Bousfield Localization

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This talk is concerned with the construction of Bousfield localization functors. In the first part, I'll make an attempt to motivate and introduce the notion of categorical localization. The formulation of Bousfield localization does not require any advanced categorical notions. The existence of these localizations is a subtle issue. In the second part, I'll show the existence of localization functors on the homotopy category of spectra, since this is the primary objects that we are interested in. One can generalize this proof to model categories if certain appropriate hypotheses are made. These generalization are in [H].

1 . A Tour through Categorical Localizations

Quite often in topology and algebra one encounters a situation where there are certain maps that are not isomorphism, but it would be expedient if they were. For instance, in $\mathcal{T}\mathrm{op}_*$ the weak equivalences are not invertible up to homotopy, but they induce isomorphisms with respect to all the homology and cohomology theories (sometimes by design), i.e. for all we care they are isomorphisms. In these situations one chooses to pass to a setting where these maps become genuine isomorphisms. Speaking more concretely, suppose we are given a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ in it that we would like to invert. We create a category $\mathcal{C}[\mathcal{W}^{-1}]$ and a functor $F : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ that sends the elements of $\mathcal{W}$ to isomorphisms, and it is universal with respect to this property. We would call such a category the localization of $\mathcal{C}$ with respect to $\mathcal{W}$. There is a simple formal construction, which we will discuss shortly, that accomplishes it. However, with this simplicity comes a price to pay. There are several set-theoretic issues that one needs to overcome.

The idea of the simple construction is to forcefully invert the morphisms while trying to preserve the categorical structure. Here is what we do. Let the objects of the category $\mathcal{C}[\mathcal{W}^{-1}]$ be the same as those of $\mathcal{C}$. Let $\mathcal{W}^{-1}$ denote the class of symbols of form σ^{-1}, where $\sigma \in \mathcal{W}$. We will call the source (resp. target) of σ, the target (resp. source) of σ^{-1}. The morphisms between A and B in $\mathcal{C}[\mathcal{W}^{-1}]$ are the sequences $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ with $\{\sigma_i\} \subset \mathrm{Mor}(\mathcal{C}) \coprod \mathcal{W}^{-1}$, the source of σ_0 being A, target of σ_n being B, such that the target of σ_i is the source of σ_{i+1}, subject to the following equivalence relations:

1) if σ_i is an identity we omit it

2) if $\sigma_{i+1} = \sigma_i^{-1}$ or $\sigma_i = \sigma_{i+1}^{-1}$, then we can omit the entries

3) if $\sigma_i, \sigma_{i+1} \in \mathrm{Mor}(\mathcal{C})$, then we replace them by an entry containing $\sigma_i\sigma_{i+1}$

4) similarly if $\sigma_i, \sigma_{i+1} \in \mathcal{W}^{-1}$, such that $\sigma_i = \mu_i^{-1}$, $\sigma_{i+1} = \mu_{i+1}^{-1}$ and $\mu_{i+1}\mu_i \in \mathcal{W}$, then we can replace the entries by one containing $(\mu_{i+1}\mu_i)^{-1}$.

We think of these morphsisms as zigzags between objects A and B, e.g.

$$\begin{tikzcd} & C_1 & & C_3 & \\ A \arrow[ur] & & C_2 \arrow[ul] \arrow[ur] & & B \arrow[ul] \end{tikzcd}$$

If we define the composition to be the concatenation, then we will obtain the unital and associativity axioms. The catch is that the morphisms between A and B may no longer form a set. We may avoid discussing this issue if we deside to talk about universes. However, I will avoid talking about this. We will call this construction *the category of fractions of* $\mathcal{C}$ *with respect to* $\mathcal{W}$.

Remark 1.1. We can embed $\mathcal{C}$ into $\mathcal{C}[\mathcal{W}^{-1}]$ in the obvious way. This embedding possesses a universal property: given any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, such that $F\sigma$ is invertible for any $\sigma \in \mathcal{W}$, then there is a *unique* factorization of F through $\mathcal{C}[\mathcal{W}^{-1}]$. As one can note this universal property is very strict. In fact, we possibly wouldn't mind if a localized category that factorized F uniquely up to a categorical equivalence, or something like that. Thus, we still have some hope of constructing a localization and staying in our standard, cozy set-theoretic world. In fact, if we impose enough restrictions on $\mathcal{C}$ and $\mathcal{W}$, this construction will yield a category equivalent to a set-theoretic category.

Example 1.2. Let R be a commutative, unital ring, and let S be a multiplicative set in R. Then it is well-known from commutative algebra, how to construct for any R-module M, another R-module $S^{-1}M$ of formal fractions with denominators in S modulo certain relations. This construction is natural, in the sense that there is an obvious way to extend this to an endofunctor of $\mathcal{M}\mathrm{od}_R$. However, let us forget the construction and recall the universal property that $S^{-1}M$ possesses. For each M there is a map to $S^{-1}M$, such that if we are given an R-linear map $M \longrightarrow N$, where N has the property that multiplication by elements of S are invertible (i.e. N is S-local), then the map factors *uniquely* through $S^{-1}M$. Since, each $S^{-1}M$ is an $S^{-1}R$-module, the previous statement amount to saying that there is an adjunction $S^{-1} : \mathcal{M}\mathrm{od}_R \rightleftarrows \mathcal{M}\mathrm{od}_{S^{-1}R} : I$, where I denotes the inclusion functor. In any case, let $\mathcal{S}$ denote the class of morphisms in $\mathcal{M}\mathrm{od}_R$ that become isomorphisms after applying S^{-1}. Clearly, the multiplication by elements of S is in $\mathcal{S}$. Tautologically, we obtain a map $F : \mathcal{M}\mathrm{od}_R[\mathcal{S}^{-1}] \longrightarrow \mathcal{M}\mathrm{od}_{S^{-1}R}$. Is this map an equivalence? Actually, it is and the inverse is provided by the inclusion: $G : \mathcal{M}\mathrm{od}_{S^{-1}R} \hookrightarrow \mathcal{M}\mathrm{od}_R \longrightarrow \mathcal{M}\mathrm{od}_R[\mathcal{S}^{-1}]$. Clearly, FG is the identity functor on $\mathcal{M}\mathrm{od}_{S^{-1}R}$. On the other hand, $GF(M) = S^{-1}M$. However, in $\mathcal{M}\mathrm{od}_R[\mathcal{S}^{-1}]$ the map $M \longrightarrow S^{-1}M$ is invertible. These maps provide a natural isomorphism $\iota : \mathbb{1}_{\mathcal{M}\mathrm{od}_R[\mathcal{S}^{-1}]} \longrightarrow GF$. We conclude that $\mathcal{M}\mathrm{od}_R[\mathcal{S}^{-1}]$ is far less set-theoretically forbidding that it might have originally looked.

Example 1.3. It was mentioned previously the desire to invert weak homotopy equivalences in $\mathcal{T}\mathrm{op}_*$. Topologists do it as follows. We first replace spaces with a CW-complex that is weakly-equivalent to it in a functorial fashion. This results in the following effect: if we are given a weak equivalence in spaces, then after the replacement, we have a weak equivalence between CW-complexes. We then discover, using the Whitehead's theorem, that there is homotopy inverse to our map in this category. Thus, if we create a category where the objects are the same and the morphisms are replaced by the homotopy classes maps between their CW-replacements, we will be done. This idea was generalized to a great extent by Quillen to something called homotopical algebra. Quillen showed that if the category possesses a certain amount of extra structure, $\mathcal{W}$ being a part of it, then the "inversion" of $\mathcal{W}$ can be performed via this homotopical procedure. Such categories are referred to as model categories. In fact, in this setting the localization of the category is the homotopy category of the model category. In the cases that we are interested there is a possibility of using homotopical algebra. This will be the subject of the third section.

Essentially the purpose of this section is to generalize example 1.2. Let $\mathcal{C}$ be a category, and let L be an endofunctor for $\mathcal{C}$, with a natural transformation $\iota : \mathbb{1} \longrightarrow L$, and such that $L\iota = \iota_L$ with the common value an isomorphism. We will call such functors *Bousfield localizations*. Then we can define for such a functor a collection $\mathcal{W}$ of morphisms that it inverts: $\sigma \in \mathcal{W}$ if $L(\sigma)$ is an isomorphism. These morphisms will be called *L-equivalences*. We can also define the category $\mathcal{C}_{\mathcal{W}}$, the full subcategory of $\mathcal{W}$-local objects in $\mathcal{C}$. An object Z is *$\mathcal{W}$-local* if for any morphism $f : X \longrightarrow Y$ in $\mathcal{W}$ the induced morphism $f^* : \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)$ is an isomorphism. This definition applies even if do not have a Bousfield localization from which the class of morphisms comes from.

This category $\mathcal{C}_{\mathcal{W}}$ is a candidate for a set-theoretic model for $\mathcal{C}[\mathcal{W}^{-1}]$. However, there is a problem. We do not yet have a functor $\mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{W}}$ factoring L. This amounts to saying that LZ is $\mathcal{W}$-local for all $Z \in \mathcal{C}$. Clearly, once we have this we automatically get that this functor inverts the elements We show something tiny bit more general.

Proposition 1.4. *An object Z is in $\mathcal{C}_{\mathcal{W}}$ if and only if ι_Z is an isomorphism.*

Proof. The *only if* we consider the morphism $\iota_Z : Z \longrightarrow LZ$, which is in $\mathcal{W}$. Then the induced map $\iota_Z^* : \mathrm{Hom}(LZ, Z) \longrightarrow \mathrm{Hom}(Z, Z)$ is an isomorphism. Thus, we have a morphism $(\iota_Z^*)^{-1}(\mathbb{1}_Z)$, which is clearly the inverse of ι_Z.

For the *if* part, suppose we have $f : X \longrightarrow Y$ in $\mathcal{W}$ and $g : X \longrightarrow Z$. We need to show that there is a unique map $h : Y \longrightarrow Z$, such that $hf = g$. We test $\iota_Z^{-1} Lg (Lf)^{-1} \iota_Y$ as candidate for $h - \iota_Z^{-1} Lg (Lf)^{-1} \iota_Y f = \iota_Z^{-1} Lg (Lf)^{-1} (Lf) \iota_X = \iota_Z^{-1} Lg \iota_X = \iota_Z^{-1} \iota_Z g = g$. Now suppose $\widetilde{h} : Y \longrightarrow Z$ is such that $\widetilde{h} f = hf = g$. This implies that $L\widetilde{h} = Lh$, since Lf is an isomorphism. Thus, $\iota_Z h = Lh\iota_Y = L\widetilde{h}\iota_Y = \iota_Z \widetilde{h}$, so $h = \widetilde{h}$, since ι_Z is an isomorphism. □

Proposition 1.5. *The natural map $G : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}_{\mathcal{W}}$ is an equivalence.*

Proof. An inverse functor can be given by the composition $F : \mathcal{C}_{\mathcal{W}} \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$. One can easily check that ι defines a natural isomorphism from $\mathbb{1}_{\mathcal{C}_{\mathcal{W}}}$ to GF. The natural transformation ι can be also defined for $\mathcal{C}[\mathcal{W}^{-1}]$, and it is an isomorphism, since ι_X is in $\mathcal{W}$. □

We see that having a Bousfield localization is quite useful for producing manageable localization categories. However, we have another problem as well. Most of the time we will start by having a class of morphisms $\mathcal{W}$ without having a Bousfield localization. It will be of use to formulate certain set of basic axioms that these sets $\mathcal{W}$ need to satisfy in order to come from a Bousfield localization.

Let $\mathcal{C}$ be a category and $\mathcal{W}$ a class of morphisms. We would like to understand the properties that it ought to have in order to come from a localizing endofunctor. There are a couple of obvious things that need to hold. We assemble those into the following definition.

Definition 1.6. *We say a class of morphisms $\mathcal{W}$ of $\mathcal{C}$ is* weakly closed *if the following conditions hold:*
1) the isomorphisms of $\mathcal{C}$ are in $\mathcal{W}$
2) $\mathcal{W}$ is closed under composition
3) if $\mathcal{W}$ contains two elements of the $\{\psi_1, \psi_2, \psi_1\psi_2\}$, then it contains the third.

Remark 1.7. These axioms are clearly satisfied if $\mathcal{W}$ arises from a Bousfield localization. In fact, if we have any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, then the class of morphisms σ, such that $F(\sigma)$ is an isomorphisms, is weakly closed. Since these are closed conditions on $\mathcal{W}$, for any $\mathcal{W}$ we can define the notion of *weak closure*, $\overline{\mathcal{W}}$, by taking the intersection of all weakly closed classes containing $\mathcal{W}$. Note that $\mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ will invert everything in $\overline{\mathcal{W}}$. This observation basically demonstrates that $\mathcal{C}[\mathcal{W}^{-1}] \simeq \mathcal{C}[\overline{\mathcal{W}}^{-1}]$. Therefore, even in the most general context one can safely assume that $\mathcal{W}$ is weakly closed.

The previous definition had to do more with functors that Bousfield localizations. We need to augment these set of axioms. The following definition is taken from [GZ].

Definition 1.8. *A class of morphisms $\mathcal{W}$ of $\mathcal{C}$ admit a calculus of left fractions if:*
1) it is weakly closed
2) given a diagram of solid arrows, with $\beta \in \mathcal{W}$, we can extend it to a commutative square, with $\gamma \in \mathcal{W}$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Z \arrow[d, dashed, "\gamma"] \\ Y \arrow[r, dashed, "\delta"'] & W \end{tikzcd}$$

3) if we are given $\phi, \psi : X \longrightarrow Z$ in $\mathcal{C}$ and there is $\sigma : Y \longrightarrow X$ in $\mathcal{W}$, such that $\phi\sigma = \psi\sigma$, then there is $\gamma : Z \longrightarrow W$ in $\mathcal{W}$, such that $\gamma\phi = \gamma\psi$.

Remark 1.9. One of the effects of these axioms also allow the possibility of permuting elements in $\mathrm{Mor}(\mathcal{C}) \coprod \mathcal{W}^{-1}$ in the sequences that represent morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$. One can show that we can express any morphism between A and B as a single stage zigzag, i.e. something of the form (β^{-1}, α). This does not quite resolve the set-theoretic issues, but it does simplify the description of our category.

Proposition 1.10. *If L is a Bousfield localization, then the class of L-equivalences, $\mathcal{W}$, admits a calculus of fractions.*

Proof. $\mathcal{W}$ is weakly closed, since L is a functor. To show 2) and 3), we choose $W = LZ$, $\gamma = \iota_Z$. The existence of δ follows from the requirement that LZ is $\mathcal{W}$-local. Since $L\sigma$ is an isomorphism, we see that $L\psi = L\phi$, whence $L(\iota_Z\psi) = L(\iota_Z\phi)$. Hence, $\iota_{LZ}\iota_Z\psi = \iota_X L(\iota_Z\psi) = \iota_X L(\iota_Z\phi) = \iota_{LZ}\iota_Z\phi$, so $\iota_Z\psi = \iota_Z\phi$, since ι_{LZ} is an isomorphism. □

Thus, we see that admitting a calculus of fractions is important for a class of morphisms $\mathcal{W}$. Now we will proceed to define Bousfield localization starting from $\mathcal{W}$ that admits a calculus of fractions.

Definition 1.11. *A morphism $\sigma : A \longrightarrow B$ is called a $\mathcal{W}$-localization of A if $\sigma \in \mathcal{W}$ and $B \in \mathcal{C}_{\mathcal{W}}$.*

It is clear that if a $\mathcal{W}$-localization exists for A then it is unique up to unique isomorphism. This means that if $\mathcal{C}^{\mathcal{W}}$ denotes the full subcategory of objects in $\mathcal{C}$ that admit a $\mathcal{W}$-localization and fix a $\mathcal{W}$-localization for each object, then we can extend it uniquely to a functor $L : \mathcal{C}^{\mathcal{W}} \longrightarrow \mathcal{C}_{\mathcal{W}}$. The ideal situation is when $\mathcal{C}^{\mathcal{W}} = \mathcal{C}$. In this situation we can take the composition functor $\mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{W}} \hookrightarrow \mathcal{C}$, which, by the abuse of notation, we will denote by L. It is not a coincidence that we write it as L as the following proposition suggests.

Proposition 1.12. *If $\mathcal{C}^{\mathcal{W}} = \mathcal{C}$ and $\mathcal{W}$ is weakly closed, then L, defined above, is a Bousfield localization on $\mathcal{C}$ and the class of L-equivalences is precisely $\mathcal{W}$.*

Proof. Note that the $\mathcal{W}$-localizations that we chose for objects in $\mathcal{C}$ a natural transformation $\iota : \mathbb{1} \longrightarrow L$. We need to show that $L\iota = \iota_L$ and that the common value is an isomorphism. For each $A \in \mathcal{C}$ we have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "\iota_A"'] & LA \arrow[d, "\iota_{LA}"] \\ LA \arrow[r, "L\iota_A"'] & L^2A \end{tikzcd}$$

Since L^2A is in $\mathcal{C}_{\mathcal{W}}$ and $\iota_{LA}\iota_A = L\iota_A\iota_A$, then we conclude that $\iota_{LA} = L\iota_A$. Furthermore, $\iota_{LA}, \iota_A \in \mathcal{W}$, whence, $\iota_{LA}\iota_A \in \mathcal{W}$. This means that, in fact $\iota_{LA}\iota_A$ is a $\mathcal{W}$-localization. Therefore, ι_{LA} is an isomorphism.

Suppose we are given $\sigma : A \longrightarrow B$, such that $L\sigma$ is an isomorphism. The identity $\iota_B\sigma = L\sigma\iota_A$ shows that $\sigma \in \mathcal{W}$. Conversely, suppose $\sigma \in \mathcal{W}$. Then $\iota_B\sigma \in \mathcal{W}$ and $LB \in \mathcal{C}_{\mathcal{W}}$, so $\iota_B\sigma$ is a $\mathcal{W}$-localization. Therefore, $L\sigma$ is an isomorphism. □

Corollary 1.13. *If $\mathcal{C} = \mathcal{C}^{\mathcal{W}}$, then $\mathcal{W}$ admits a calculus of left fractions.* □

Finally, as a closing remark let us note that the only part that requires additional information that $\mathcal{C} = \mathcal{C}^{\mathcal{W}}$. This may not hold in general, and most of the time this is proven by direct means. However, there is a way of cramming extra axioms that end up giving us Bousfield localization. The next section directly proves the existence of this functor for the homotopy category of spectra. The third section is essentially a generalization of section 2.

2. The Construction of E-Localization for Spectra

In this section we will solve a specific localization problem. The following discussion is due to Bousfield, [B]. The category that we are interested in in this section is the homotopy category of spectra, $\mathrm{Ho}\,\mathcal{S}\mathrm{p}$. One ought to feel free to choose any model category for spectra. We will take $\mathcal{S}\mathrm{p}$ the model category of symmetric spectra. We fix a spectrum E. We would like to find a Bousfield localization for maps of spectra that are isomorpism in E-homology. We will denote these by $\mathcal{W}$. Note that $\mathcal{W}$ is weakly closed; therefore, if we find a localizing endofunctor, which we'll denote by L_E, then $\mathcal{W}$ will be precisely the class of L_E-equivalences by 1.12. We will also call $\mathcal{W}$-local spectra E-local, and $\mathcal{W}$-localization E-localization.

We can reinterpret what it means to be E-local in the context of spectra. Let $f : A \longrightarrow B$ be an E-equivalence. We can take the cofiber of this map $\mathrm{C}f$, and using the long exact sequence in E-homology, we conclude that $E_*(\mathrm{C}f) = 0$. We will call a spectrum C, *E-acyclic* if $E_*(C) = 0$. Then one can see that a spectrum X is E-local if and only if $[C, X] = 0$ for all spectra C that are E-acyclic. Let us denote the class of E-acyclic spectra by $\mathcal{N}$. Using this point of view gives us a way of understanding how to construct an E-localization. Notice that attaching an E_*-acyclic object to a spectrum does not affect its E-homology. The goal is to attach E-acyclic objects to a spectrum until we are not able to do so in a non-trivial way. The following proposition makes the idea more precise.

Proposition 2.1. *Let $\widetilde{\mathcal{N}}$ be a* subset *of $\mathcal{N}$. Then for any spectrum X there exists a spectrum $\widetilde{X}$ with an E-equivalence $i : X \longrightarrow \widetilde{X}$, such that $[A, \widetilde{X}] = 0$ for any $A \in \widetilde{\mathcal{N}}$.*

Proof. For each ordinal λ we construct a spectrum X_λ using the transfinite induction. We define X_0 to be X. Suppose we have constructed X_λ. The spectrum $X_{\lambda+1}$ is constructed by the following cofiber sequence

$$\bigvee_{f:A \to X_\lambda} A \longrightarrow X_\lambda \longrightarrow X_{\lambda+1},$$

where the wedge is taken over $A \in \widetilde{\mathcal{N}}$. If λ is a limit ordinal, then we define X_λ to be $\mathrm{hocolim}_{\gamma<\lambda} X_\gamma$.

Let μ be the smallest limit ordinal number exceeding the cardinality of the set $\{f : \Sigma^n \mathrm{S} \longrightarrow A | A \in \widetilde{\mathcal{N}}\}$, where S denotes the sphere spectrum. We set $\widetilde{X} = X_\mu$. Let $\widehat{A}$ be the cofibrant replacement of A. The spectrum $\widehat{A}$ can be constructed as a cell complex of size strictly less than μ. Therefore, we have factorization $\widehat{A} \longrightarrow X_\lambda \longrightarrow \widetilde{X}$ for any map of from spectrum $\widehat{A}$ to $\widetilde{X}$. The composition $A \longrightarrow X_\lambda \longrightarrow X_{\lambda+1}$ is null by design; therefore, so is $\widehat{A} \longrightarrow X_\lambda \longrightarrow X_{\lambda+1}$. Thus we conclude that $[A, \widetilde{X}] = [\widehat{A}, \widetilde{X}] = 0$.

We have a map $X \longrightarrow \widetilde{X}$ and we need to show that it is an E-equivalence. The fact that $X_\lambda \longrightarrow X_{\lambda+1}$ is an E-equivalence follows immediately from the cofiber sequence in E-homology. This shows that if $X \longrightarrow X_\lambda$ is an E-equivalence, then so is $X \longrightarrow X_{\lambda+1}$. Now suppose that $X \longrightarrow X_\gamma$ is an E-equivalence for all $\gamma < \lambda$, where λ is limit ordinal. We can see that $X \longrightarrow X_\lambda$ is also an E-equivalence, since $E_*(X_\lambda) = E_*(\mathrm{hocolim}_{\gamma<\lambda} X_\gamma) \simeq$

colim $_{\gamma<\lambda} E_*(X_\gamma)$. The last statement follows from the fact that the higher derived functors for colimit over a directed category are trivial. $\square$

The fact that $\widetilde{\mathcal{N}}$ was a set was crucial for the proof of proposition. Unfortunately, $\mathcal{N}$ is a class. Even if we take a single representative from each homotopy type in $\mathcal{N}$ it may still be a class. This motivates the next goal: we would like to find a subset $\widetilde{\mathcal{N}} \subset \mathcal{N}$, such that if $[A, X] = 0$ for all $A \in \widetilde{\mathcal{N}}$, then the same statement holds for all $A \in \mathcal{N}$. Let us fix for each ordinal λ, a set $\mathcal{N}_\lambda$, which is a set of homotopy type representatives of spectra in $A \in \mathcal{N}$, such that the cardinality of $\pi_* A$ is less than λ. The correct choice for $\widetilde{\mathcal{N}}$ to accomplish the goal is $\mathcal{N}_\mu$, where μ is the minimal infinite ordinal bounding the cardinality of $\pi_* E$. So we fix this choice for the rest of the section. Before stating a proposition we perform the following construction.

Construction 2.2. Let $\mathcal{S}$ denote the class of all objects in $X \in \mathcal{S}\mathrm{p}$, such that $|\pi_* X| < \mu$. We claim that for any $f : X \longrightarrow A$, such that $A \in \mathcal{N}$, there exists $X^{(1)} \in \mathcal{S}$ with a map $X \longrightarrow X^{(1)}$ which is E-trivial and which factors f. First assume that X is finite. Then for every $\sigma \in E_*(X)$, there is a finite spectrum F_σ over A and a map $g_\sigma : X \longrightarrow F_\sigma$, such that $g_{\sigma *}(\sigma) = 0$. We define $X^{(1)}$ as the big homotopy pushout of all g_σ. Note that $|\pi_* X^{(1)}| < \mu$, since the pushout was taken over a set of maps of order less than μ, and clearly, the induced map $X \longrightarrow X^{(1)}$ is E-trivial. Unfortunately, this choice is not unique and $^{(1)}$ is not a functor.

Now suppose that $X \in \mathcal{S}$. Then we can write it as a homotopy colimit of directed system of finite spectra of size less than μ, hocolim F_α. Suppose we have constructed a sequence $\{F_\alpha^{(1)}\}$ for all $\alpha < \lambda$. We construct $F_\lambda^{(1)}$, by first construting $\overline{F}_\lambda^{(1)}$ using the the procedure in the previous paragraph, and then we consider the following homotopy pushout diagram

$$\begin{tikzcd}
\operatorname{hocolim}_{\alpha<\lambda} F_\alpha \arrow[r] \arrow[d] & F_\lambda \arrow[r] \arrow[d] & \overline{F}_\lambda^{(1)} \arrow[d] \\
\operatorname{hocolim}_{\alpha<\lambda} F_\alpha^{(1)} \arrow[r] & G_\lambda \arrow[r] & F_\lambda^{(1)}
\end{tikzcd}$$

Thus, we have two ordered $\{F_\alpha\}$ and $\{F_\alpha^{(1)}\}$ with a map between them. Thus, by taking the homotopy colimit we obtain $X \simeq \operatorname{hocolim} F_\alpha \longrightarrow \operatorname{hocolim} F_\alpha^{(1)}$, which is E-trivial and clearly hocolim $F_\alpha^{(1)}$ is in $\mathcal{S}$. So we define $X^{(1)}$ to be the latter.

Then we define inductively, $X^{(n)}$ to be $(X^{(n-1)})^{(1)}$, and also $X^{(\infty)}$ as the homotopy colimit of the resulting sequence $\{X^{(n)}\}$. Since, all the maps in the sequence were E-trivial, then $X^{(\infty)}$ is E-acyclic. It is also clear that $X^{(\infty)}$ is in $\widetilde{\mathcal{N}}$. Thus, we conclude that for any $f : X \longrightarrow A$ with $X \in \mathcal{S}$ and $A \in \mathcal{N}$, there is a factorization $X \longrightarrow X^{(\infty)} \longrightarrow A$, with $X^{(\infty)} \in \widetilde{\mathcal{N}}$. The following proposition makes a use of this construction

Proposition 2.3. *Any E-acyclic spectrum A can be represented as a homotopy colimit of spectra in $\widetilde{\mathcal{N}}$.*

Proof. We write A as a homotopy limit of a directed system of finite spectra, hocolim F_α. Furthermore, we require that in the indexing set there are only finitely many $\alpha < \lambda$ for any λ. Then we can form a directed system of $F_\alpha^{(\infty)}$, just like we did in the construction except replacing $^{(1)}$ with $^{(\infty)}$. Thus, we have a $A \simeq \operatorname{hocolim} F_\alpha \longrightarrow \operatorname{hocolim} F_\alpha^{(\infty)} \longrightarrow A$ and the composite is the identity. This shows $\pi_*(\operatorname{hocolim} F_\alpha^{(\infty)}) \longrightarrow \pi_*(A)$ is surjective. Conversely, suppose that $\sigma \in \pi_n(F_\alpha^{(\infty)})$ is null in $\pi_n(A)$. We choose $s : \Sigma^n \mathrm{S} \longrightarrow F_\alpha^{(\infty)}$ to represent σ. Then the composite $\Sigma^n \mathrm{S} \longrightarrow A$ factors through a nullhomotopic $\Sigma^n \mathrm{S} \longrightarrow F_\beta$, where β can be chosen to be greater than α. This implies that the image of σ in $\pi_n(F_\beta^{(\infty)})$ is 0. Thus the map $\pi_*(\operatorname{hocolim} F_\alpha^{(\infty)}) \simeq \operatorname{colim} \pi_*(F_\alpha^{(\infty)}) \longrightarrow \pi_*(A)$ is injective. $\square$

Proposition 2.4. *If for any $C \in \widetilde{\mathcal{N}}$, $F^*(C)$ is trivial, then the same holds for all $C \in \mathcal{N}$.*

Proof. We write $C \in \mathcal{N}$ as a homotopy colimit of directed system of spectra in $\widetilde{\mathcal{N}}$, hocolim N_α. Using the Milnor exact sequence, $\lim^1 F^{*-1}(N_\alpha) \longrightarrow F^*(C) \longrightarrow \lim F^*(N_\alpha)$, we instantly conclude that $F^*(C)$ is trivial. $\square$

Combining the proposition 3.1 and 3.4, we obtain the following corollary.

Corollary 2.5. *The spectrum $\widetilde{X}$ (proposition 3.1) is an E-localization.*

This completes the construction of E-localization. We conclude that we have functor and E-localization functor $L_E : \operatorname{Ho} \mathcal{S}\mathrm{p} \longrightarrow \operatorname{Ho} \mathcal{S}\mathrm{p}$ for any spectrum E.

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