, \( E_{\infty}^{1,2} \cong E_2^{1,2} \cong \mathbb{Z}/(2) \) and therefore, \( \pi_*^{1} \cong \mathbb{Z}/(2) \). The generator of \( \pi_*^1 \) is the Hopf element \( \eta \), since we have shown that it is essential. This result lets us update the spectral sequence by replacing \( E_{1,n}^{1} = \pi_{2n}(S^{2n-1}) \) with \( \mathbb{Z}/(2) \) for \( n \geq 2 \), since we are already in the stable range. In particular, we get that \( E_2^{2,2} \simeq \mathbb{Z}/(2) \) and the representative is a permanent cycle. We need to make sure that this cycle does not vanish, i.e. no differentials hit it. We can show this by directly looking at the \( d_1 \) differential coming out of \( \pi_{2n}(S^{2n-1}) \) as \( E_{1,n}^{1} \to E_{1,2n-3}^{1} \). This differential factors as a composite of the boundary \( \Delta : \pi_{2n}(S^{2n-1}) \to \pi_{2n-2}(S^{2n-2}) \) and the Hopf map \( H_* : \pi_{2n-2}(S^{2n-2}) \to \pi_{2n-2}(S^{2n-3}) \). We need to note that we can factor \( \eta_{2n-3} \) as \( S^{2n-2} \to S^{2n-3} \to \Omega^2 S^{2n-1} \), i.e. \( \eta_{2n-3} \) composed with the adjoint of the identity. Thus if we compose it with the boundary map \( \Omega^2 S^{2n-1} \to S^{n-1} \) we simply get \( [\lambda_{n-1}, \epsilon_{n-1}] \eta_{2n-3} \). Then \( d_1(\eta_{2n-3}) = H_*([\lambda_{n-2}, \epsilon_{n-2}] \eta_{2n-3}) = H_*([\lambda_{n-2}, \epsilon_{n-2}]) \eta_{2n-3} \). Recall that \( H_*([\lambda_{n-1}, \epsilon_{n-1}]) = 0 \) if \( n \) is odd and \( 2\eta_{2n-3} \) if \( n \) is even, if \( n \geq 1 \). If \( n \geq 3 \), the element \( \eta_{2n-3} \) is in 2-torsion, so \( d_1(\eta_{2n-3}) = 0 \) in any case. This is all nice, but it does not guarantee that \( (2,2) \) does not get hit by \( (3,4) \) on \( E_2 \)-page. This differential
too can be shown to vanish. The way one does is as follows. As we know $E_1^{3,4}$ is the same thing as $\pi_7(S^7)$. Since the outgoing $d_1$ differential is null, then each element in $\pi_7(S^7)$ represents an element in $E_2^{3,4}$. We can compute the image of $[\tau_7] \in E_2^{3,4}$. We first take $\Delta(\tau_7) \in \pi_3(S^3)$. This element necessarily is a suspension of some $\alpha \in \pi_4(S^2)$. The element $d_2([\tau_7])$ is precisely $H_3(\alpha)$, and it is independent of the choice of $\alpha$. The catch is that $\Delta(\tau_7) = [\iota_3, \iota_3]$, which is actually 0, since $S^3$ is an H-space. Thus, $\alpha$ can be chosen to be 0, and as a consequence $d_2([\tau_7]) = 0$. As a consequence of this we conclude that $\pi_2^S = \mathbb{Z}/(2)$.

Alternatively we can cheat and use the Steenrod algebra. We claim that $\eta^2 \in \pi_2^S$ is non-trivial. Suppose that $\eta_{n+1}\eta_n$ is trivial. Then we can form the cofiber of $C\eta_{n+1}$. We have $\eta_n$ that maps its bottom cell, $S^{n+1}$, to $S^n$, and this map can be extended to a map from $C\eta_{n+1}$, since $\eta_{n+1}\eta_n$ is null. Call this map $\chi : C\eta_{n+1} \rightarrow S^n$ and consider its cofiber, $C\chi$. One can check that $H^*(C\chi; \mathbb{F}_2) \simeq \mathbb{F}_2(x, y, z)$, where $|x| = n$, $|y| = n + 2$ and $|z| = n + 4$. We get an obvious map $C\chi \rightarrow \Sigma C\eta_{n+1}$ inducing an isomorphism in cohomology in dimensions $n + 2$ and $n + 4$. Therefore, we conclude that $Sq^2y = z$. On the other hand we have the diagram on the right and one can easily check that the rightmost vertical arrow induces isomorphism in cohomology in dimensions $n$ and $n + 2$, whence $Sq^2x = y$. This implies that $z = Sq^2Sq^2x = Sq^3Sq^1x = 0$, which is impossible. This makes us conclude that $\eta_{n+1}\eta_n$ cannot be null for any $n$. The EHP spectral sequence demonstrates that $\eta^2$ is the only essential element in $\pi_2^S$.

We have just a single leftover from the 2nd stem, $\pi_4(S^2)$. The EHP sequence gives an exact sequence

$$\pi_0(S^5) \rightarrow \pi_4(S^2) \rightarrow \pi_5(S^3) \rightarrow \pi_5(S^5) \simeq \mathbb{Z} \rightarrow \pi_3(S^2) \simeq \mathbb{Z} \rightarrow \pi_4(S^3) \simeq \mathbb{Z}/(2).$$

This shows that $\pi_4(S^2) \rightarrow \pi_5(S^3)$ is surjective. The image of the $\eta_3$ is $[\iota_2, \iota_2]\eta_3$. On the other hand $\iota_2$ gets sent to $[\iota_2, \iota_2]$ which has to be the same as $2\eta_2$. Thus, we see that the image of $\eta_3$ in $\pi_4(S^2)$ is $2\eta_2\eta_3$, i.e. 0. This implies that $\pi_4(S^2) \simeq \pi_5(S^3) \simeq \mathbb{Z}/(2)$.

References

[Ha1] A. Hatcher, Spectral Sequences in Algebraic Topology