Nerses Aramyan

Research Statement

Introduction

I am interested in geometric constructions of extended topological field theories, which implement the machinery of higher category theory. Recently, I have been focusing on topological field theories taking values in the deloopings of the circle. The work of Brylinski-McLaughlin, [BM1], [BM2], Gomi-Terashima, [GT], Lipsky, [Li], and Schreiber, [Sch], implicitly indicated that there should be a procedure of constructing extended topological field theories from the data of differential cohomology. Fix a positive integer n. One model for differential cohomology is the smooth n-truncated Deligne cohomology, which is the hypercohomology of the smooth n-truncated Deligne complex. Using a version of the Dold-Kan construction, we create from the Deligne complex a homotopy sheaf of spaces, $|\mathfrak{D}_n|^+$, on the site of smooth manifolds. We can also construct, following Jacob Lurie, the sheaf of oriented n-dimensional topological field theories taking values in $\mathrm{B}^n U(1)$, which we will write as $\mathrm{TFT}_n^{\mathrm{or}}$. The main result of my research is the following theorem.

Theorem. (A) *There is a morphism of homotopy sheaves*

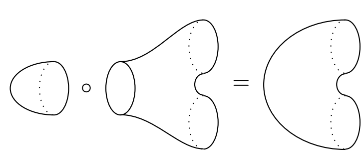
$$\int : |\mathfrak{D}_n|^+ \longrightarrow \mathrm{TFT}_n^{\mathrm{or}},$$

called the adjoint integration pairing. The composition of the adjoint integration pairing with the restriction to space of framed topological field theories $\mathrm{TFT}_n^{\mathrm{or}} \longrightarrow \mathrm{TFT}_n^{\mathrm{fr}}$ *is an equivalence of homotopy sheaves.*

The adjoint integration pairing can be viewed as a mathematical abstraction for Schwarz-type topological field theories from physics, i.e. ones where the action functional is constructed from manipulations of differential forms without additional data. One of my hopes is to understand whether the construction can be used to understand the well-known Schwarz-type topological field theories from physics, e.g. the topological Wess-Zumino-Witten model, Dijkgraaf-Witten theory, Chern-Simons theory. What this theorem states is that the physical Schwarz-type field theories are somehow linked to the framed field theories from mathematics. From physics we know that there are also Witten-type field theories, in which the action functionals are constructed by making a choice of non-topological entities, but can be shown to be independent of the choice. This suggests a possibility of detecting Witten-type topological field theories in the non-framed sector.

Topological quantum field theories were first defined by Atiyah in [A]. The definition was inspired by Segal's definition of conformal field theories in [S]. They are important to mathematicians because they can be used to capture invariants of manifolds. Despite having their origins in physics, they can be understood without any knowledge of physics.

For each integer $n \geq 1$, there is an n-dimensional bordism category denoted as $\mathrm{Bord}_n^{\mathrm{or}}$, and defined as follows. The objects of the category are the $(n-1)$-dimensional oriented compact closed manifolds. A morphism from oriented manifold Σ_1 to oriented manifold Σ_2, is an equivalence class of compact oriented n-dimensional manifolds M along with an orientation preserving diffeomorphism $\partial M \simeq \overline{\Sigma_1} \coprod \Sigma_2$. Two such oriented manifolds M and N are equivalent if they are diffeomorphic (in the oriented sense) relative to their boundaries. Whenever specifying a morphism we will simply write the manifold with boundary, i.e. we will omit writing the specified diffeomorphism on the boundary and the fact that it is really a representative of an equivalence class. Given morphisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Sigma_2 \longrightarrow \Sigma_3$, the composition is given by the manifold $M \cup_{\Sigma_2} N$. The following is the example of composition of the "cap" bordism with the "pair of pants" bordism,



The composition defined this way is associative. The identity of M is given by the cylinder $M \times I$, where I denotes the closed unit interval. Finally, we note that the disjoint union, $\coprod$, specifies a symmetric monoidal product on $\mathrm{Bord}_n^{\mathrm{or}}$, and it has the empty manifold, $\emptyset$, as its unit.

As you might have already guessed the superscript "or" on top of Bord_n, stands for "orientable". There are other flavors of bordism categories that implement other structures on manifolds, such as spin structure, framing. We will label these categories $\mathrm{Bord}_n^{\mathrm{spin}}$, $\mathrm{Bord}_n^{\mathrm{fr}}$, respectively. In addition to this we may put no structure at all, in which case we obtain the unoriented bordism category, Bord_n. Here we will denote a general structured bordism category as $\mathrm{Bord}_n^{\mathrm{str}}$, where "structured" can mean unoriented, oriented, spin, framed.

Now we are ready to phrase Atiyah's definition of topological field theory.

Definition. *A structured n-dimensional topological field theory is a symmetric monoidal functor from* $\left(\mathrm{Bord}_n^{\mathrm{str}}, \coprod\right)$ *to a symmetric monoidal category* $(\mathcal{C}, \otimes)$.

In low dimensions topological field theories admit very nice descriptions. In the following examples all the topological field theories are taking values in $\mathcal{C}$. An oriented 1-dimensional topological field theory is equivalent to a dualizable object in $\mathcal{C}$. An unoriented 1-dimensional topological field theory is equivalent to a dualizable object X in $\mathcal{C}$ along with choice of isomorphism $X \simeq X^\vee$. A 2-dimensional (un)oriented topological field theory is equivalent to a commutative Frobenius algebra in $\mathcal{C}$, which is again a dualizable object. These observations come from our ability to decompose manifolds into simpler pieces in these dimensions. For instance, we know that oriented compact 0-dimensional manifolds are disjoint unions of positively and negatively oriented points, and the 1-dimensional bordisms can be constructed from gluing disjoint unions of intervals. If we move a dimension up then we find that oriented closed compact 1-dimensional manifolds are disjoint unions of circles, and the 2-dimensional bordism can be constructed using "caps" and "pair of pants".

The discussion in the previous paragraph underlines a strategy for understanding higher dimensional topological field theories: find a set of elementary bordisms that generate the rest and find their relations. It, however, becomes quite cumbersome for $n \geq 3$. The essential difficulty is the fact that the axioms of topological field theories only allow one to cut and paste along codimension 1 submanifolds. We would be in a significantly better situation if we could cut along manifolds of higher codimension. This could be accomplished if we allow the consideration of manifolds with corners. In fact, if we use higher category theory, we can assemble structured n-dimensional manifolds with corners into a higher categorical widget called an (∞, n)-category, which we will write as $\mathbf{Bord}_n^{\mathrm{str}}$. This object is a lot more sophisticated as compared to its classical counterpart, and we will not delve into the details here. However, from higher categorical point of view they admit simple characterizations. We will phrase the result for $\mathbf{Bord}_n^{\mathrm{fr}}$. This is a symmetric monoidal (∞, n)-category, because we can take disjoint unions of manifolds.

Definition. *An* extended *n-dimensional framed topological field theory is a symmetric monoidal functor from* $\mathbf{Bord}_n^{\mathrm{fr}}$ *to a symmetric monoidal* (∞, n)*-category* $\mathcal{C}$.

As we could see from low dimensional examples for ordinary categories topological field theories landed in dualizable object of the target. One can formulate a higher categorical definition of dualizability. In fact, for extended topological field theories there are no other obstructions.

Theorem. (Lurie) *Giving an extended n-dimensional framed topological field theory with target* $\mathcal{C}$ *is equivalent to giving a dualizable object in* $\mathcal{C}$.

This is the celebrated Baez-Dolan cobordism hypothesis, which was described in [BD], and proved by Lurie in [Lu1]. The theorem can be rephrased as saying that $\mathbf{Bord}_n^{\mathrm{fr}}$ is the free symmetric monoidal (∞, n)-category with duals generated by a point. As a consequence of this theorem we observe that there are extended topological field theories exist in abundance. As remarkable as this theorem is, it is an existence result—it does not provide an explicit construction of extended topological field theories from the data familiar to physicists.

In the classical context there is an explicit construction of topological field theories with a background manifold. More specifically, we can talk about the bordism category over an oriented manifold M, which we will write as $\mathrm{Bord}_n^{\mathrm{or}}(M)$. This category is the same $\mathrm{Bord}_n^{\mathrm{or}}$ except all the manifolds and bordism are equipped with a smooth map to the background manifold M. We describe a procedure that produces a 1-dimensional topological field theory with background M. Suppose that we are given a hermitian line bundle $\mathcal{L} \longrightarrow M$, and a hermitian connection on it ∇. For each point of M, $m : * \longrightarrow M$, set $Z_{(\mathcal{L},\nabla)}(m)$ to equal to $\mathcal{L}_m$ if $*$ is positively oriented, and to $\mathcal{L}_m^\vee$ otherwise. If we are given a smooth path $f : I \longrightarrow M$, we can use the parallel transport provided by the connection ∇ to define a linear isomorphism $Z_{(\mathcal{L},\nabla)}(f) : \mathcal{L}_{f(0)} \longrightarrow \mathcal{L}_{f(1)}$. One can check that this prescription can be extended to a topological field theory with target category of hermitian lines $\mathcal{H}\mathrm{erm}\mathcal{L}\mathrm{ines}$, $Z_{(\mathcal{L},\nabla)} : \mathrm{Bord}_1^{\mathrm{or}}(M) \longrightarrow \mathcal{H}\mathrm{erm}\mathcal{L}\mathrm{ines}$. The element $Z(S^1 \longrightarrow M) \in U(1)$ coincides with the holonomy of this path.

The work of Brylinski and McLaughlin, [BM1] and [BM2], helps to generalize this process to dimension $n = 2$. The line bundles are replaced by objects called gerbes, and they are endowed with something called a connective structure and curving. Gerbes are a particular type of stacks on a manifold; namely, the ones with connected stalks. The field theories coming from these objects are a little trickier to describe, so I'll omit doing it. In [Li], Lipsky generalizes this procedure for all n using the smooth n-truncated Deligne complex. However, the construction is lengthy and requires many choices.

In my work I put the results in the previous paragraph in framework of Lurie's paper, and demonstrate that the construction of Lipsky can be interpreted as an extended topological field theory. Furthermore, I achieved great simplifications of methods that Lipsky used by observing that some of the procedures are part of homotopy sheafification. The theorem in beginning of the introduction is obtained by the use of Stokes' theorem—the difficulty that Lipsky and others had to deal with comes from the fact that some higher categorical procedures are difficult to describe explicitly.

Below we present directions for future research. The first direction concerns the applications of this project to theoretical physics. In this direction certain more refined version of the construction is necessary. The second concerns a sector of topological field theories that is not detected by the adjoint integration pairing. The last concerns a further generalization of the construction to settings, where the field theories take values in objects other than the circle.

Topological Field Theories from Physics

The topological Wess-Zumino-Witten model, Dijkgraaf-Witten theory, Chern-Simons theory are all topological field theories in the physics sense. This means that they are given by specifying an action functional that does not depend on any dynamic quantities (such as metrics or conformal structures). I intend to investigate on how these physical field theories can be interpreted as mathematical topological field theories as described above. As far as my preliminary research shows one needs to look at slightly more refined source than $|\mathfrak{D}_n|^+$. The correct object to look at is the discrete Deligne complex, $|\mathfrak{D}_n^{\mathrm{disc}}|^+$ and the discrete topological field theories, $\mathrm{TFT}_n^{\mathrm{or,disc}}$, i.e. ones taking values in $\mathrm{B}^n U(1)_{\mathrm{disc}}$. The construction of integration pairing is very point-set theoretic, which allows us to construct it for the discrete versions as well. Thus we have a functor

$$\int : |\mathfrak{D}_n^{\mathrm{disc}}|^+ \longrightarrow \mathrm{TFT}_n^{\mathrm{or,disc}}.$$

After composing it with $\mathrm{TFT}_n^{\mathrm{or,disc}} \longrightarrow \mathrm{TFT}_n^{\mathrm{fr,disc}}$ we no longer have an equivalence, but if we precompose with the so called flat discrete Deligne complex $|\mathfrak{D}_n^{\mathrm{disc},\flat}|^+ \longrightarrow |\mathfrak{D}_n^{\mathrm{disc}}|^+$ we do get an equivalence. For simply-connected, compact, semisimple Lie group G, there is an invariant 3-form μ coming from the 3-form $\langle -, [-,-]\rangle$ on the Lie algebra of G. The form μ is integral. There is a curvature map $|\mathfrak{D}_2^{\mathrm{disc}}|^+(G) \longrightarrow \mathcal{A}^3(G)$. The image of the curvature map is the integral lattice in $\mathcal{A}^3(G)$. For any $m \in \mathbb{Z}$ we can lift $m\mu$ to $|\mathfrak{D}_2^{\mathrm{disc}}|^+(G)$ and push it to $\mathrm{TFT}_2^{\mathrm{or,disc}}(G)$. The resulting topological field theory is the topological Wess-Zumino-Witten model in the sense of [Sch].

Another thing that we can do with the refined integration pairing is to evaluate it on stacks. The stack of interest in the case of Dijkgraaf-Witten and Chern-Simons theories is $\mathrm{B}G_\nabla$ of principal G-bundles with connections. In the case of Dijkgraaf-Witten theory, the group G is discrete, so $\mathrm{B}G_\nabla$ is the same as $\mathrm{B}G$ as there is no connection data. The space $|\mathfrak{D}_n^{\mathrm{disc}}|^+(\mathrm{B}G)$ is linked to group cohomology with coefficients in $U(1)_{\mathrm{disc}}$. If G is a connected and compact Lie group, then $|\mathfrak{D}_n^{\mathrm{disc}}|^+(\mathrm{B}G_\nabla)$ contains not only the group cohomology data, but also the Lie algebra cohomology data in it, which can be used to obtain the Chern-Simons functional. The integration pairing construction has the correct input and the correct output for both theories, but does it produce the theories that we want? If not, then how can we modify the construction to obtain these theories?

Non-Framed Field Theories

In this section we describe a mysterious sector of field theories not detected by adjoint integration pairing. The theorem of the introduction demonstrates that the framed field theories split off of oriented field theories. Indeed, what we have is a triangle

$$\begin{array}{ccc} & \mathrm{TFT}_n^{\mathrm{or}} & \\ {\scriptstyle\int}\nearrow & & \searrow \\ |\mathfrak{D}_n|^+ & \xrightarrow[\simeq]{} & \mathrm{TFT}_n^{\mathrm{fr}} \end{array}$$

In fact, all the maps can be augmented to those of sheaves of infinite loopspaces. This implies that $\mathrm{TFT}_n^{\mathrm{fr}}$ is a direct summand of $\mathrm{TFT}_n^{\mathrm{or}}$ under the product. Thus, there is a sheaf $\mathrm{TFT}_n^{\not{\mathrm{fr}}}$, such that $\mathrm{TFT}_n^{\mathrm{or}} \simeq \mathrm{TFT}_n^{\mathrm{fr}} \times \mathrm{TFT}_n^{\not{\mathrm{fr}}}$.

We will call this sheaf the sheaf of *non-framed field theories*. This sheaf is non-trivial and encodes the information that is not detected by adjoint integration pairing construction. The non-triviality can be seen from topological considerations. If $\mathrm{Map}^{\otimes}(-,-)$ denotes the mapping space of infinite loopspace maps and $\mathrm{MTSO}(n)$ denotes the oriented tangential Thom spectrum in dimension n, then we have the following equivalences

$$\mathrm{TFT}_n^{\mathrm{or}}(X) \simeq \mathrm{Map}^{\otimes}(\Omega^{\infty}(\mathrm{MTSO}(n) \wedge \Sigma_+^{\infty} X), \mathrm{B}^n S^1) \text{ and } \mathrm{TFT}_n^{\mathrm{fr}}(X) \simeq \mathrm{Map}^{\otimes}(\Omega^{\infty}\Sigma_+^{\infty} X, \mathrm{B}^n S^1).$$

Then we see that

$$\mathrm{TFT}_n^{\not{\mathrm{fr}}}(X) \simeq \mathrm{Map}^{\otimes}(\Omega^{\infty}(C \wedge \Sigma_+^{\infty} X), \mathrm{B}^n S^1),$$

where C is the cofiber of the natural map $\mathrm{S} \longrightarrow \mathrm{MTSO}(n)$, where S denotes the sphere spectrum. The splitting is not very surprising from the topological considerations: all the topological field theories above can be modeled to be topological abelian groups, hence, products of Eilenberg-MacLane spaces for discrete groups; the splitting follows from the fact that $\mathrm{H}_0(\mathrm{MTSO}(n)) \simeq \mathbb{Z}$. A similar splitting occurs, when we consider the discrete version of the integration pairing construction.

This raises a very interesting question: is there a geometric construction, similar to adjoint integration pairing, that "detects" the non-framed topological field theories. We made a point that my thesis research deals with Schwarz-type topological field theories. Do the Witten-type topological field theories fall into non-framed sector or not?

Extraordinary Differential Cohomology & Stokes' Theorem

There are two driving facts behind the result of my thesis: 1) we can "resolve" an abelian group objects using differential forms; 2) Stokes' theorem. There are not very many abelian group objects over manifolds $U(1)$ and $\mathbb{R}$ and their products. As is well-known when we move to higher categorical framework abelian groups turn into E_{∞}-spaces. The question is: how one can build an analogue of the n-truncated Deligne complex from an E_{∞}-object in the ∞-topos of simplicial sheaves over the site of smooth manifolds? This complex should be built in a way that an analogue of Stokes' theorem works for it. This would allow us to construct a version of adjoint integration pairing with the target being topological field theories taking values in extraordinary cohomology theories. Of special interest to me is differential K-theory, where procedures similar to the description above exist, [BS].

References

[A] M. Atiyah, Topological Quantum Field Theories, Publications Mathématiques de l'IHÉS 68, 175-186, 1989

[BD] J. Baez, J. Dolan. Higher-Dimensional Algebra and Topological Quantum Field Theory. J. Math. Phys. 36 (11), 6073-6105, 1995

[BM1] J.-L. Brylinski, D. A. McLaughlin, The Geometry of Degree-Four Characteristic Classes and of Line Bundles on Loop Spaces I, Duke Mathematical Journal, Vol. 75, No. 3, 603-638, 1994

[BM2] J.-L. Brylinski, D. A. McLaughlin, The geometry of degree-4 characteristic classes and of line bundles on loop spaces II, Duke Mathematical Journal, Vol. 83, No. 3, 105-139, 1996

[BS] U. Bunke, T. Schick, Differential K-Theory. A Survey, Global differential geometry, Springer, Heidelberg, 303-358, 2012

[GT] K. Gomi, Y. Terashima, A Fiber Integration Formula for the Smooth Deligne Cohomology, IMRN No. 13, 2000

[Li] D. Lipsky, Cocycle Constructions for Topological Field Theories, UIUC Thesis, 2010

[Lu1] J. Lurie, On the Classification of Topological Field Theories, 2010

[Lu2] J. Lurie, Higher Topos Theory, Annals of Mathematics Studies 170, 2012

[S] G. Segal, The Definition of Conformal Field Theory, Differential geometrical methods in theoretical physics, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 250, Kluwer Acad. Publ., Dordrecht, 165-171, 1988

[Sch] U. Schreiber, Differential Cohomology in a Cohesive ∞-Topos, 21$^{\mathrm{st}}$ Century

[T] R. Thom, Quelques propriétés globales des variétés différentiables, Commentarii Mathematici Helvetici 28, 1786, 1954