PROBLEM SET

First Order Logic and Gödel Incompleteness

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Problem 1. Define appropriate signatures for
(a) vector spaces over $\mathbb{Q}$;
(b) metric spaces.

Problem 2.
(a) Show by an example that in the signature $\tau_{\text{group}} = (1, \cdot)$, a substructure of a group need not be a subgroup.
(b) Define a signature for groups different from the one above so that a substructure of a group is a subgroup.

Problem 3. If $h : \mathbf{A} \to \mathbf{B}$ is a τ-homomorphism then the image $h(A)$ is a universe of a substructure of $\mathbf{B}$.

Problem 4. Prove that if τ does not contain any relation symbols, then any bijective τ-homomorphism is a τ-isomorphism. (That's why this happens with groups and rings, but not with graphs or orderings.)

Problem 5. A structure is called *rigid* if it has no automorphisms (isomorphisms with itself) other than the identity. Show that the structures $\mathbf{N} = (\mathbb{N}, 0, S, +, \cdot)$ and $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, \cdot)$ are rigid.

Problem 6. A structure $\mathbf{M}$ is called *ultrahomogeneous* if given any isomorphism between two finitely generated substructures, it extends to an automorphism of the whole structure, i.e. if $\mathbf{A}, \mathbf{B}$ are finitely generated substructures of $\mathbf{M}$ and $h : \mathbf{A} \to \mathbf{B}$ is an isomorphism, then there is an automorphism $\overline{h}$ of $\mathbf{M}$ with $\overline{h} \sqsupseteq h$. Show that $(\mathbb{Q}, <)$ is ultrahomogeneous. The same proof should also work to show that $(\mathbb{R}, <)$ is ultrahomogeneous.

Problem 7. Show that an isomorphism between structures is an elementary map, i.e. if $h : \mathbf{A} \to \mathbf{B}$ is an isomorphism, then for any formula $\phi(v_1, ..., v_n)$ and $(a_1, ..., a_n) \in A^n$,

$$\mathbf{A} \models \phi(a_1, ...a_n) \iff \mathbf{B} \models \phi(h(a_1), ..., h(a_n)).$$

Problem 8.
(a) Suppose that $\phi_1, ..., \phi_n$ are τ-formulas and ψ is a Boolean combination of them. Show that there is $\mathcal{S} \subseteq \mathcal{P}(\{1, ..., n\})$ such that

$$\models \psi \leftrightarrow \bigvee_{X \in \mathcal{S}} \left(\bigwedge_{i \in X} \phi_i \wedge \bigwedge_{i \notin X} \neg\phi_i \right).$$

(b) Show that every formula is equivalent to one of the form

$$Q_1 v_1 \dots Q_n v_n \psi,$$

where ψ is a quantifier-free formula (i.e. has no quantifiers) and each Q_i is either $\forall$ or $\exists$.

Problem 9. Define theories for

(a) vector spaces over $\mathbb{Q}$;
(b) metric spaces.

Problem 10.

(a) Show that there is a τ_{group}-sentence ϕ such that

$$\mathbf{M} \vDash \phi \iff \mathbf{M} \cong \mathbb{Z}/2\mathbb{Z}.$$

(b) More generally, let τ be a finite signature and $\mathbf{A}$ be a finite τ-structure. Show that there is a τ-sentence ϕ such that for any τ-structure $\mathbf{B}$,

$$\mathbf{B} \vDash \phi \iff \mathbf{B} \cong \mathbf{A}.$$

In particular,

$$\mathbf{B} \equiv \mathbf{A} \iff \mathbf{B} \cong \mathbf{A}.$$

Problem 11. A formula is called *universal* (*existential*) if it of the form $\forall x \psi$ ($\exists x \psi$), where ψ is quantifier free. Let $\mathbf{A} \subseteq \mathbf{B}$ be τ-structures, $\phi(\vec{v})$ be a τ-formula and $\vec{a} \in A^n$. Show that

(a) if ϕ is quantifier free, then $\mathbf{A} \vDash \phi(\vec{a}) \iff \mathbf{B} \vDash \phi(\vec{a})$;
(b) if ϕ is universal, then $\mathbf{B} \vDash \phi(\vec{a}) \implies \mathbf{A} \vDash \phi(\vec{a})$;
(c) if ϕ is existential, then $\mathbf{A} \vDash \phi(\vec{a}) \implies \mathbf{B} \vDash \phi(\vec{a})$.

Problem 12. Find a sentence that is true in $(\mathbb{N}, <)$ and false in $(\mathbb{Q}, <)$.

Problem 13. Let $\vec{v_1}, \dots, \vec{v_n} \in \mathbb{Q}^m$. Show that $\{\vec{v_1}, \dots, \vec{v_n}\}$ is linearly independent over $\mathbb{Q}$ if and only if it is linearly independent over $\mathbb{R}$.
HINT: Show that linear independence can be expressed by both universal and existential formulas.

Problem 14. Let ϕ be a τ-sentence. The *finite spectrum* of ϕ is the set

$$\{n \in \mathbb{N}^+ : \text{there is } \mathbf{M} \vDash \phi \text{ with } M = n\},$$

where $\mathbb{N}^+$ is the set of positive integers.

(a) Let $\tau = (E)$, where E is a binary relation symbol, and let ϕ be the sentence that asserts that E is an equivalence relation each class of which has exactly 2 elements. Show that the finite spectrum of ϕ is all positive even numbers.
(b) For each of the following subsets of $\mathbb{N}^+$, show that it is the finite spectrum of some sentence ϕ in some signature τ:
  (i) $\{2^n 3^m : n, m \in \mathbb{N}^+\}$,
  (ii) $\{n \in \mathbb{N}^+ : n \text{ is composite}\}$,
  (iii) $\{n^2 : n \in \mathbb{N}^+\}$,
  (iv) $\{p^n : p \text{ is prime and } n \in \mathbb{N}^+\}$,
  (v) $\{p : p \text{ is prime}\}$.

Problem 15. Show that any definable set in $\mathbf{N}$ is 0-definable.

Problem 16. Determine whether the following are 0-definable:
(a) The set $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$.
HINT: You need a nontrivial fact from elementary number theory.
(b) The set of non-negative numbers in $(\mathbb{Q}, +, \cdot)$.
(c) The set of non-negative numbers in $(\mathbb{Q}, +)$.
(d) The set of positive numbers in $(\mathbb{R}, <)$.
(e) The function $\max(x, y)$ in $(\mathbb{R}, <)$.
(f) The function $\operatorname{mean}(x, y) = \frac{x+y}{2}$ in $(\mathbb{R}, <)$.
(g) 2 in $(\mathbb{R}, +, \cdot)$.
(h) The relation $d(x, y) \le 2$ in an undirected graph (with no loops) (Γ, E), where $d(x, y)$ denotes the edge distance function.
(i) The relation $d(x, y) = 2$ in (Γ, E) (as above).

HINT: To prove your negative answers use Problem 7.

Problem 17. Let $\mathbf{C}_{\exp} = (\mathbb{C}, 0, 1, +, \cdot, \exp)$, where exp is the usual exponentiation $z \mapsto e^z$. Show that $\mathbb{Z}$ is definable in $\mathbf{C}_{\exp}$. Conclude that so is $\mathbb{N}$.

Problem 18. Show that the relation

$$P(x, y) \iff x \text{ and } y \text{ are connected}$$

is not 0-definable in the graph (Γ, E) that consists of two bi-infinite paths, i.e. two disjoint copies of $\mathbb{Z}$.
HINT: Show that any formula $\phi(v_1, ..., v_k)$ of length n does not distinguish between k-tuples of vertices that are "sufficiently spread out", where for $(a_1, ..., a_k) \in \Gamma^k$ the latter means that the edge-distance between a_i and a_j is at least 2^n (maybe infinite), for all $i \ne j$.

Problem 19. Prove the Tarski-Vaught test.

Problem 20. Let $\mathbf{A} \subseteq \mathbf{B}$ and assume that for any finite $S \subseteq A$ and $b \in B$, there exists an automorphism f of $\mathbf{B}$ that fixes S pointwise (i.e. $f(a) = a$ for all $a \in S$) and $f(b) \in A$. Show that $\mathbf{A} \prec \mathbf{B}$.

Problem 21. Show that $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$, but $(\mathbb{Q}, <) \ncong (\mathbb{R}, <)$.
HINT: Use Problems 20 and 6.

Problem 22. Let $\mathbf{A}_0, \mathbf{A}_1, ...$ be τ-structures with $\mathbf{A}_0 \prec \mathbf{A}_1 \prec ....$ Show that $A = \bigcup_{n \in \mathbb{N}} A_n$ is a universe of a τ-structure $\mathbf{A}$ and that $\mathbf{A}_n \prec \mathbf{A}$ for all n.

Problem 23. Prove the Löwenheim-Skolem theorem for countable signatures. The proof for uncountable signatures is the same, but it involves some very elementary cardinal arithmetic, so you don't have to do it.
HINT: Construct a substructure containing S that satisfies the Tarski-Vaught test.

Problem 24. Conclude from the Löwenheim-Skolem theorem that any satisfiable theory T has a model of cardinality at most $\max\{|\tau|, \aleph_0\}$. In particular, if ZFC is satisfiable, then it has a countable model (although that model $\mathbf{M}$ would believe it contains sets of uncountable cardinality, e.g. $\mathbb{R}^{\mathbf{M}}$). Explain why this DOES NOT imply that ZFC is not satisfiable.

Problem 25. Show that for any τ-formulas ϕ, ψ, $\vdash (\phi \wedge \neg\phi) \to \psi$.

Problem 26. Prove the Deduction theorem. You may assume that any tautology is provable, where a τ-formula is called a *tautology* if it is a Boolean combination of subformulas $\phi_1, ..., \phi_n$ such that if we replace the latter by propositional variables $P_1, ..., P_n$ taking values in $\{\texttt{true, false}\}$, then ϕ would hold for all possible values of $P_1, ..., P_n$ (2^n possibilities). HINT: Induction on the length of the formal proof.

Problem 27. Prove the Constant Substitution theorem.

Problem 28. Show the following:
(a) (Associativity of +) $\mathsf{PA} \vdash \forall x \forall y \forall z[(x+y)+z = x+(y+z)]$,
(b) $\mathsf{PA} \vdash \forall x(0+x = x)$,
(c) (Commutativity of +) $\mathsf{PA} \vdash \forall x \forall y(x+y = y+x)$.

Problem 29. Prove the lemma about consistency.

Problem 30. Show that any consistent τ-theory T has a completion, i.e. there exists a consistent complete τ-theory $T' \supseteq T$. If you are not familiar with Zorn's lemma or transfinite induction, you may assume τ is countable.

Problem 31. Show that a τ-theory T is semantically complete if and only if for any $\mathbf{A}, \mathbf{B} \vDash T$, $\mathbf{A} \equiv B$.

Problem 32. Consider the topological space $\mathcal{T}$ of all consistent complete theories as described in the notes.
(a) Show that the statement "$T \vdash \phi \implies \exists$ finite $T_0 \subseteq T$ with $T_0 \vdash \phi$" is equivalent to $\mathcal{T}$ being compact.
(b) Find an appropriate topological space whose compactness is equivalent to the statement of the Compactness theorem. Obviously, you CANNOT use the Completeness theorem in proving the equivalence since otherwise $\mathcal{T}$ would work.

HINT: Use the equivalent definition of compactness of a topological space, which states that every family of closed sets with the finite intersection property has a nonempty intersection.

Problem 33. Let $\mathcal{T}$ be as in Problem 32. For every τ-sentence ϕ, put

$$<\phi> = \{T \in \mathcal{T} : T \vdash \phi\}.$$

(a) Prove that for $A \subseteq \mathcal{T}$, if

$$A = \bigcap_{i \in I} <\phi_i> = \bigcup_{j \in J} <\psi_j>,$$

then there are finite $I_0 \subseteq I, J_0 \subseteq J$ such that

$$A = \bigcap_{i \in I_0} <\phi_i> = \bigcup_{j \in J_0} <\psi_j> .$$

(b) Conclude from (a) that the only clopen (closed and open) sets in $\mathcal{T}$ are of the form $<\phi>$, where ϕ is a τ-sentence.

Problem 34. A τ-theory T' is called an *axiomatization* of a τ-theory T if for every τ-sentence ϕ,

$$T \vdash \phi \iff T' \vdash \phi.$$

T is called *finitely axiomatizable* if it has a finite axiomatization.

(a) Prove that a τ-theory T is finitely axiomatizable if and only if there is a finite subset of T that is an axiomatization of T.
(b) Let $T_\infty = \{\phi_n : n \in \mathbb{N}\}$, where

$$\phi_n \equiv \exists x_1 \ldots \exists x_n \left(\bigwedge_{1 \le i < j \le n} x_i \neq x_j \right).$$

Show that T_∞ is not finitely axiomatizable.

Problem 35. Fill in the left out details in the proof of the Completeness theorem.

Problem 36. Show that if a theory has arbitrarily large finite models, then it has an infinite model.

Problem 37.

(a) Show that for any model $\mathbf{M}$ of PA, there is a unique homomorphism $f : \mathbf{N} \to \mathbf{M}$ and this f is one-to-one (i.e. f is a τ_{a}-embedding).
(b) In the notation of (a), define the standard part of $\mathbf{M}$ by

$$\overline{\mathbb{N}} = f[\mathbb{N}].$$

Show that if $\mathbf{M}$ is nonstandard then $\overline{\mathbb{N}}$ is not definable in $\mathbf{M}$.
(c) (Overspill) Let $\mathbf{M}$ be a nonstandard model of PA, let $\phi(x, \vec{y})$ be a τ_{a}-formula, where $\vec{y}$ is a k-vector, and $\vec{a} \in M^k$. Show that if $\mathbf{M} \vDash \phi(n, \vec{a})$ for infinitely many $n \in \overline{\mathbb{N}}$, then there is $w \in M \setminus \overline{\mathbb{N}}$ such that $\mathbf{M} \vDash \phi(w, \vec{a})$. In other words, if a statement is true about infinitely many natural numbers, then it is true about an infinite number.

Problem 38.

(a) Show that there is a countable nonstandard model of $\mathsf{Th}(\mathbf{N})$ (True Arithmetic).
(b) Show that if $\mathbf{M}$ is a model of $\mathsf{Th}(\mathbf{N})$, then the unique homomorphism $f : \mathbb{N} \to \mathbf{M}$ is an elementary embedding. In other words, the standard part of $\mathbf{M}$ is (the universe of) an elementary substructure of $\mathbf{M}$.

Problem 39. Give an example of structures $\mathbf{A} \prec \mathbf{B}$ that demonstrate that the converse of Problem 20 fails.

Problem 40. Give an example of a structure $\mathbf{A}$ (in some signature) and a definable binary relation $Q(x, y)$ in it such that the unary relation

$$P(x) \iff \text{(for infinitely many } y\text{) } Q(x, y)$$

is NOT definable in $\mathbf{A}$.

Problem 41. Let $\mathbf{M}$ be a nonstandard model of PA and let $\overline{\mathbb{N}}$ be its standard part. By replacing it with $\mathbb{N}$, we can assume without loss of generality that $\overline{\mathbb{N}} = \mathbb{N}$.

(a) For all $a, b \in M$, define

$$a \sim b \iff |a - b| \in \mathbb{N},$$

where $z = |a - b|$ is the unique element in M such that $x + z = y$ or $y + z = x$. Show that $\sim$ is an equivalence relation on M and that it is NOT definable in $\mathbf{M}$.

(b) Put $Q = M/\sim$, so $Q = \{[a] : a \in M\}$, where $[a]$ denotes the equivalence class of a. Define the relation $\leq_Q$ on Q as follows: for all $[a], [b] \in Q$,

$$[a] <_Q [b] \iff \exists c \in M \setminus \mathbb{N} \text{ such that } a + c = b$$

Show that $<_Q$ is well-defined (does not depend on the representatives a, b) and is a linear ordering on Q.

(c) Show that the ordering $(Q, \leq_Q)$ has a least element but no greatest element and is a dense (in itself), i.e. for all $u, v \in Q$,

$$u <_Q v \implies \exists w(u <_Q w <_Q v),$$

where $u <_Q v$ stands for $u \leq_Q v \wedge u \neq v$.

Problem 42. Prove that a graph is 3-colorable if and only if so is every finite subgraph of it.

Problem 43. Solve Problem 18 using the Compactness theorem.

Problem 44.

(a) Show that the class of connected graphs is not axiomatizable.

(b) Show that the class of disconnected graphs is not axiomatizable.
HINT: Assume for contradiction that such theory T exists. Take two fresh constant symbols a, b and note that the theory constructed in part (a) implies S. Use Compactness (or Completeness) to get rid of the constant symbols a, b and arrive at a contradiction.

Problem 45. (Weak Lefschetz Principle) Let ϕ be a τ_{ring}-sentence. Show that if $\mathsf{ACF}_0 \models \phi$, then for large enough prime p, $\mathsf{ACF}_p \models \phi$.

Problem 46. The following is a well known theorem of additive combinatorics:

Theorem (van der Waerden)**.** *Suppose $\mathbb{Z}$ is finitely colored. Then one of the color classes contains arbitrarily long arithmetic progressions.*

Use this theorem and the Compactness theorem to derive the following finitary version:

Theorem. *Given any positive integers m and k, there exists $N \in \mathbb{N}$ such that whenever $\{0, 1, ..., N-1\}$ is colored with m colors, one of the color classes contains an arithmetic progression of length k.*

Problem 47. Show that the theory of vector spaces over $\mathbb{Q}$ is κ-categorical, for any uncountable cardinal κ.

Problem 48. A τ-theory T is called *absolutely categorical* if any two models of T are isomorphic (in particular, all models have the same cardinality).

(a) Show that if T is absolutely categorical then every model of T is finite.

(b) Let $\tau = (f)$, where f is a binary function symbol. Give an example of a τ-theory T, all models which are infinite (in particular T is not absolutely categorical according to (a)).

(c) Show that for any signature τ (not necessarily finite) and any τ-structures $\mathbf{A}, \mathbf{B}$, if $\mathbf{A}$ is finite, then

$$\mathbf{A} \equiv \mathbf{B} \iff \mathbf{A} \cong \mathbf{B}.$$

Conclude that the theory of any finite structure is absolutely categorical.
HINT: See Problem 10.

Problem 49.

(a) Show that the theory DLO of dense linear orderings without end points is $\aleph_0$-categorical. In particular, $\mathbb{Q}$ is the only countable dense linear ordering without end points up to isomorphism.
HINT: Enumerate the two models and recursively construct a sequence of finite partial isomorphisms by going back and forth between the models.

(b) Let $\tau_n = (<, c_1, ..., c_n)$, where c_i are constant symbols. Show that the theory
$$\mathrm{DLO}_n = \mathrm{DLO} \cup \{c_i < c_{i+1} : i < n\}$$
is $\aleph_0$-categorical. Conclude that DLO_n is complete.

(c) Let $\tau_n = (<, c_1, c_2, ...)$, where c_i are constant symbols. Show that the theory
$$\mathrm{DLO}_\infty = \mathrm{DLO} \cup \{c_i < c_{i+1} : i \in \mathbb{N}\}$$
is complete.

(d) Show that DLO_∞ has exactly three countable nonisomorphic models.

Problem 50. Let K be a field and let $\overline{K}$ be an algebraic closure of K. A nonconstant polynomial $f \in K[X_1, ..., X_n]$ is called *irreducible* if whenever $f = gh$ for some $g, h \in K[X_1, ..., X_n]$, either $\deg(g) = 0$ or $\deg(h) = 0$. Furthermore, f is called *absolutely irreducible* if it is irreducible in $\overline{K}[X_1, ..., X_n]$ (view f as an element of $\overline{K}[X_1, ..., X_n]$).

For example, the polynomial $X^2+1 \in \mathbb{R}[X]$ is irreducible, but it is not absolutely irreducible since $X^2 + 1 = (X + i)(X - i)$ in $\mathbb{C}[X]$. On the other hand, $XY - 1 \in \mathbb{Q}[X, Y]$ is absolutely irreducible.

Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and prove the following:

Theorem (Noether-Ostrowski Irreducibility Theorem)**.** *For $f \in \mathbb{Z}[X_1, ..., X_n]$ and prime p, let f_p denote the polynomial in $\mathbb{F}_p[X_1, ..., X_n]$ obtained by applying the canonical map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ to the coefficients of f (i.e. mod-ing out the coefficients by p). For all $f \in \mathbb{Z}[X_1, ..., X_n]$, f is absolutely irreducible (as an element of $\mathbb{Q}[X_1, ..., X_n]$) if and only if f_p is absolutely irreducible (as an element of $\mathbb{F}_p[X_1, ..., X_n]$), for all sufficiently large primes p.*

HINT: Your proof should be shorter than the statement of the problem.
REMARK: The original algebraic proof of this theorem is quite involved!

Problem 51. (Ax's theorem) Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, i.e. $f = (f_1, ..., f_n)$, where each $f_i(z_1, ..., z_n)$ is a polynomial in $z_1, ..., z_n$ with coefficients in $\mathbb{C}$. Prove that if f is injective then it is surjective.
HINT: Prove the statement for fixed n and fixed degree $d = \max_i\{\deg(f_i)\}$. Use without proof that $\overline{\mathbb{F}}_p = \bigcup_{l\geq 1} \mathbb{F}_{p^l}$, where $\mathbb{F}_{p^l}$ denotes the field of p^l elements (unique up to isomorphism) and $\overline{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Use the pigeon-hole principle.

Problem 52. Prove Tarski's theorem that $\mathsf{Th}(\mathbf{N})$ is not arithmetical.

Problem 53.

(a) Prove part (e) of Lemma 5.8.

(b) Prove parts (a)-(c) of Lemma 5.14.

(c) Rigorously prove that the Pair function in part (d) of Lemma 5.14 is a bijection.

Problem 54.

(a) Prove that the functions in (R1) of the definition of recursive functions are primitive recursive.
(b) Prove Lemmas 5.8, 5.14 and 5.15 with "recursive" replaced by "primitive recursive".

Problem 55. For $f : \mathbb{N}^{k+1} \to \mathbb{N}$, define
$$\bar{f}(\vec{a}, n) =< f(\vec{a}, 0), f(\vec{a}, 1), ..., f(\vec{a}, n-1) > .$$
Given $g : \mathbb{N}^{k+2} \to \mathbb{N}$, let $f : \mathbb{N}^2 \to \mathbb{N}$ be defined by the identity
$$f(\vec{a}, n) = g(\vec{a}, \bar{f}(\vec{a}, n)).$$
Show that if g is primitive recursive, then so is f.

Problem 56. Let $g_0, g_1 : \mathbb{N}^k \to \mathbb{N}$, $h_0, h_1 : \mathbb{N}^{k+3} \to \mathbb{N}$. We say that $f_0, f_1 : \mathbb{N}^{k+1} \to \mathbb{N}$ are defined by *simultaneous recursion* from g_0, g_1, h_0, h_1 if
$$\begin{array}{ll} f_0(\vec{a}, 0) = g_0(\vec{a}); & f_1(\vec{a}, 0) = g_1(\vec{a}); \\ f_0(\vec{a}, x+1) = h_0(\vec{a}, x, f_0(\vec{a}, x), f_1(\vec{a}, x)); & f_1(\vec{a}, x+1) = h_1(\vec{a}, x, f_0(\vec{a}, x), f_1(\vec{a}, x)). \end{array}$$
Show that if g_0, g_1, h_0, h_1 are primitive recursive, then so are f_0, f_1.

Problem 57. Let $g : \mathbb{N} \to \mathbb{N}$, $h : \mathbb{N}^3 \to \mathbb{N}$, $\tau : \mathbb{N}^2 \to \mathbb{N}$. We say that $f : \mathbb{N}^2 \to \mathbb{N}$ is defined by *nested recursion* from g, h, τ if
$$\begin{array}{rcl} f(0, y) & = & g(y) \\ f(x+1, y) & = & h(x, y, f(x, \tau(x, y))). \end{array}$$
Show that if g, h, τ are primitive recursive, then so is f.

Problem 58. Prove the following proposition (an equivalent version of Proposition 5.18) using the outline below:

Proposition. *There exists a recursive function $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that for every n, $\phi_n := \phi(n, \cdot)$ is primitive recursive and for every $k \in \mathbb{N}$ and every primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there is n such that $\forall \vec{a} \in \mathbb{N}^k$,*
$$f(\vec{a}) = \phi_n(<\vec{a}>).$$
In the latter case, we say that ϕ_n corresponds to f.

Let $\phi : \mathbb{N}^2 \to \mathbb{N}$ be a function with the following property: for every $n \in \mathbb{N}$, (below a is equal to the arity of the function corresponding to ϕ_n)

- if $n =<1, a, m>$ then ϕ_n corresponds to
  - the function $+ : \mathbb{N}^2 \to \mathbb{N}$ if $m = 0$ and $a = 2$,
  - the function $\cdot : \mathbb{N}^2 \to \mathbb{N}$ if $m = 1$ and $a = 2$,
  - the function $\chi_\le : \mathbb{N}^2 \to \mathbb{N}$ if $m = 2$ and $a = 2$,
  - the function $P^k_i : \mathbb{N}^k \to \mathbb{N}$ if $m =<k, i>$ and $a = k$.
- if $n =<2, a, m>$, where
  - $m =<<n_0, ..., n_k>$ for some $k \ge 1$,
  - $(n_0)_1 = k$,
  - $(n_i)_1 = a$ for $i = 1, ..., k$,

  then letting $g : \mathbb{N}^k \to \mathbb{N}$ be the function corresponding to ϕ_{n_0} and $h_i : \mathbb{N}^a \to \mathbb{N}$ the functions corresponding to ϕ_{n_i}, (for $i = 1, ..., k$), ϕ_n corresponds to the function obtained by *composition* from g and $h_0, ..., h_{k-1}$.
- if $n =<3, a, m>$, where

- $a \geq 1$
- $m = \langle n_0, n_1 \rangle$
- $(n_0)_1 = a - 1$,
- $(n_1)_1 = a + 1$,

then letting $g : \mathbb{N}^{a-1} \to \mathbb{N}$ be the function corresponding to ϕ_{n_0} and $h : \mathbb{N}^{a+1} \to \mathbb{N}$ the function corresponding to ϕ_{n_1}, ϕ_n corresponds to the function obtained by *primitive recursion* from g and h.

Note that to calculate the value $\phi(n, l)$, one needs to know $\phi(n', l')$ for only finitely many (n', l') with either $n' < n$ or $l' < l$. Use this and an idea similar to Dedekind's analysis of recursion to show that ϕ is recursive (i.e. one can define a recursive ϕ satisfying the property above).

Problem 59. Show that the Ackermann function is recursive.

Problem 60. Prove the following facts about the Ackermann function:
(a) $A(n, x + y) \geq A(n, x) + y$;
(b) $n \geq 1 \implies A(n + 1, y) > A(n, y) + y$;
(c) $A(n + 1, y) \geq A(n, y + 1)$;
(d) $2A(n, y) < A(n + 2, y)$;
(e) $x < y \implies A(n, x + y) \leq A(n + 2, y)$.

Problem 61. Show that the graph of the Ackermann function is primitive recursive.
HINT: Use a similar argument to the one for Problem 59, but bound your search using the value of the function.
NOTE: This implies that the Ackermann function is recursive.

Problem 62.
(a) Show that the Ackermann function grows faster than any primitive recursive function; more precisely, prove that for any primitive recursive function $f : \mathbb{N}^k \to \mathbb{N}$, there exists $n_f \in \mathbb{N}$ such that $f(\vec{x}) \leq A(n_f, |x|_1)$ for all $\vec{x} \in \mathbb{N}^k$, where $|\vec{x}|_1 = x_1 + \ldots + x_n$.
(b) Conclude that the Ackermann function is not primitive recursive.

Problem 63. Prove the following:
(a) $\mathsf{Q} \nvdash (x + y) + z = x + (y + z)$;
(b) $\mathsf{Q} \nvdash x + y = y + x$;
(c) $\mathsf{Q} \nvdash \forall x(0 + x = x)$.

Problem 64. Let $n, m \in \mathbb{N}$. Show the following:
(a) $\neg n \leq m \iff \mathsf{Q} \vdash \neg\Delta(n) \leq \Delta(m)$;
HINT: For $\Longleftarrow$ show the contrapositive.
(b) $\mathsf{Q} \vdash x \leq \Delta(n) \vee \Delta(n + 1) \leq x$.
HINT: Prove by induction on n.

Problem 65.
(a) Show that the representability of recursive functions in Q implies that recursive functions/relations are arithmetical.
(b) Give a direct proof that recursive functions/relations are arithmetical (without using their representability in Q).

Problem 66.

(a) Show that we can replace "recursive" by "arithmetical" in the statement of Gödel's Incompleteness theorem for $T \subseteq \mathsf{Th}(\mathbf{N})$, i.e. prove that if $T \subseteq \mathsf{Th}(\mathbf{N})$ is arithmetical, then it is incomplete.

(b) Show that there exists an arithmetical completion of PA, i.e. there is a complete τ_{a}-theory $T \supseteq PA$ such that $\ulcorner T \urcorner = \{\ulcorner \phi \urcorner : \phi \in T\}$ is an arithmetical subset of $\mathbb{N}$. Conclude that we CANNOT replace "recursive" by "arithmetical" in Rosser's form of the First Incompleteness theorem.

Problem 67. For a recursive theory T in any finite signature, recall that the relation

$$\mathsf{Proof}_T(a,b) \iff b \text{ is a code of a sentence and } a \text{ is a code of a proof of it from } T$$

is recursive, and let $\mathbf{Proof}_T(x,y)$ be a τ_{a}-formula representing it in Q. Put

$$\mathbf{Provable}_T(y) = \exists x \mathbf{Proof}_T(x,y).$$

Determine which of the following statements are true for an arbitrary τ_{a}-sentence θ and prove your answers:

(a) $\mathsf{PA} \vdash \theta \iff \mathbf{N} \vDash \mathbf{Provable}_{\mathsf{PA}}([\theta])$,
(b) $\mathsf{PA} \vdash \theta \to \mathbf{Provable}_{\mathsf{PA}}([\theta])$,
(c) $\mathsf{PA} \vdash \theta \implies \mathsf{PA} \vdash \mathbf{Provable}_{\mathsf{PA}}([\theta])$.

Problem 68. Let ϕ and θ be τ_a-sentences. Consider the following statements:

(1) $\mathsf{PA} \vdash \phi \implies \mathsf{PA} \vdash \theta$;
(2) $\mathsf{PA} \vdash \phi \to \theta$.

Are they equivalent for all ϕ, θ? If not, which implication holds and which may fail? Prove all your answers.

Problem 69. Let $\mathbf{Provable}_T$ be as in Problem 67. Determine which of the following statements are provable in PA for an arbitrary τ_{a}-sentence θ:

(a) $\mathbf{Provable}_{\mathsf{PA}}([\theta]) \to \theta$.
(b) $\mathbf{Provable}_{\mathsf{PA} \cup \{\neg\theta\}}([\theta]) \to \mathbf{Provable}_{\mathsf{PA}}([\theta])$,
(c) $\mathbf{Provable}_{\mathsf{PA}}([\theta]) \to \neg\mathbf{Provable}_{\mathsf{PA}}([\neg\theta])$,
(d) $\mathbf{Provable}_{\mathsf{PA}}(\mathbf{Provable}_{\mathsf{PA}}([\theta])) \to \mathbf{Provable}_{\mathsf{PA}}([\theta])$,
(e) $\mathbf{Provable}_{\mathsf{PA}}([\theta]) \to \mathbf{Provable}_{\mathsf{PA}}(\mathbf{Provable}_{\mathsf{PA}}([\theta]))$.

Prove all your answers except for part (d). For the latter, just make a guess.

Problem 70. Let τ be a finite signature. State and prove Löb's theorem for any recursive τ-theory T that interprets PA.

Problem 71. Show that the set of Σ_1^0 relations is closed under finite unions/intersections and taking projections, i.e. under the operations $\vee, \wedge, \exists$.

Problem 72. For $A \subseteq \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, we say that f enumerates A if the image of f is A, i.e. $f[\mathbb{N}] = A$.

Definition. *A set $A \subseteq \mathbb{N}$ is called recursively enumerable (r.e. for short) if there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ enumerating A.*

Prove the following characterizations of recursive and Σ_1^0 sets:

(a) A set $A \subseteq \mathbb{N}$ is recursive if and only if it is either finite or enumerable by a *strictly increasing* recursive function.
(b) For a set $A \subseteq \mathbb{N}$, the following are equivalent:
  (1) A is Σ^0_1
  (2) A is either finite or enumerable by an *injective* recursive function
  (3) A is r.e.

Problem 73. (Craig's lemma) Show that any Σ^0_1 theory T (in a finite signature τ) has a recursive axiomatization. Conclude that we CAN replace "recursive" by "Σ^0_1" in Rosser's form of the First Incompleteness theorem.
HINT: For a sentence ϕ, ϕ being in T is "recursively witnessed" by a number $n_\phi \in \mathbb{N}$. Modify ϕ into a logically equivalent sentence that encodes the witness n_ϕ.
REMARK 1: Compare this with Problem 66.
REMARK 2: One can in fact show that any Σ^0_1 theory has a *primitive recursive* axiomatization. This follows from Kleene's Normal Form theorem (see (c) of Problem 74).

Problem 74.
(a) (Weak representability of Σ^0_1) Show that for any Σ^0_1 relation $P \subseteq \mathbb{N}^k$, there is a τ_{a}-formula $\phi(\vec{x})$ such that for all $\vec{a} \in \mathbb{N}^k$:
$$P(\vec{a}) \iff \mathsf{Q} \vdash \phi(\Delta(\vec{a})).$$
NOTE: This is weaker than representability in Q.
(b) (Universal Σ^0_1 set) Recall the relation $U_{\mathsf{Q}} \subseteq \mathbb{N}^2$ defined as follows: for every $e, n \in \mathbb{N}$,
$$U_{\mathsf{Q}}(e,n) \iff \mathsf{Sub}_0(e,n) \in \ulcorner \mathsf{Thm}(\mathsf{Q}) \urcorner.$$
Show that for every Σ^0_1 relation $P \subseteq \mathbb{N}$, there is $e \in \mathbb{N}$ with $P = U_{\mathsf{Q}}(e)$. Thus U_{Q} is a universal Σ^0_1 relation.
(c) (Kleene's Normal Form theorem) Conclude that there is a *primitive recursive* relation $R \subseteq \mathbb{N}^3$ such that for any Σ^0_1 relation $P \subseteq \mathbb{N}^k$, there is $e \in \mathbb{N}$ such that $\forall \vec{a} \in \mathbb{N}^k$
$$P(\vec{a}) \iff \exists x R(e, <\vec{a}>, x).$$
In particular, every Σ^0_1 relation $P \subseteq \mathbb{N}^k$ is of the form
$$P(\vec{a}) \iff \exists x S(\vec{a}, x),$$
for some *primitive recursive* relation $S \subseteq \mathbb{N}^{k+1}$.

Problem 75. Let $A, B \subseteq \mathbb{N}^k$. Show the following:
(a) (Reduction property for Σ^0_1) If A, B are Σ^0_1, then there are disjoint Σ^0_1 sets $A^*, B^* \subseteq \mathbb{N}^k$ such that $A^* \subseteq A$, $B^* \subseteq B$ and $A^* \cup B^* = A \cup B$.
(b) (Separation property for Π^0_1) If A, B are disjoint Π^0_1 sets, then there is a Δ^0_1 (and hence recursive) set $S \subseteq \mathbb{N}^k$ such that $S \supseteq A$ and $S \cap B = \varnothing$.

Problem 76. Let $\tau = (0, 1, +, -, \cdot, <)$ and $\mathbf{Q} = (\mathbb{Q}, 0, 1, +, -, \cdot, <)$.
(a) Show that for every subset $S \subseteq \mathbb{Q}$ that is definable in $\mathbf{Q}$ by a quantifier free formula, there is $q \in \mathbb{Q}$ such that $(q, \infty) \subseteq S$ or $(q, \infty) \cap S = \varnothing$.
HINT: Prove this by induction on the construction (length) of the quantifier-free formula defining S.
(b) Use (a) to show that $\mathsf{Th}(\mathbf{Q})$ does NOT admit quantifier elimination.

Problem 77. (Diagram lemma) Let τ be a signature and let $\mathbf{A}, \mathbf{B}$ be τ-structures. Show that the following are equivalent:

(1) There is a τ-embedding $\mathbf{A} \hookrightarrow \mathbf{B}$;
(2) $\mathbf{A}$ can be expanded (not extended!) to a $\tau(B)$-structure that is a model of $\mathsf{Diag}(\mathbf{B})$.