INTRODUCTION TO DESCRIPTIVE SET THEORY

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Mathematicians in the early 20th century discovered that the Axiom of Choice implied the existence of pathological subsets of the real line lacking desirable regularity properties (for example nonmeasurable sets). This gave rise to descriptive set theory, a systematic study of classes of sets where these pathologies can be avoided, including, in particular, the definable sets. In the first half of the course, we will use techniques from analysis and set theory, as well as infinite games, to study definable sets of reals and their regularity properties, such as the perfect set property (a strong form of the continuum hypothesis), the Baire measurability, and measurability.

Descriptive set theory has found applications in harmonic analysis, dynamical systems, functional analysis, and various other areas of mathematics. Many of the recent applications are via the theory of definable equivalence relations (viewed as sets of pairs), which provides a framework for studying very general types of classification problems in mathematics. The second half of this course will give an introduction to this theory, culminating in a famous dichotomy theorem, which exhibits a minimum element among all problems that do not admit concrete classification.

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Part 1. Polish spaces

1. Definition and examples

Definition 1.1. A topological space is called *Polish* if it is separable and completely metrizable (i.e. admits a complete compatible metric).

We work with Polish topological spaces as opposed to Polish metric spaces because we don't want to fix a particular complete metric, we may change it to serve different purposes; all we care about is that such a complete compatible metric exists. Besides, our maps are homeomorphisms and not isometries, so we work in the category of topological spaces and not metric spaces.

Examples 1.2.

(a) For all $n \in \mathbb{N}$, $n = \{0, 1, \dots, n-1\}$ is Polish with discrete topology; so is $\mathbb{N}$;

(b) $\mathbb{R}^n$ and $\mathbb{C}^n$, for $n \geqslant 1$;

(c) Separable Banach spaces; in particular, separable Hilbert spaces, $\ell^p(\mathbb{N})$ and $L^p(\mathbb{R})$ for $0 < p < \infty$.

The following lemma, whose proof is left as an exercise, shows that when working with Polish spaces, we may always take a complete compatible metric $d \leqslant 1$.

Observation 1.3. *If X is a topological space with a compatible metric d, then the following metric is also compatible: for $x, y \in X$, $D(x, y) = \min(d(x, y), 1)$.*

Proposition 1.4.

(a) *Completion of any separable metric space is Polish.*

(b) *A closed subset of a Polish space is Polish (with respect to relative topology).*

(c) *A countable disjoint union*[1] *of Polish spaces is Polish.*

(d) *A countable product of Polish spaces is Polish (with respect to the product topology).*

Proof. (a) and (b) are obvious. We leave (c) as an exercise and prove (d). To this end, let $X_n, n \in \mathbb{N}$ be Polish spaces and let $d_n \leqslant 1$ be a complete compatible metric for X_n. For $x, y \in \prod_{n \in \mathbb{N}} X_n$, define

$$d(x, y) := \sum_{n \in \mathbb{N}} 2^{-n} d_n(x(n), y(n)).$$

It is easy to verify that d is a complete compatible metric for the product topology on $\prod_{n \in \mathbb{N}} X_n$. □

Examples 1.5.

(a) $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$;

(b) The Cantor space $\mathcal{C} := 2^{\mathbb{N}}$, with the discrete topology on 2;

[1] Disjoint union of topological spaces $\{X_i\}_{i \in I}$ is the space $\bigsqcup_{i \in I} X_i := \bigcup_{i \in I} \{i\} \times X_i$ equipped with the topology generated by sets of the form $\{i\} \times U_i$, where $i \in I$ and $U_i \subseteq X_i$ is open.

(c) The Baire space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$, with the discrete topology on $\mathbb{N}$.

(d) The Hilbert cube $\mathbb{I}^{\mathbb{N}}$, where $\mathbb{I} = [0,1]$.

In addition, Polish spaces are closed under fiber products, which we now define.

Definition 1.6. For a sequence $(X_i)_{i\in I}$ of topological spaces, a topological space Y, and continuous maps $f_i : X_i \to Y$, the *fiber product* of the (X_i, f_i) over Y is the set

$$\prod_{i\in I}(X_i, f_i) := \left\{(x_i)_{i\in I} \in \prod_{i\in I} X_i : f_i(x_i) = f_j(x_j) \text{ for all } i,j \in I\right\}$$

with the subspace topology of $\prod_{i\in I} X_i$.

The proof of the following is left as an exercise.

Proposition 1.7. *Any fiber product over a Hausdorff space is a closed subset of the product. In particular, a fiber product of countably-many Polish spaces over a Hausdorff space is Polish.*

As mentioned in Proposition 1.4, closed subsets of Polish spaces are Polish. What other subsets have this property? The proposition below answers this question, but first we recall here that countable intersections of open sets are called G_δ sets, and countable unions of closed sets are called F_σ.

Lemma 1.8. *If X is a metric space, then closed sets are G_δ; equivalently, open sets are F_σ.*

Proof. Let $C \subseteq X$ be a closed set and let d be a metric for X. For $\varepsilon > 0$, define $U_\varepsilon := \{x \in X : d(x, C) < \varepsilon\}$, and we claim that $C = \bigcap_n U_{1/n}$. Indeed, $C \subseteq \bigcap_n U_{1/n}$ is trivial, and to show the other inclusion, fix $x \in \bigcap_n U_{1/n}$. Thus, for every n, we can pick $x_n \in C$ with $d(x, x_n) < 1/n$, so $x_n \to x$ as $n \to \infty$, and hence $x \in C$ by the virtue of C being closed. □

Proposition 1.9. *For every Polish space X, there is an injection $c : X \hookrightarrow \mathcal{C}$ such that the c-preimages of the sets $V_n := \{x \in \mathcal{C} : x(n) = 1\}$, $n \in \mathbb{N}$, are open. In particular, the c-preimages of open sets are F_σ.*

Proof. Fixing a countable basis $(U_n)_{n\in\mathbb{N}}$, define $c : X \to \mathcal{C}$ by $c(x)(n) = 1 :\Leftrightarrow x \in U_n$. We leave it as an exercise to verify that c is as desired. □

Proposition 1.10. *A subset of a Polish space is Polish if and only if it is G_δ.*

Proof. Let X be a Polish space and let d_X be a complete compatible metric on X.
$\Leftarrow$: Considering first an open set $U \subseteq X$, we exploit the fact that it does not contain its boundary points to define a compatible metric for the topology of U that makes the boundary of U "look like infinity" in order to prevent sequences that converge to the boundary from being Cauchy. In fact, instead of defining a metric explicitly[2], we define a homeomorphism of U with a closed subset of $X \times \mathbb{R}$ by

$$x \mapsto \left(x, \frac{1}{d_X(x, \partial U)}\right),$$

where d_X is a complete compatible metric for X. It is, indeed, easy to verify that this map is an embedding and its image is closed.

[2] This version of the proof was suggested by Anton Bernshteyn.

Combining countably-many instances of this gives a proof for G_δ sets: given $Y := \bigcap_{n\in\mathbb{N}} U_n$ with U_n open, the map $Y \to X \times \mathbb{R}^{\mathbb{N}}$ defined by

$$x \mapsto \left(x, \left(\frac{1}{d_X(x, \partial U_n)}\right)_n\right)$$

is a homeomorphism of Y with a closed subset of $X \times \mathbb{R}^{\mathbb{N}}$.
$\Rightarrow$ (Alexandrov): Let $Y \subseteq X$ be completely metrizable and let d_Y be a complete compatible metric for Y. Define an open set $V_n \subseteq X$ as the union of all open sets $U \subseteq X$ that satisfy

(i) $U \cap Y \neq \emptyset$,
(ii) $\mathrm{diam}_{d_X}(U) < 1/n$,
(iii) $\mathrm{diam}_{d_Y}(U \cap Y) < 1/n$.

We show that $Y = \bigcap_{n\in\mathbb{N}} V_n$. First fix $x \in Y$ and take any $n \in \mathbb{N}$. Take an open neighborhood $U_1 \subseteq Y$ of x in Y of d_Y-diameter less than $1/n$. By the definition of relative topology, there is an open set U_2 in X such that $U_2 \cap Y = U_1$. Let U_3 be an open neighborhood of x in X of d_X-diameter less than $1/n$. Then $U = U_2 \cap U_3$ satisfies all of the conditions above. Hence $x \in V_n$.

Conversely, if $x \in \bigcap_{n\in\mathbb{N}} V_n$, then for each $n \in \mathbb{N}$, there is an open (relative to X) neighborhood $U_n \subseteq X$ of x satisfying the conditions above. Condition (ii) implies that $x \in \overline{Y}$, so any open neighborhood of x has a nonempty intersection with Y; because of this, we can replace U_n by $\bigcap_{m\leqslant n} U_m$ and assume without loss of generality that $(U_n)_{n\in\mathbb{N}}$ is decreasing. Now, take $x_n \in U_n \cap Y$. Conditions (i) and (ii) imply that $(x_n)_{n\in\mathbb{N}}$ converges to x. Moreover, condition (iii) and the fact that $(U_n)_{n\in\mathbb{N}}$ is decreasing imply that $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to d_Y. Thus, since d_Y is complete, $x_n \to x'$ for some $x' \in Y$. Because limit is unique in Hausdorff spaces, $x = x' \in Y$. $\square$

As an example of a G_δ subset of a Polish space, we give the following proposition, whose proof is left to the reader.

Proposition 1.11. *The Cantor space $\mathcal{C}$ is homeomorphic to a closed subset of the Baire space $\mathcal{N}$, whereas $\mathcal{N}$ is homeomorphic to a G_δ subset of $\mathcal{C}$.*

2. Trees

2.A. **Set theoretic trees.** For a nonempty set A, we denote by $A^{<\mathbb{N}}$ the set of finite tuples of elements of A, i.e.

$$A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^n,$$

where $A^0 = \{\emptyset\}$. For $s \in A^{<\mathbb{N}}$, we denote by $|s|$ the length of s; thus, s is a function from $\{0, 1, \ldots, |s|-1\}$ to A. Recalling that functions are sets of pairs, the notation $s \subseteq t$ for $s, t \in A^{<\mathbb{N}}$ means that $|s| \leqslant |t|$ and $s(i) = t(i)$ for all $i < |s|$.

Definition 2.1. For a set A, a subset T of $A^{<\mathbb{N}}$ is called a (*set theoretic*) *tree* if it is closed downward under $\subseteq$, i.e. for all $s, t \in A^{<\mathbb{N}}$, if $t \in T$ and $s \subseteq t$, then $s \in T$.

For $s \in A^{<\mathbb{N}}$ and $a \in A$, we write $s^\frown a$ to denote the extension of s to a tuple of length $|s|+1$ that takes the value a at index $|s|$. For $s, t \in A^{<\mathbb{N}}$, we also write $s^\frown t$ to denote the tuple obtained by appending t at the end of s.

To see why the sets T in the above definitions are called trees, we now show how to obtain a graph theoretic rooted tree G_T from a set theoretic tree T. Define the vertex set of G_T to be T. Note that since T is nonempty, $\emptyset \in T$, and declare $\emptyset$ the root of G_T. Now put an edge between $s, t \in T$ if $t = s^\frown a$ for some $a \in A$.

Conversely, given a graph theoretic rooted tree G (a connected acyclic graph) with root v_0, one obtains a set theoretic tree T_G on $V(G)$ (the set of vertices of G) by identifying each vertex v with $(v_1, ..., v_n)$, where $v_n = v$ and $(v_0, v_1, ..., v_n)$ is the unique path from v_0 to v.

2.B. **Infinite branches and closed subsets of $A^{\mathbb{N}}$.** Call a tree T on A *pruned* if for every $s \in T$, there is $a \in A$ with $s^\frown a \in T$.

Given a tree T on a set A, we denote by $[T]$ the set of infinite branches through T, that is,

$$[T] = \left\{x \in A^{\mathbb{N}} : \forall n \in \mathbb{N}(x|_n \in T)\right\}.$$

Thus we obtain a subset of $A^{\mathbb{N}}$ from a tree. Conversely, given a subset $Y \subseteq A^{\mathbb{N}}$, we can obtain a tree T_Y on A by:

$$T_Y = \{x|_n : x \in Y, n \in \mathbb{N}\}.$$

Note that T_Y is a pruned tree. It is also clear that $Y \subseteq [T_Y]$, but for which subsets Y do we have $[T_Y] = Y$? To answer this question, we give A the discrete topology and consider $A^{\mathbb{N}}$ as a topological space with the product topology. Note that the sets of the form

$$N_s := \left\{x \in A^{\mathbb{N}} : s \subseteq x\right\},$$

for $s \in A^{<\mathbb{N}}$, form a basis for the product topology on $A^{\mathbb{N}}$.

Lemma 2.2. *For a tree T on A, $[T]$ is a closed subset of $A^{\mathbb{N}}$.*

Proof. We show that the complement of $[T]$ is open. Indeed, if $x \notin [T]$, then there is $n \in \mathbb{N}$ such that $s = x|_n \notin T$. But then $N_s \cap [T] = \emptyset$. □

Proposition 2.3. *A subset $Y \subseteq A^{\mathbb{N}}$ is closed if and only if $Y = [T_Y]$.*

Proof. The right-to-left direction follows form the previous lemma, so we prove left-to-right. Let Y be closed, and as $Y \subseteq [T_Y]$, we only need to show $[T_Y] \subseteq Y$. Fix $x \in [T_Y]$. By the definition of T_Y, for each $n \in \mathbb{N}$, there is $y_n \in Y$ such that $x|_n \subseteq y_n$. It is clear that $(y_n)_{n \in \mathbb{N}}$ converges pointwise to x (i.e. converges in the product topology) and hence $x \in Y$ since Y is closed. □

Note that if A is countable, then $A^{\mathbb{N}}$ is Polish. Examples of such spaces are the Cantor space and the Baire space, which, due to their combinatorial nature, are two of the most useful Polish spaces in descriptive set theory. We think of the Cantor space and the Baire space as the sets of infinite branches through the complete binary and $\mathbb{N}$-ary trees, respectively. Trees on $\mathbb{N}$ in particular play a crucial role in the subject, as we will see below.

2.C. **Compactness.** Above, we characterized the closed subsets of $A^{\mathbb{N}}$ as the sets of infinite branches through trees on A. Here we characterize the compact subsets.

For a tree T on A, and $s \in T$, put

$$T(s) := \{a \in A : s^\frown a \in T\}.$$

We say that a tree T on A is *finitely branching* if for each $s \in T$, $T(s)$ is finite. Equivalently, in G_T every vertex has finite degree.

Lemma 2.4 (König)**.** *Any finitely branching infinite tree T on A has an infinite branch, i.e. $[T] \neq \emptyset$.*

Proof. Very easy, left to the reader. $\square$

For a tree $T \subseteq A^{<\mathbb{N}}$, sets of the form $N_s \cap [T]$, for $s \in A^{<\mathbb{N}}$, form a basis for the relative topology on $[T]$. Also, for $s \in A^{<\mathbb{N}}$, putting $T_s := \{t \in T : t \subseteq s \vee t \supseteq s\}$, it is clear that T_s is a tree and $[T_s] = N_s \cap [T]$.

Proposition 2.5. *For a tree $T \subseteq A^{<\mathbb{N}}$, if T is finitely branching, then $[T]$ is compact. For a pruned tree $T \subseteq A^{<\mathbb{N}}$, the converse also holds: if $[T]$ is compact, then T is finitely branching. Thus, a closed set $Y \subseteq A^{\mathbb{N}}$ is compact if and only if T_Y is finitely branching.*

Proof. We leave proving the first statement as an exercise. For the second statement, let T be a pruned tree and suppose that $[T]$ is compact. Assume for contradiction that T is not finitely branching, i.e. there is $s \in T$ such that $T(s)$ is infinite. Since $[T]$ is compact, so is $[T_s]$ being a closed subset, so we can focus on $[T_s]$. For each $a \in T(s)$, the set $[T_{s^\frown a}]$ is nonempty because T is pruned. But then $\{[T_{s^\frown a}]\}_{a \in T(s)}$ is an open cover of $[T_s]$ that doesn't have a finite subcover since the sets in it are nonempty and pairwise disjoint. $\square$

A subset of a topological space is called *σ-compact* or K_σ if it is a countable union of compact sets. The above proposition shows that the Baire space is not compact. In fact, we have something stronger:

Corollary 2.6. *The Baire space $\mathcal{N}$ is not σ-compact.*

Proof. For $x, y \in \mathcal{N}$, we say that *y dominates* (resp. *eventually dominates*) x, if $y(n) \geqslant x(n)$ for all (resp. sufficiently large) $n \in \mathbb{N}$. Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact subsets of $\mathcal{N}$. By the above proposition, T_{K_n} is finitely branching and hence there is $x_n \in \mathcal{N}$ that dominates every element of K_n. By diagonalization, we obtain $x \in \mathcal{N}$ that eventually dominates x_n for all $n \in \mathbb{N}$. Thus, x eventually dominates every element of $\bigcup_{n \in \mathbb{N}} K_n$, and hence, the latter cannot be all of $\mathcal{N}$. $\square$

The Baire space is not just an example of a topological space that is not σ-compact, it is in fact the canonical obstruction to being σ-compact as the following dichotomy shows:

Theorem 2.7 (Hurewicz)**.** *For any Polish space X, either X is σ-compact, or else, X contains a closed subset homeomorphic to $\mathcal{N}$.*

We will not prove this theorem here since its proof is somewhat long and we will not be using it below.

2.D. **Monotone tree-maps and continuous functions.** In this subsection, we show how to construct continuous functions between tree spaces (i.e. spaces of infinite branches through trees).

Definition 2.8. Let S, T be trees (on sets A, B, respectively). A map $\varphi : S \to T$ is called *monotone* if $s \subseteq t$ implies $\varphi(s) \subseteq \varphi(t)$. For such φ let

$$D_\varphi = \left\{ x \in [S] : \lim_{n \to \infty} |\varphi(x|_n)| = \infty \right\}.$$

Define $\varphi^* : D_\varphi \to [T]$ by letting

$$\varphi^*(x) := \bigcup_{n\in\mathbb{N}} \varphi(x|_n).$$

We call φ *proper* if $D_\varphi = [S]$.

Proposition 2.9. *Let $\varphi : S \to T$ be a monotone map (as above).*

(a) *The set D_φ is G_δ in $[S]$ and $\varphi^* : D_\varphi \to [T]$ is continuous.*
(b) *Conversely, if $f : G \to [T]$ is continuous, with $G \subseteq [S]$ a G_δ set, then there is monotone $\varphi : S \to T$ with $f = \varphi^*$.*

Proof. The proof of (b) is outlined as a homework problem, so we will only prove (a) here. To see that D_φ is G_δ note that for $x \in [S]$, we have

$$x \in D_\varphi \iff \forall n \exists m |\varphi(x|_m)| \geqslant n,$$

and the set $U_{n,m} = \{x \in [S] : |\varphi(x|_m)| \geqslant n\}$ is trivially open as membership in it depends only on the first m coordinates. For the continuity of φ^*, it is enough to show that for each $\tau \in T$, the preimage of $[T_\tau]$ under φ^* is open; but for $x \in D_\varphi$,

$$\begin{aligned} x \in (\varphi^*)^{-1}([T_\tau]) &\Leftrightarrow \varphi^*(x) \in [T_\tau] \\ &\Leftrightarrow \varphi^*(x) \supseteq \tau \\ &\Leftrightarrow (\exists \sigma \in S \text{ with } \varphi(\sigma) \supseteq \tau)\ x \in S_\sigma, \end{aligned}$$

and in the latter condition, $\exists$ is a union, and $x \in S_\sigma$ defines a basic open set. □

Using this machinery, we easily derive the following useful lemma.

A closed set C in a topological space X is called a *retract of X* if there is a continuous function $f : X \to C$ such that $f|_C = \mathrm{id}|_C$ (i.e. $f(x) = x$, for all $x \in C$). This f is called a *retraction of X to C*.

Lemma 2.10. *Any nonempty closed subset $X \subseteq A^{\mathbb{N}}$ is a retract of $A^{\mathbb{N}}$. In particular, for any two nonempty closed subsets $X \subseteq Y$, X is a retract of Y.*

Proof. Noting that T_X is a nonempty pruned tree, we define a monotone map $\varphi : A^{<\mathbb{N}} \to T_X$ such that $\varphi(s) = s$ for $s \in T_X$ and thus φ^* will be a retraction of $A^{\mathbb{N}}$ to X. For $s \in A^{<\mathbb{N}}$, we define $\varphi(s)$ by induction on $|s|$. Let $\varphi(\emptyset) := \emptyset$ and assume $\varphi(s)$ is defined. Fix $a \in A$. If $s^\frown a \in T_X$, put $\varphi(s^\frown a) := s^\frown a$. Otherwise, there is $b \in A$ with $\varphi(s)^\frown b \in T_X$ because T_X is pruned, and we put $\varphi(s^\frown a) := \varphi(s)^\frown b$. □

3. Compact metrizable spaces

3.A. **Basic facts and examples.** Recall that a topological space is called *compact* if every open cover has a finite subcover. By taking complements, this is equivalent to the statement that every (possibly uncountable) family of closed sets with the finite intersection property[3] has a nonempty intersection. In the following proposition we collect basic facts about compact spaces, which we won't prove (see Sections 0.6, 4.1, 4.2, 4.4 of [Fol99]).

Proposition 3.1.

(a) *Closed subsets of compact topological spaces are compact.*

[3] We say that a family $\{F_i\}_{i\in I}$ of sets has the *finite intersection property* if for any finite $I_0 \subseteq I$, $\bigcap_{i\in I_0} F_i \neq \emptyset$.

(b) *Compact (in the relative topology) subsets of Hausdorff topological spaces are closed.*
(c) *Union of finitely many compact subsets of a topological space is compact. Finite sets are compact.*
(d) *Continuous image of a compact space is compact. In particular, if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f maps closed (resp. F_σ) sets to closed (resp. F_σ) sets.*
(e) *A continuous injection from a compact space into a Hausdorff space is an embedding (i.e. a homeomorphism with the image).*
(f) *Disjoint union of finitely many compact spaces is compact.*
(g) *(Tychonoff's Theorem) Product of compact spaces is compact.*

Definition 3.2. For a metric space (X,d) and $\varepsilon > 0$, a set $F \subseteq X$ is called an *ε-net* if any point in X is within ε distance from a point in F, i.e. $X = \bigcup_{y\in F} B(y,\varepsilon)$, where $B(y,\varepsilon)$ is the open ball of radius ε centered at y. Metric space (X,d) is called *totally bounded* if for every $\varepsilon > 0$, there is a finite ε-net.

Lemma 3.3. *Totally bounded metric spaces are separable.*

Proof. For every n, let F_n be a finite $\frac{1}{n}$-net. Then, $D = \bigcup_n F_n$ is countable and dense. □

Proposition 3.4. *Let (X,d) be a metric space. The following are equivalent:*
(1) *X is compact.*
(2) *Every sequence in X has a convergent subsequence.*
(3) *X is complete and totally bounded.*

In particular, compact metrizable spaces are Polish.

Proof. Outlined in a homework exercise. □

Examples of compact Polish spaces include $\mathcal{C}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\mathbb{I} = [0,1]$, $\mathbb{I}^{\mathbb{N}}$. A more advanced example is the space $P(X)$ of Borel probability measures on a compact Polish space X under the weak*-topology. In the next two subsections we will see however that the Hilbert cube and the Cantor space play special roles among all the examples.

3.B. **Universality of the Hilbert Cube.**

Theorem 3.5. *Every separable metrizable space embeds into the Hilbert cube $\mathbb{I}^{\mathbb{N}}$. In particular, the Polish spaces are, up to homeomorphism, exactly the G_δ subspaces of $\mathbb{I}^{\mathbb{N}}$, and the compact metrizable spaces are, up to homeomorphism, exactly the closed subspaces of $\mathbb{I}^{\mathbb{N}}$.*

Proof. Let X be a separable metrizable space. Fix a compatible metric $d \leqslant 1$ and a dense subset $(x_n)_{n\in\mathbb{N}}$. Define $f : X \to \mathbb{I}^{\mathbb{N}}$ by setting

$$f(x) := \Big(d(x,x_n)\Big)_{n\in\mathbb{N}},$$

for $x \in X$. It is straightforward to show that f is injective and that f, f^{-1} are continuous. □

Corollary 3.6. *Every Polish space can be embedded as a dense G_δ subset into a compact metrizable space.*

Proof. By the previous theorem, any Polish space is homeomorphic to a G_δ subset Y of the Hilbert cube and Y is dense in its closure, which is compact. □

As we just saw, Polish spaces can be thought of as G_δ subsets of a particular Polish space. Although this is interesting on its own, it would be more convenient to have Polish spaces as closed subsets of some Polish space because the set of closed subsets of a Polish space has a nice structure, as we will see later on. This is accomplished in the following theorem.

Theorem 3.7. *Every Polish space is homeomorphic to a closed subspace of $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let X be a Polish space and, by the previous theorem, we may assume that X is a G_δ subspace of $\mathbb{I}^{\mathbb{N}}$. Letting d be a complete compatible metric on $\mathbb{I}^{\mathbb{N}}$, we use a trick similar to the one used in Proposition 1.10 when we made a G_δ set "look closed" by changing the metric. Let $X = \bigcap_n U_n$, where U_n are open in $\mathbb{I}^{\mathbb{N}}$, and put $F_n = \mathbb{I}^{\mathbb{N}} \setminus U_n$. Define $f : X \to \mathbb{R}^{\mathbb{N}}$ as follows: for $x \in X$,

$$f(x)(n) := \begin{cases} x(k) & \text{if } n = 2k \\ \frac{1}{d(x,F_k)} & \text{if } n = 2k+1 \end{cases}.$$

Even coordinates ensure that f is injective and the odd coordinates ensure that the image is closed. The continuity of f follows from the continuity of the coordinate functions $x \mapsto f(x)(n)$, for all $n \in \mathbb{N}$. Thus, f is an embedding as $\mathbb{I}^{\mathbb{N}}$ is compact (see (e) of Proposition 3.1); in fact, f^{-1} is just the projection onto the even coordinates and hence is obviously continuous. $\square$

3.C. **Continuous images of the Cantor space.** Theorem 3.5 characterizes all compact metrizable spaces as closed subspaces of $\mathbb{I}^{\mathbb{N}}$. Here we give another characterization using surjections instead of injections (reversing the arrows).

Theorem 3.8. *Every nonempty compact metrizable space is a continuous image of $\mathcal{C}$.*

Proof. First we show that $\mathbb{I}^{\mathbb{N}}$ is a continuous image of $\mathcal{C}$. For this it is enough to show that $\mathbb{I}$ is a continuous image of $\mathcal{C}$ since $\mathcal{C}$ is homeomorphic to $\mathcal{C}^{\mathbb{N}}$ (why?). But the latter is easily done via the map $f : \mathcal{C} \to \mathbb{I}$ given by

$$x \mapsto \sum_n x(n) 2^{-n-1}.$$

Now let X be a compact metrizable space, and by Theorem 3.5, we may assume that X is a closed subspace of $\mathbb{I}^{\mathbb{N}}$. As we just showed, there is a continuous surjection $g : \mathcal{C} \to \mathbb{I}^{\mathbb{N}}$ and thus, $g^{-1}(X)$ is a closed subset of $\mathcal{C}$, hence a retract of $\mathcal{C}$ (Lemma 2.10). $\square$

3.D. **The hyperspace of compact sets.** In this subsection we discuss the set of all compact subsets of a given Polish space and give it a natural topology, which turns out to be Polish.

Notation 3.9. For a set X and a collection $\mathcal{A} \subseteq \mathscr{P}(X)$, define the following for each subset $U \subseteq X$:

$$(U)_{\mathcal{A}} := \{A \in \mathcal{A} : A \subseteq U\},$$
$$[U]_{\mathcal{A}} := \{A \in \mathcal{A} : A \cap U \neq \emptyset\}.$$

Now let X be a topological space. We denote by $\mathcal{K}(X)$ the collection of all compact subsets of X and we equip it with the *Vietoris topology*, namely, the one generated by the sets of the form $(U)_{\mathcal{K}}$ and $[U]_{\mathcal{K}}$, for open $U \subseteq X$. Thus, a basis for this topology consists of the sets

$$\begin{aligned}\langle U_0; U_1, \ldots, U_n\rangle_{\mathcal{K}} :=& (U_0)_{\mathcal{K}} \cap [U_1]_{\mathcal{K}} \cap [U_2]_{\mathcal{K}} \cap \ldots \cap [U_n]_{\mathcal{K}} \\ =& \{K \in \mathcal{K}(X) : K \subseteq U_0 \wedge K \cap U_1 \neq \emptyset \wedge \ldots \wedge K \cap U_n \neq \emptyset\},\end{aligned}$$

for $U_0, U_1, \ldots, U_n$ open in X. By replacing U_i with $U_i \cap U_0$, for $i \leqslant n$, we may and will assume that $U_i \subseteq U_0$ for all $i \leqslant n$.

Now we assume further that (X, d) is a metric space with $d \leqslant 1$. We define the *Hausdorff metric* d_H on $\mathcal{K}(X)$ as follows: for $K, L \in \mathcal{K}(X)$, put

$$\delta(K, L) := \max_{x \in K} d(x, L),$$

with convention that $d(x, \emptyset) = 1$ and $\delta(\emptyset, L) = 0$. Thus, letting $B(L, r) := \{x \in X : d(x, L) < r\}$, we have

$$\delta(K, L) = \inf_r [K \subseteq B(L, r)].$$

$\delta(K, L)$ is not yet a metric because it is not symmetric. So we symmetrize it: for arbitrary $K, L \in \mathcal{K}(X)$, put

$$d_H(K, L) = \max\{\delta(K, L), \delta(L, K)\}.$$

Thus,

$$d_H(K, L) = \inf_r [K \subseteq B(L, r) \text{ and } L \subseteq B(K, r)].$$

Proposition 3.10. *Hausdorff metric is compatible with the Vietoris topology.*

Proof. Left as an exercise. ☐

Proposition 3.11. *If X is separable, then so is $\mathcal{K}(X)$.*

Proof. Let D be a countable dense subset of X, and put

$$\mathrm{Fin}(D) = \{F \subseteq D : F \text{ is finite}\}.$$

Clearly $\mathrm{Fin}(D) \subseteq \mathcal{K}(X)$ and is countable. To show that it is dense, fix a nonempty basic open set $\langle U_0; U_1, \ldots, U_n\rangle_{\mathcal{K}}$, where we may assume that $\emptyset \neq U_i \subseteq U_0$ for all $i \leqslant n$. By density of D in X, there is $d_i \in U_i \cap D$ for each $1 \leqslant i \leqslant n$, so $\{d_i : 1 \leqslant i \leqslant n\} \in \langle U_0; U_1, \ldots, U_n\rangle_{\mathcal{K}} \cap \mathrm{Fin}(D)$. ☐

Next we will study convergence in $\mathcal{K}(X)$. Given $K_n \to K$, we will describe what K is in terms of K_n without referring to Hausdorff metric or Vietoris topology. To do this, we discuss other notions of limits for compact sets.

Given any topological space X and a sequence $(K_n)_{n \in \mathbb{N}}$ in $\mathcal{K}(X)$, define its *topological upper limit*, $\overline{\mathrm{T}\lim}_n K_n$ to be the set

$$\{x \in X : \text{Every open nbhd of } x \text{ meets } K_n \text{ for infinitely many } n\},$$

and its *topological lower limit*, $\underline{\mathrm{T}\lim}_n K_n$, to be the set

$$\{x \in X : \text{Every open nbhd of } x \text{ meets } K_n \text{ for all but finitely many } n\}.$$

It is immediate from the definitions that $\underline{\mathrm{T}\lim}_n K_n \subseteq \overline{\mathrm{T}\lim}_n K_n$ and both sets are closed (but may not be compact). It is also easy to check that

$$\overline{\mathrm{T}\lim}_n K_n = \bigcap_{i \in \mathbb{N}} \overline{\bigcup_{j \geqslant i} K_j}.$$

If $\underline{\mathrm{T\,lim}}_n K_n = \overline{\mathrm{T\,lim}}_n K_n$, we call the common value the *topological limit of* $(K_n)_{n\in\mathbb{N}}$, and denote it by $\mathrm{T\,lim}_n K_n$. If $d \leqslant 1$ is a compatible metric for X, then one can check (left as an exercise) that $K_n \to K$ in Hausdorff metric implies $K = \mathrm{T\,lim}_n K_n$. However, the converse may fail as Example 3.12(b) shows.

Examples 3.12.

(a) Let $X = \mathbb{R}^m$ and $K_n = \bar{B}(\vec{0}, 1 + \frac{(-1)^n}{n})$. Then $K_n \to \bar{B}(\vec{0}, 1)$ in Hausdorff metric.

(b) Let $X = \mathbb{R}$ and $K_n = [0,1] \cup [n, n+1]$. The Hausdorff distance between different K_n is 1, so the sequence $(K_n)_n$ does not converge in Hausdorff metric. Nevertheless $\mathrm{T\,lim}_n K_n$ exists and is equal to $[0,1]$.

(c) Let $X = \mathbb{R}$, $K_{2n} = [0,1]$ and $K_{2n+1} = [1,2]$, for each $n \in \mathbb{N}$. In this case, we have $\overline{\mathrm{T\,lim}}_n K_n = [0,2]$, whereas $\underline{\mathrm{T\,lim}}_n K_n = \{1\}$.

Finally, note that if X is first-countable (for example, metrizable) and $K_n \neq \emptyset$ then the topological upper limit consists of all $x \in X$ that satisfy:

$$\exists (x_n)_{n\in\mathbb{N}}[\forall n(x_n \in K_n) \text{ and } x_{n_i} \to x \text{ for some subsequence } (x_{n_i})_{i\in\mathbb{N}}],$$

and the topological lower limit consists of all $x \in X$ that satisfy:

$$\exists (x_n)_{n\in\mathbb{N}}[\forall n(x_n \in K_n) \text{ and } x_n \to x].$$

The relation between these limits and the limit in Hausdorff metric is explored more in homework problems. Here we just use the upper topological limit merely to prove the following theorem.

Theorem 3.13. *If X is completely metrizable then so is $\mathcal{K}(X)$. In particular, if X is Polish, then so is $\mathcal{K}(X)$.*

Proof. Let $d \leqslant 1$ be a complete compatible metric on X and let $(K_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{K}(X)$, where we assume without loss of generality that $K_n \neq \emptyset$. Setting $K = \overline{\mathrm{T\,lim}}_n K_n = \bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\geqslant N} K_n}$, we will show that $K \in \mathcal{K}(X)$ and $d_H(K_n, K) \to 0$.

Claim. K is compact.

Proof of Claim. Since K is closed and X is complete, it is enough to show that K is totally bounded. For this, we will verify that given $\varepsilon > 0$ there is a finite set $F \subseteq X$ such that $K \subseteq B(F, \varepsilon)$. Let N be such that $d_H(K_n, K_m) < \varepsilon/4$ for all $n, m \geqslant N$, and let F be an $\varepsilon/4$-net for K_N, i.e. $K_N \subseteq B(F, \varepsilon/4)$. Thus, for each $n \geqslant N$,

$$K_n \subseteq B(K_N, \varepsilon/4) \subseteq B(F, \varepsilon/4 + \varepsilon/4)$$

so $\bigcup_{n\geqslant N} K_n \subseteq B(F, \varepsilon/2)$, and hence

$$K \subseteq \overline{\bigcup_{n\geqslant N} K_n} \subseteq \overline{B(F, \varepsilon/2)} \subseteq B(F, \varepsilon). \qquad \boxtimes$$

It remains to show that $d_H(K_n, K) \to 0$. Fix $\varepsilon > 0$ and let N be such that $d_H(K_n, K_m) < \varepsilon/2$ for all $n, m \geqslant N$. We will show that if $n \geqslant N$, $d_H(K_n, K) < \varepsilon$.

Proof of $\delta(K, K_n) < \varepsilon$. By the choice of N, we have $K_m \subseteq B(K_n, \varepsilon/2)$ for each $m \geqslant N$. Thus $\bigcup_{m\geqslant N} K_m \subseteq B(K_n, \varepsilon/2)$ and hence $K \subseteq \overline{\bigcup_{m\geqslant N} K_m} \subseteq B(K_n, \varepsilon)$.

Proof of $\delta(K_n, K) < \varepsilon$. Fix $x \in K_n$. Using the fact that $(K_m)_{m \geqslant n}$ is Cauchy, we can find $n = n_0 < n_1 < \ldots < n_i < \ldots$ such that $d_H(K_{n_i}, K_m) < \varepsilon 2^{-(i+1)}$ for all $m \geqslant n_i$. Then, define $x_{n_i} \in K_{n_i}$ as follows: $x_{n_0} := x$ and for $i > 0$, let $x_{n_i} \in K_{n_i}$ be such that $d(x_{n_i}, x_{n_{i+1}}) < \varepsilon 2^{-(i+1)}$. It follows that $(x_{n_i})_{i \in \mathbb{N}}$ is Cauchy, so $x_{n_i} \to y$ for some $y \in X$. By the definition of K, $y \in K$. Moreover, $d(x, y) \leqslant \sum_{i=0}^{\infty} d(x_{n_i}, x_{n_{i+1}}) < \sum_{i=0}^{\infty} \varepsilon 2^{-(i+1)} = \varepsilon$. □

Proposition 3.14. *If X is compact metrizable, so is $\mathcal{K}(X)$.*

Proof. It is enough to show that if d is a compatible metric for X, with $d \leqslant 1$, then $(\mathcal{K}(X), d_H)$ is totally bounded. Fix $\varepsilon > 0$. Let $F \subseteq X$ be a finite ε-net for X. Then it is easy to verify that $\mathscr{P}(F)$ is an ε-net for $\mathcal{K}(X)$, i.e. $\mathcal{K}(X) \subseteq \bigcup_{S \subseteq F} B_{d_H}(S, \varepsilon)$. □

4. Perfect Polish spaces

Recall that a point x of a topological space X is called *isolated* if $\{x\}$ is open; call x a *limit point* otherwise. A space is *perfect* if it has no isolated points. If P is a subset of a topological space X, we call P *perfect in* X if P is closed and perfect in its relative topology. For example, $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^n$, $\mathbb{C}^{\mathbb{N}}$, $\mathbb{I}^{\mathbb{N}}$, $\mathcal{C}$, $\mathcal{N}$ are perfect. Another example of a perfect space is $C(X)$, where X compact metrizable.

Caution 4.1. $\mathbb{Q}$ is a perfect topological space, but it is not a perfect subset of $\mathbb{R}$ because it isn't closed.

4.A. **Embedding the Cantor space.**

The following definition gives a construction that is used when embedding the Cantor space.

Definition 4.2. A *Cantor scheme* on a set X is a family $(A_s)_{s \in 2^{<\mathbb{N}}}$ of subsets of X such that:

(i) $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$, for $s \in 2^{<\mathbb{N}}$;

(ii) $A_{s^\frown i} \subseteq A_s$, for $s \in 2^{<\mathbb{N}}, i \in \{0, 1\}$.

If (X, d) is a metric space and we additionally have

(iii) $\lim_{n \to \infty} \operatorname{diam}(A_{x|_n}) = 0$, for $x \in \mathcal{C}$,

we say that $(A_s)_{s \in 2^{<\mathbb{N}}}$ has *vanishing diameter*. In this case, we let

$$D = \left\{ x \in \mathcal{C} : \bigcap_{n \in \mathbb{N}} A_{x|_n} \neq \emptyset \right\}$$

and define $f : D \to X$ by $\{f(x)\} := \bigcap A_{x|_n}$. This f is called the *associated map*. Note that f is injective.

Theorem 4.3 (Perfect Set)**.** *Let X be a nonempty perfect Polish space. Then there is an embedding of $\mathcal{C}$ into X.*

Proof. Fix a complete compatible metric for X. Using that X is nonempty perfect, define a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ on X by induction on $|s|$ so that

(i) U_s is nonempty open;

(ii) $\operatorname{diam}(U_s) < 1/|s|$;

(iii) $\overline{U_{s^\frown i}} \subseteq U_s$, for $i \in \{0, 1\}$.

We do this as follows: let $U_\emptyset = X$ and assume U_s is defined. Since X does not have isolated points, U_s must contain at least two points $x \neq y$. Using the fact that X is Hausdorff, take two disjoint open neighborhoods $U_{s^\frown 0} \ni x$ and $U_{s^\frown 1} \ni y$ with small enough diameter so that the conditions (ii) and (iii) above are satisfied. This finishes the construction.

Now let $f : D \to X$ be the map associated with the Cantor scheme. It is clear that $D = \mathcal{C}$ because for $x \in \mathcal{C}$, $\bigcap_{n \in \mathbb{N}} U_{x|_n} = \bigcap_{n \in \mathbb{N}} \overline{U_{x|_n}} \neq \emptyset$ by the completeness of X. It is also clear that f is injective, so it is enough to prove that it is continuous (since continuous injections from compact to Hausdorff are embeddings). To this end, let $x \in \mathcal{C}$ and, for $\varepsilon > 0$, take an open ball $B \subseteq X$ of radius ε around $f(x)$. Because $\operatorname{diam}(U_{x|_n}) \to 0$ as $n \to \infty$ and $f(x) \in U_{x|_n}$ for all n, there is n such that $U_{x|_n} \subseteq B$. But then

$$f(N_{x|_n}) \subseteq U_{x|_n} \subseteq B. \qquad \square$$

Corollary 4.4. *Any nonempty perfect Polish space has cardinality continuum.*

Corollary 4.5. *For a perfect Polish space X, the generic compact subset of X is perfect. More precisely, the set*

$$\mathcal{K}_p(X) = \{K \in \mathcal{K}(X) : K \text{ is perfect}\}$$

is a dense G_δ subset of $\mathcal{K}(X)$.

Proof. Left as a homework exercise. $\square$

4.B. **The Cantor–Bendixson Theorem, Derivatives and Ranks.** The following theorem shows that Continuum Hypothesis holds for Polish spaces.

Theorem 4.6 (Cantor–Bendixson)**.** *Let X be a Polish space. Then X can be uniquely written as $X = P \cup U$, with P a perfect subset of X and U countable open.*

The perfect set P is called the *perfect kernel of X*.

We will give two different proofs of this theorem. In both, we define a *notion of smallness* for open sets and throw away the small basic open sets from X to get P. However, the small open sets in the first proof are "larger" than those in the second proof, and thus, in the first proof, after throwing small basic open sets away once, we are left with no small open set, while in the second proof, we have to repeat this process transfinitely many times to get rid of all small open sets. This transfinite analysis provides a very clear picture of the structure of X and allows for defining notions of derivatives of sets and ranks.

Call a point $x \in X$ a *condensation point* if it does not have a small open neighborhood, i.e. every open neighborhood of x is uncountable.

Proof of Theorem 4.6. We will temporarily call an open set *small* if it is countable. Let P be the set of all condensation points of X; in other words, $P = X \setminus U$, where U is the union of all small open sets. Thus it is clear that P is closed and doesn't contain isolated points. Also, since X is second countable, U is a union of countably many small basic open sets and hence, is itself countable. This finishes the proof of the existence.

For the uniqueness, suppose that $X = P_1 \cup U_1$ is another such decomposition. Thus, by definition, U_1 is small and hence $U_1 \subseteq U$. So it is enough to show that $P_1 \subseteq P$, which follows from the fact that in any perfect Polish space Y, all points are condensation points. This is because if $U \subseteq Y$ is an open neighborhood of a point $x \in Y$, then U itself is a nonempty perfect Polish space by Proposition 1.10 and hence is uncountable, by the Perfect Set Theorem. $\square$

Corollary 4.7. *Any uncountable Polish space contains a homeomorphic copy of $\mathcal{C}$ and in particular has cardinality continuum.*

In particular, it follows from Proposition 1.10 that every uncountable G_δ or F_σ set in a Polish space contains a homeomorphic copy of $\mathcal{C}$ and so has cardinality continuum; thus, the Continuum Hypothesis holds for such sets.

To give the second proof, we temporarily declare an open set small if it is a singleton.

Definition 4.8. For any topological space X, let

$$X' = \{x \in X : x \text{ is a limit point of } X\}.$$

We call X' the *Cantor–Bendixson derivative of* X. Clearly, X' is closed since $X' = X \setminus U$, where U is the union of all small open sets. Also X is perfect iff $X = X'$.

Using transfinite recursion we define the iterated Cantor–Bendixson derivatives X^α, $\alpha \in \mathrm{ON}$, as follows:

$$\begin{aligned} X^0 &:= X, \\ X^{\alpha+1} &:= (X^\alpha)', \\ X^\lambda &:= \bigcap_{\alpha<\lambda} X^\alpha, \text{ if } \lambda \text{ is a limit.} \end{aligned}$$

Thus $(X^\alpha)_{\alpha\in\mathrm{ON}}$ is a decreasing transfinite sequence of closed subsets of X. The following theorem provides an alternative way of constructing the perfect kernel of a Polish space.

Theorem 4.9. *Let X be a Polish space. For some countable ordinal α_0, $X^\alpha = X^{\alpha_0}$ for all $\alpha > \alpha_0$, and X^{α_0} is the perfect kernel of X.*

Proof. Since X is second countable and $(X^\alpha)_{\alpha\in\mathrm{ON}}$ is a decreasing transfinite sequence of closed subsets of X, it must stabilize in countably many steps, i.e. there is a countable ordinal α_0, such that $X^\alpha = X^{\alpha_0}$ for all $\alpha > \alpha_0$. Thus $(X^{\alpha_0})' = X^{\alpha_0}$ and hence X^{α_0} is perfect.

To see that $X \setminus X^{\alpha_0}$ is countable, note that for any second countable space Y, $Y \setminus Y'$ is equal to a union of small basic open sets, and hence is countable. Thus, $X \setminus X^{\alpha_0} = \bigcup_{\alpha<\alpha_0}(X^\alpha \setminus X^{\alpha+1})$ is countable since such is α_0. □

Definition 4.10. For any Polish space X, the least ordinal α_0 as in the above theorem is called the *Cantor–Bendixson rank of* X and is denoted by $|X|_{\mathrm{CB}}$. We also let

$$X^\infty = X^{|X|_{\mathrm{CB}}} = \text{the perfect kernel of } X.$$

Clearly, for X Polish, X is countable $\iff X^\infty = \emptyset$.

5. Zero-dimensional spaces

5.A. **Definition and examples.** A topological space X is called *disconnected* if it can be partitioned into two nonempty open sets; otherwise, call X *connected*. In other words, X is connected if and only if the only clopen (closed and open) sets are $\emptyset, X$.

For example, $\mathbb{R}^n, \mathbb{C}^n, \mathbb{I}^{\mathbb{N}}, \mathbb{T}$ are connected, but $\mathcal{C}$ and $\mathcal{N}$ are not. In fact, in the latter spaces, not only are there nontrivial clopen sets, but there is a basis of clopen sets; so these spaces are in fact very disconnected.

We call topological spaces that admit a basis of clopen sets *zero-dimensional*; the name comes from a general notion of dimension (small inductive dimension) being 0 for exactly these spaces. It is clear (why?) that Hausdorff zero-dimensional spaces are *totally disconnected*, i.e. the only connected subsets are the singletons. However the converse fails in general even for metric spaces.

The proposition below shows that zero-dimensional second-countable topological spaces have a countable basis consisting of clopen sets. To prove this proposition, we need the following.

Lemma 5.1. *Let X be a second-countable topological space. Then every open cover $\mathcal{V}$ of X has a countable subcover.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis and put

$$I = \{n \in \mathbb{N} : \exists V \in \mathcal{V} \text{ such that } V \supseteq U_n\}.$$

Note that $(U_n)_{n\in I}$ is still an open cover of X since $\mathcal{V}$ is a cover and every $V \in \mathcal{V}$ is a union of elements from $(U_n)_{n\in I}$. For every $n \in I$, choose (by AC) a set $V_n \in \mathcal{V}$ such that $V_n \supseteq U_n$. Clearly, $(V_n)_{n\in I}$ covers X, since so does $(U_n)_{n\in I}$. □

Proposition 5.2. *Let X be a second-countable topological space. Then every basis $\mathcal{B}$ of X has a countable subbasis.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis and for each $n \in \mathbb{N}$, choose a cover $\mathcal{B}_n \subseteq \mathcal{B}$ of U_n (exists because $\mathcal{B}$ is a basis). Since each U_n is a second-countable topological space, the lemma above gives a countable subcover $\mathcal{B}'_n \subseteq \mathcal{B}_n$. Put $\mathcal{B}' = \bigcup_n \mathcal{B}'_n$ and note that $\mathcal{B}'$ is a basis because every U_n is a union of sets in $\mathcal{B}'$ and $(U_n)_{n\in\mathbb{N}}$ is a basis. □

5.B. **Luzin schemes.**

Definition 5.3. A *Luzin scheme* on a set X is a family $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of X such that

(i) $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ if $s \in \mathbb{N}^{<\mathbb{N}}$, $i \neq j$;

(ii) $A_{s^\frown i} \subseteq A_s$, for $s \in \mathbb{N}^{<\mathbb{N}}, i \in \mathbb{N}$.

If (X, d) is a metric space and we additionally have

(iii) $\lim_{n\to\infty} \operatorname{diam}(A_{x|_n}) = 0$, for $x \in \mathcal{N}$,

we say that $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ has *vanishing diameter*. In this case, we let

$$D = \left\{ x \in \mathcal{N} : \bigcap_{n\in\mathbb{N}} A_{x|_n} \neq \emptyset \right\}$$

and define $f : D \to X$ by $\{f(x)\} := \bigcap A_{x|_n}$. This f is called the *associated map*.

From now on, for $s \in \mathbb{N}^{<\mathbb{N}}$, we will denote by N_s the basic open sets of $\mathcal{N}$, i.e.

$$N_s = \{x \in \mathcal{N} : x \supseteq s\}.$$

Here are some useful facts about Luzin schemes.

Proposition 5.4. *Let $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ be a Luzin scheme on a metric space (X, d) that has vanishing diameter and let $f : D \to X$ be the associated map.*

(a) *f is injective and continuous.*

(b) *If $A_\emptyset = X$ and $A_s = \bigcup_i A_{s^\frown i}$ for each $s \in \mathbb{N}^{<\mathbb{N}}$, then f is surjective.*

(c) *If each A_s is open, then f is an embedding.*

(d) *If (X,d) is complete and $\overline{A_{s^\frown i}} \subseteq A_s$ for each $s \in \mathbb{N}^{<\mathbb{N}}, i \in \mathbb{N}$, then D is closed. If moreover, each A_s is nonempty, then $D = \mathcal{N}$.*

Also, same holds for a Cantor scheme.

Proof. In part (a), injectivity follows from (i) of the definition of Luzin scheme, and continuity follows from vanishing diameter because if $x_n \to x$ in D, then for every $k \in \mathbb{N}$, $f(x_n) \in A_{x|_k}$ for large enough n, so $d(f(x_n), f(x)) \leqslant \operatorname{diam}(A_{x|_k})$, but the latter goes to 0 as $k \to \infty$.

Part (b) is straightforward, and part (c) follows from the fact that $f(N_s \cap D) = A_s \cap f(D)$.

For (d), we will show that D^c is open. Fix $x \in D^c$ and note that the only reason why $\bigcap_n A_{x|_n}$ is empty is because $A_{x|_n} = \emptyset$ for some $n \in \mathbb{N}$ since otherwise, $\bigcap_n A_{x|_n} = \bigcap_n \overline{A_{x|_n}} \neq \emptyset$ by the completeness of the metric. But then the entire $N_{x|_n}$ is contained in D^c, so D^c is open. □

5.C. Topological characterizations of the Cantor space and the Baire space.

Theorem 5.5 (Brouwer)**.** *The Cantor space $\mathcal{C}$ is the unique, up to homeomorphism, perfect nonempty, compact metrizable, zero-dimensional space.*

Proof. It is clear that $\mathcal{C}$ has all these properties. Now let X be such a space and let d be a compatible metric. We will construct a Cantor scheme $(C_s)_{s \in 2^{<\mathbb{N}}}$ on X that has vanishing diameter such that for each $s \in 2^{<\mathbb{N}}$,

(i) C_s is nonempty;

(ii) $C_\emptyset = X$ and $C_s = C_{s^\frown 0} \cup C_{s^\frown 1}$;

(iii) C_s is clopen.

Assuming this can be done, let $f : D \to X$ be the associated map. Because X is compact and each C_s is nonempty closed, we have $D = \mathcal{C}$. Therefore, f is a continuous injection of $\mathcal{C}$ into X by (a) of Proposition 5.4. Moreover, because $\mathcal{C}$ is compact and X is Hausdorff, f is actually an embedding. Lastly, (ii) implies that f is onto.

As for the construction of $(C_s)_{s \in 2^{<\mathbb{N}}}$, partition $X = \bigcup_{i<n} X_i$, $n \geqslant 2$, into nonempty clopen sets of diameter $< 1/2$ (how?) and put $C_{0^i} = X_i \cup \ldots \cup X_{n-1}$ for $i < n$, and $C_{0^i{}^\frown 1} = X_i$ for $i < n-1$. Now repeat this process within each X_i, using sets of diameter $< 1/3$, and so on (recursively). □

Theorem 5.6 (Alexandrov–Urysohn)**.** *The Baire space $\mathcal{N}$ is the unique, up to homeomorphism, nonempty Polish zero-dimensional space, for which all compact subsets have empty interior.*

Proof. Outlined in a homework problem. □

Corollary 5.7. *The space of irrational numbers is homeomorphic to the Baire space.*

5.D. Closed subspaces of the Baire space.

Theorem 5.8. *Every zero-dimensional separable metrizable space can be embedded into both $\mathcal{N}$ and $\mathcal{C}$. Every zero-dimensional Polish space is homeomorphic to a closed subset of $\mathcal{N}$ and a G_δ subset of $\mathcal{C}$.*

Proof. The assertions about $\mathcal{C}$ follow from those about $\mathcal{N}$ and Proposition 1.11. To prove the results about $\mathcal{N}$, let X be as in the first statement of the theorem and let d be a compatible metric for X. Then we can easily construct Luzin scheme $(C_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ on X with vanishing diameter such that for each $s \in \mathbb{N}^{<\mathbb{N}}$,

(i) $C_\emptyset = X$ and $C_s = \bigcup_{i\in\mathbb{N}} C_{s^\frown i}$;
(ii) C_s is clopen.

(Some C_s may, however, be empty.) Let $f : D \to X$ be the associated map. By (i) and (ii), f is a surjective embedding, thus a homeomorphism between D and X. Finally, by (d) of Proposition 5.4, D is closed if d is complete. □

5.E. **Continuous images of the Baire space.**

Theorem 5.9. *Any nonempty Polish space X is a continuous image of $\mathcal{N}$. In fact, for any Polish space X, there is a closed set $C \subseteq \mathcal{N}$ and a continuous bijection $f : C \to X$.*

Proof. The first assertion follows from the second and Lemma 2.10. For the second assertion, fix a complete compatible metric d on X.

By (a), (b) and (d) of Proposition 5.4, it is enough to construct a Luzin scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ on X with vanishing diameter such that for each $s \in \mathbb{N}^{<\mathbb{N}}$,
(i) $F_\emptyset = X$ and $F_s = \bigcup_{i\in\mathbb{N}} F_{s^\frown i}$;
(ii) $\overline{F_{s^\frown i}} \subseteq F_s$, for $i \in \mathbb{N}$.

We put $F_\emptyset := X$ and attempt to define $F_{(i)}$ for $i \in \mathbb{N}$ as follows: take an open cover $(U_j)_{j\in\mathbb{N}}$ of X such that $\mathrm{diam}(U_j) < 1$, and put $F_{(i)} := U_i \setminus (\bigcup_{n<i} U_n)$. This clearly works. We can apply the same process to each $F_{(i)}$ and obtain $F_{(i,j)}$, but now (ii) may not be satisfied. To satisfy (ii), we need to use that $F_{(i)}$ is F_σ (it is also G_δ, but that's irrelevant) as we will see below. Thus, we add the following auxiliary condition to our Luzin scheme:
(iii) F_s is F_σ.

To construct such $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, it is enough to show that for every F_σ set $F \subseteq X$ and every $\varepsilon > 0$, we can write $F = \bigcup_{n\in\mathbb{N}} F_n$, where F_n are pairwise disjoint F_σ sets of diameter $< \varepsilon$ such that $\overline{F_n} \subseteq F$. To this end, write $F = \bigcup_{i\in\mathbb{N}} C_i$, where $(C_i)_{i\in\mathbb{N}}$ is an increasing sequence of closed sets with $C_0 = \emptyset$. Then $F = \bigcup_{i\in\mathbb{N}} (C_{i+1} \setminus C_i)$ and, as above, we can write $C_{i+1} \setminus C_i = \bigcup_{j\in\mathbb{N}} E_{i,j}$, where $E_{i,j}$ are disjoint F_σ sets of diameter $< \varepsilon$. Thus $F = \bigcup_{i,j} E_{i,j}$ works since $\overline{E_{i,j}} \subseteq \overline{C_{i+1} \setminus C_i} \subseteq C_{i+1} \subseteq F$. □

6. Baire category

6.A. **Nowhere dense sets.** Let X be a topological space. A set $A \subseteq X$ is said to be *dense in* $B \subseteq X$ if $A \cap B$ is dense in B.

Definition 6.1. Let X be a topological space. A set $A \subseteq X$ is called *nowhere dense* if there is no nonempty open set $U \subseteq X$ in which A is dense.

Proposition 6.2. *Let X be a topological space and $A \subseteq X$. The following are equivalent:*
(1) *A is nowhere dense;*
(2) *A misses a nonempty open subset of every nonempty open set (i.e. for every open set $U \neq \emptyset$ there is a nonempty open subset $V \subseteq U$ such that $A \cap V = \emptyset$);*
(3) *The closure $\overline{A}$ has empty interior.*

Proof. Follows from definitions. □

Proposition 6.3. *Let X be a topological space and $A, U \subseteq X$.*
(a) *A is nowhere dense if and only if $\overline{A}$ is nowhere dense.*

(b) *If U is open, then $\partial U := \overline{U} \setminus U$ is closed nowhere dense.*
(c) *If U is open dense, then U^c is closed nowhere dense.*
(d) *Nowhere dense subsets of X form an ideal*[4].

Proof. Part (a) immediately follows from (2) of Proposition 6.2. For (b) note that ∂U is disjoint from U so its interior cannot be nonempty. Since it is also closed, it is nowhere dense by (3) of Proposition 6.2, again. As for part (c), it follows directly from (b) because by the density of U, $\partial U = U^c$. Finally, we leave part (d) as an easy exercise. □

For example, the Cantor set is nowhere dense in $[0,1]$ because it is closed and has empty interior. Also, any compact set K is nowhere dense in $\mathcal{N}$ because it is closed and the corresponding tree T_K is finitely branching. Finally, in a perfect Hausdorff (T_2 is enough) space, singletons are closed nowhere dense.

6.B. **Meager sets.**

Definition 6.4. Let X be a topological space. A set $A \subseteq X$ is *meager* if it is a countable union of nowhere dense sets. The complement of a meager set is called *comeager*.

Note that the family $\mathrm{MGR}(X)$ of meager subsets of X is a σ-ideal[5] on X; in fact, it is precisely the σ-ideal generated by nowhere dense sets. Consequently, comeager sets form a countably closed filter[6] on X.

Meager sets often have properties analogous to those enjoyed by the null sets in $\mathbb{R}^n$ (with respect to the Lebesgue measure). The following proposition lists some of them.

Proposition 6.5. *Let X be a topological space and $A \subseteq X$.*

(a) *A is meager if and only if it is contained in a countable union of closed nowhere dense sets. In particular, every meager set is contained in a meager F_σ set.*
(b) *A is comeager if and only if it contains a countable intersection of open dense sets. In particular, dense G_δ sets are comeager.*

Proof. Part (b) follows from (a) by taking complements, and part (a) follows directly from the corresponding property of nowhere dense sets proved above. □

An example of a meager set is any σ-compact set in $\mathcal{N}$. Also, any countable set in a nonempty perfect Hausdorff space is meager, so, for example, $\mathbb{Q}$ is meager in $\mathbb{R}$.

As an application of some of the statements above, we record the following random fact:

Proposition 6.6. *Every second countable space X contains a dense G_δ (hence comeager) subset Y that is zero-dimensional in the relative topology.*

Proof. Indeed, if $\{U_n\}_{n\in\mathbb{N}}$ is a basis for X, then $F = \bigcup_n (\overline{U_n} \setminus U_n)$ is meager F_σ and $Y = X \setminus F$ is zero-dimensional. □

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[4] An *ideal* on a set X is a collection of subsets of X containing $\emptyset$ and closed under subsets and finite unions.

[5] An *σ-ideal* on a set X is an ideal that is closed under countable unions.

[6] A *filter* on a set X is the dual to an ideal on X, more precisely, it is a collection of subsets of X containing X and closed under supersets and finite intersections. If moreover, it is closed under countable intersections, we say that it is *countably closed*.

6.C. **Relativization of nowhere dense and meager.** Let X be a topological space and P be a property of subsets of X (e.g. open, closed, compact, nowhere dense, meager). We say that property P is *absolute between subspaces* if for every subspace $Y \subseteq X$ and every object O built out of Y (e.g., a subset $O \subseteq Y$ or a sequence $O := (y_n) \subseteq Y$), O has property P within the space Y iff it has property P within the space X. Examples of properties for subsets that are absolute between subspaces are compactness and connectedness[7] (why?). Another example, for sequences $(y_n) \subseteq Y$ together with a designated point $y \in Y$, is convergence $\lim_n y_n \to y$.

On the other hand, it is clear that properties like open or closed (for subsets) are not absolute. Furthermore, being nowhere dense is not absolute: let $X = \mathbb{R}$ and $A = Y = \{0\}$. Now A is clearly nowhere dense in $\mathbb{R}$ but in Y all of a sudden it is, in fact, open, and hence not nowhere dense. Thus being nowhere dense does not *transfer downward* (from a bigger space to a smaller subspace); same goes for meager. However, the following proposition shows that it *transfers upward* and that it is absolute between open subspaces.

Proposition 6.7. *Let X be a topological space, $Y \subseteq X$ be a subspace and $A \subseteq Y$.*

(a) *If A is nowhere dense (resp. meager) in Y, it is still nowhere dense (resp. meager) in X.*
(b) *If Y is open, then A is nowhere dense (resp. meager) in Y iff it is nowhere dense (resp. meager) in X.*

Proof. Straightforward, using (2) of Proposition 6.2. □

6.D. **Baire spaces.** Being a σ-ideal is a characteristic property of many notions of "smallness" of sets, such as being countable, having measure 0, etc, and meager is one of them. However, it is possible that a topological space X is such that X itself is meager, so the σ-ideal of meager sets trivializes, i.e. is equal to $\mathscr{P}(X)$. The following definition isolates a class of spaces where this doesn't happen.

Definition 6.8. A topological space is said to be *Baire* if every nonempty open set is nonmeager.

Proposition 6.9. *Let X be a topological space. The following are equivalent:*

(a) *X is a Baire space, i.e. every nonempty open set is nonmeager.*
(b) *Every comeager set is dense.*
(c) *The intersection of countably many dense open sets is dense.*

Proof. Follows from the definitions. □

As mentioned above, in any topological space, dense G_δ sets are comeager. Moreover, by the last proposition, we have that in Baire spaces any comeager set contains a dense G_δ set. So we get:

Corollary 6.10. *In Baire spaces, a set is comeager if and only if it contains a dense G_δ set.*

Proposition 6.11. *If X is a Baire space and $U \subseteq X$ is open, then U is a Baire space.*

Proof. Follows from (b) of Proposition 6.7. □

Theorem 6.12 (Baire Category)**.** *Every completely metrizable space is Baire. Every locally compact Hausdorff space is Baire.*

[7] Thanks to Lou van den Dries for pointing out that connectedness is absolute.

Proof. We will only prove for completely metrizable spaces and leave the locally compact Hausdorff case as an exercise (outlined in a homework problem). So let (X,d) be a complete metric space and let $(U_n)_{n\in\mathbb{N}}$ be dense open. Let U be nonempty open and we show that $\bigcap_n U_n \cap U \neq \emptyset$. Put $V_0 = U$ and since $U_0 \cap V_0 \neq \emptyset$, there is a nonempty open set V_1 of diameter < 1 such that $\overline{V_1} \subseteq U_0 \cap V_0$. Similarly, since $U_1 \cap V_1 \neq \emptyset$, there is a nonempty open set V_2 of diameter $< 1/2$ such that $\overline{V_2} \subseteq U_1 \cap V_1$, etc. Thus there is a decreasing sequence $(\overline{V_n})_{n\geqslant 1}$ of nonempty closed sets with vanishing diameter ($\mathrm{diam}(V_n) < 1/n$) and such that $\overline{V_n} \subseteq U_n \cap U$. By the completeness of X, $\bigcap_n \overline{V_n}$ is nonempty (is, in fact, a singleton) and hence so is $\bigcap_n U_n \cap U$. □

Thus, Polish spaces are Baire and hence comeager sets in them are "truly large", i.e. they are not meager! This immediately gives:

Corollary 6.13. *In nonempty Polish spaces, dense meager sets are not G_δ. In particular, $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$.*

Proof. If a subset is dense G_δ, then it is comeager, and hence nonmeager. □

Definition 6.14. Let X be a topological space and $P \subseteq X$. If P is comeager, we say that P *holds generically* or that *the generic element of X is in P*. (Sometimes the word *typical* is used instead of generic.)

In a nonempty Baire space X, if $P \subseteq X$ holds generically, then, in particular, $P \neq \emptyset$. This leads to a well-known method of existence proofs in mathematics: in order to show that a given set $P \subseteq X$ is nonempty, where X is a nonempty Baire space, it is enough to show that P holds generically. Although the latter task seems harder, the proofs are often simpler since having a notion of largeness (like nonmeager, uncountable, positive measure) allows using pigeon hole principles and counting, whereas constructing a concrete object in P is often complicated. The first example of this phenomenon was due to Cantor who proved the existence of transcendental numbers by showing that there are only countably many algebraic ones, whereas reals are uncountable, and hence, "most" real numbers are transcendental. Although the existence of transcendental numbers was proved by Liouville before Cantor, the simplicity of Cantor's proof and the apparent power of the idea of counting successfully "sold" Set Theory to the mathematical community.

Part 2. Regularity properties of subsets of Polish spaces

In this part we will discuss various desirable properties for subsets of Polish spaces and in the next part we will discuss classes of subsets that have them. In some sense the strongest regularity property for a subset A is that of being *determined*; it is based on infinite games associated with A and roughly speaking implies the other regularity properties. Thus, we will first start with infinite games.

7. Infinite games and determinacy

Let A be a nonempty set and $D \subseteq A^{\mathbb{N}}$. We associate with D the following game:

$$\begin{array}{llllll} \text{I} & a_0 & & a_2 & & \\ & & & & & \dots \\ \text{II} & & a_1 & & a_3 & \end{array}$$

Player I plays $a_0 \in A$, II then plays $a_1 \in A$, I plays $a_2 \in A$, etc. Player I wins iff $(a_n)_{n\in\mathbb{N}} \in D$. We call D the *payoff set*.

We denote this game by $G(A,D)$ or $G(D)$ if A is understood. We refer to $s \in A^{<\mathbb{N}}$ as a *position* in the game, and we refer to $x \in A^{\mathbb{N}}$ as a *run* of the game. A *strategy* for Player I is a "rule" by which Player I determines what to play next based on Player II's previous moves; formally, it is just a map $\varphi : A^{<\mathbb{N}} \to A$, and we say that Player I *follows the strategy* φ if he plays $a_0 = \varphi(\emptyset)$, $a_2 = \varphi((a_1))$, $a_4 = \varphi((a_1,a_3)),\dots$, when Player II plays $a_1, a_3, \dots$.

Equivalently (but often more conveniently), we can define a *strategy for Player I* as a tree $\sigma \subseteq A^{<\mathbb{N}}$ such that

(i) σ is nonempty;
(ii) if $(a_0,a_1,\dots,a_{2n-1}) \in \sigma$, then for exactly one $a_{2n} \in A$, $(a_0,a_1,\dots,a_{2n-1},a_{2n}) \in \sigma$;
(iii) if $(a_0,a_1,\dots,a_{2n}) \in \sigma$, then for all $a_{2n+1} \in A$, $(a_0,a_1,\dots,a_{2n},a_{2n+1}) \in \sigma$.

Note that σ must necessarily be a pruned tree. Again, this is interpreted as follows: I starts with the unique $a_0 \in A$ such that $(a_0) \in \sigma$. If II next plays $a_1 \in A$, then $(a_0,a_1) \in \sigma$, and Player I plays the unique $a_2 \in A$ such that $(a_0,a_1,a_2) \in \sigma$, etc.

The notion of a *strategy for Player II* is defined analogously.

A strategy for Player I is *winning* in $G(A,D)$ if for every run of the game $(a_n)_{n\in\mathbb{N}}$ in which I follows this strategy, $(a_n)_{n\in\mathbb{N}} \in D$. Similarly, one defines a *winning* strategy for Player II. Note that it cannot be that both I and II have a winning strategy in $G(A,D)$.

Definition 7.1. We say that the game $G(A,D)$, or just the set $D \subseteq A^{\mathbb{N}}$, is *determined* if one of the two players has a winning strategy.

Proposition 7.2. *If $|A| \geqslant 2$ and $\sigma \subseteq A^{<\mathbb{N}}$ is a strategy for one of the players in $G(A,D)$, then $[\sigma]$ is a nonempty perfect subset of $A^{\mathbb{N}}$. Moreover, if σ is a winning strategy for Player I (Player II), then $[\sigma] \subseteq D$ ($[\sigma] \subseteq D^c$).*

Proof. Clear from the definitions. $\square$

7.A. **Nondetermined sets and** AD. Not all subsets $D \subseteq A^{\mathbb{N}}$ are determined: the Axiom of Choice (AC) allows construction of pathological sets which are not determined. Here is an example.

Example 7.3. (AC) Let A be a countable set containing at least two elements. If $\sigma \subseteq A^{<\mathbb{N}}$ is a winning strategy for one of the players in the game $G(D)$ for some $D \subseteq A^{\mathbb{N}}$, then by the proposition above, $[\sigma]$ is a nonempty perfect subset of either D or D^c, and hence either D or D^c (maybe both) contains a nonempty perfect subset. Hence, to construct a set that is nondetermined, it is enough to construct a set $B \subseteq A^{\mathbb{N}}$ such that neither B, nor B^c, contains a nonempty perfect subset. Such a set is called a *Bernstein set*.

The construction uses AC and goes as follows: assuming that A is countable, there are at most $2^{\aleph_0}$ (=continuum) many perfect subsets (why?), and hence, by AC, there is a transfinite enumeration $(P_\xi)_{\xi<2^{\aleph_0}}$ of all nonempty perfect subsets of $A^{\mathbb{N}}$. Now by transfinite recursion, pick distinct points $a_\xi, b_\xi \in P_\xi$ (by AC, again) so that $a_\xi, b_\xi \notin \{a_\lambda, b_\lambda : \lambda < \xi\}$. This can always be done since the cardinality of the latter set is $2|\xi| \leqslant \max\{|\xi|, \aleph_0\} < 2^{\aleph_0}$, while $|P_\xi| = 2^{\aleph_0}$.

Now put $B = \left\{b_\xi\right\}_{\xi<2^{\aleph_0}}$ and thus $\left\{a_\xi\right\}_{\xi<2^{\aleph_0}} \subseteq B^c$. It is clear that there is no $\xi < 2^{\aleph_0}$ such that $P_\xi \subseteq B$ or $P_\xi \subseteq B^c$. Thus, B is a Bernstein set.

It perhaps shouldn't be surprising that sets that come from AC (out of nowhere) have pathologies. However, the sets that are "definable" (constructed from open sets using certain operations such as countable unions, complements, projections) are expected to have nice properties, for example, be determined. An important class of definable sets is that of Borel sets[8]. We will see later on that Borel sets are determined, but ZFC cannot possibly prove determinacy of definable sets beyond Borel.

The *Axiom of Determinacy* (AD) is the statement that all subsets of $\mathbb{N}^{\mathbb{N}}$ are determined. As we just saw, AC contradicts AD. However ZF +AD is believed to be consistent, although one cannot prove it in ZF since AD implies the consistency of ZF, so it would contradict Gödel's Incompleteness theorem.

7.B. **Games with rules.** It is often convenient to consider games in which the players do not play arbitrary $a_0, a_1, \ldots$ from a given set A, but have to also obey certain rules. Formally, this means that we are given A and a nonempty pruned tree $T \subseteq A^{<\mathbb{N}}$, which determines the *legal positions*. For $D \subseteq [T]$ consider the game $G(T, D)$ played as follows:

$$\begin{array}{lccccc} \mathrm{I} & a_0 & & a_2 & & \\ & & & & & \cdots \\ \mathrm{II} & & a_1 & & a_3 & \end{array}$$

Players I and II take turns playing $a_0, a_1, \ldots$ so that $(a_0, \ldots, a_n) \in T$ for each n. I wins iff $(a_n)_{n\in\mathbb{N}} \in D$.

Thus if $T = A^{<\mathbb{N}}$ and $D \subseteq A^{<\mathbb{N}}$, $G(A^{<\mathbb{N}}, D) = G(A, D)$ in our previous notation.

The notions of strategy, winning strategy, and determinacy are defined as before. So, for example, a strategy for I would now be a nonempty pruned subtree $\sigma \subseteq T$ satisfying conditions (ii) and (iii) as before, as long as in (iii) a_{2n+1} is such that $(a_0, a_1, \ldots, a_{2n}, a_{2n+1}) \in T$.

[8]The Borel σ-algebra $\mathcal{B}(X)$ of a topological space X is the smallest σ-algebra containing all the open sets. A subset $A \subseteq X$ is called *Borel* if $A \in \mathcal{B}$.

Note that the game $G(T, D)$ is equivalent to the game $G(A, D')$, where $D' \subseteq A^{\mathbb{N}}$ is defined by

$$x \in D' \iff \big(\exists n (x|_n \notin T) \wedge (\text{the least } n \text{ such that } x|_n \notin T \text{ is even})\big) \text{ or } \big(x \in [T] \wedge x \in D\big),$$

and two games G, G' are said to be *equivalent* if Player I (resp. II) has a winning strategy in G iff I (resp. II) has a winning strategy in G'. Thus the introduction of "games with rules" does not really lead to a wider class of games.

8. The perfect set property

Let X be a Polish space.

Definition 8.1. A set $A \subseteq X$ is said to have the *perfect set property* (PSP) if it is either countable or contains a nonempty perfect subset (and thus has cardinality continuum).

By the perfect set theorem, X itself has the PSP and so does any G_δ subset $A \subseteq X$ since it is Polish in the relative topology. We will see later that actually all Borel sets have the PSP. However, Bernstein set B constructed in Example 7.3 does not have the PSP: indeed, it does not contain a perfect subset by definition, neither is it countable, because otherwise, B^c would be uncountable G_δ, and hence would contain a perfect set; a contradiction.

8.A. **The associated game.** We now describe a game that is associated with the PSP and explore the connection between the PSP and determinacy.

Let X be a nonempty perfect Polish space with complete compatible metric d. Fix also a countable basis $\mathcal{V} := \{V_n\}_{n \in \mathbb{N}}$ of nonempty open sets for X. Given $A \subseteq X$, consider the following game $G^*(A)$ called the *$*$-game*:

$$\begin{array}{llll} \mathrm{I} & (U_0^{(0)}, U_1^{(0)}) & (U_0^{(1)}, U_1^{(1)}) & \\ & & & \cdots \\ \mathrm{II} & \qquad\qquad i_0 & \qquad\qquad i_1 & \end{array}$$

Here, for $i \in \{0,1\}$ and $n \in \mathbb{N}$, $U_i^{(n)} \in \mathcal{V}$ with $\operatorname{diam}\left(U_i^{(n)}\right) < \frac{1}{n+1}$, $U_0^{(n)} \cap U_1^{(n)} = \emptyset$, $i_n \in \{0,1\}$, and $\overline{U_0^{(n+1)}} \cup \overline{U_1^{(n+1)}} \subseteq U_{i_n}^{(n)}$. Note that because X is nonempty perfect, each nonempty open $U \subseteq X$ contains two disjoint nonempty basic open sets, and therefore, the game above is well-defined (will never get stuck at a finite step). Let $x \in X$ be defined by $\{x\} := \bigcap_n U_{i_n}^{(n)}$. Then I wins iff $x \in A$.

Thus in this game Player I starts by playing two disjoint basic open sets of diameter < 1 and II next picks one of them. Then I plays two disjoint basic open sets of diameter $< 1/2$, whose closures are contained in the set that II picked before, and then II picks one of them, etc. (So this is a version of a cut-and-choose game.) The sets that II picked define a unique x. Then I wins iff $x \in A$.

This is clearly a game $G(T, D_A)$ with rules, where T is a tree on the set $\mathcal{V}^2 \cup \{0,1\}$ defined according to the rules described above, and $D_A \subseteq [T]$ is the set of all runs of the game such that x, defined as above, belongs to A.

Theorem 8.2. *Let X be a nonempty perfect Polish space and $A \subseteq X$.*

(a) *I has a winning strategy in $G^*(A)$ iff A contains a Cantor set.*

(b) *II has a winning strategy in $G^*(A)$ iff A is countable.*

Proof. (a) Using a winning strategy for Player I, we can easily construct a Cantor scheme $(U_s)_{s\in 2^{<\mathbb{N}}}$ with U_s open, $\overline{U_{s^\frown 0}} \cup \overline{U_{s^\frown 1}} \subseteq U_s$, $\mathrm{diam}(U_s) < 1/|s|$ for $s \neq \emptyset$, and such that for each $y \in \mathcal{C}$, if $\{x\} = \bigcap_n U_{y|_n}$, then $x \in A$. So A contains a Cantor set.

Conversely, if $C \subseteq A$ is a Cantor set (or any nonempty perfect set), we can find a winning strategy for Player I as follows: I starts with (a legal) $(U_0^{(0)}, U_1^{(0)})$ such that $U_i^{(0)} \cap C \neq \emptyset$ for all $i \in \{0,1\}$. Next II chooses one of $U_0^{(0)}, U_1^{(0)}$, say $U_0^{(0)}$ for definiteness. Since C is perfect, I can play (a legal) $(U_0^{(1)}, U_1^{(1)})$ such that $U_i^{(1)} \cap C \neq \emptyset$ for all $i \in \{0,1\}$, etc. Clearly, this is a winning strategy for I.

(b) If A is countable, say $A = \{x_n\}_{n\in\mathbb{N}}$, then a winning strategy for Player II is defined by having him choose $U_{i_n}^{(n)}$ in his n^{th} move so that $x_n \notin U_{i_n}^{(n)}$.

Finally, assume σ is a winning strategy for II. Given $x \in A$, we call a position

$$p = ((U_0^{(0)}, U_1^{(0)}), i_0, \ldots, (U_0^{(n)}, U_1^{(n)}), i_n)$$

good for x if it has been played according to σ (i.e., $p \in \sigma$) and $x \in U_{i_n}^{(n)}$. By convention, the empty position $\emptyset$ is good for x. If every good for x position p has a proper extension that is also good for x, then there is a run of the game according to σ, which produces $x \in A$, and hence Player I wins, giving a contradiction. Thus for every $x \in A$, we can pick a position $p_x \in \sigma$ that is maximal good for x.

We claim that the map $x \mapsto p_x$ is injective, in other words, a position p cannot be maximal good for two distinct $x, y \in A$. Indeed, otherwise, there are disjoint open sets $U_0^{(n+1)} \ni x$ and $U_1^{(n+1)} \ni y$ small enough so that $p' = p^\frown(U_0^{(n+1)}, U_1^{(n+1)})$ is a legal move. But then no matter what $i_{n+1} \in \{0,1\}$ is, $p'^\frown i_{n+1}$ is a good position for one of x and y, contradicting the maximality of p. Thus we have an injective map from A into σ and hence A is countable. □

Note that AD implies the determinacy of all games on a countable set C, including those with rules (i.e. on trees $T \subseteq C^{\mathbb{N}}$). In particular, it implies that the $*$-game above is determined. And thus we have:

Corollary 8.3 (AD)**.** *All subsets of an arbitrary Polish space X have the* PSP.

Proof. The $*$-game and the above theorem are for perfect Polish spaces, while X may not be perfect. However, using the Cantor–Bendixson theorem, we can apply the $*$-game to the perfect kernel of X. □

9. Baire measurability

9.A. **The definition and closure properties.** Let $\mathcal{I}$ be a σ-ideal on a set X. For $A, B \subseteq X$, we say that A and B are *equal mod* $\mathcal{I}$, noted $A =_{\mathcal{I}} B$, if the symmetric difference $A \Delta B = (A \setminus B) \cup (B \setminus A) \in \mathcal{I}$. This is clearly an equivalence relation that respects complementation and countable unions/intersections.

When $\mathcal{I} := \mathrm{MGR}(X)$ for a topological space X, we say that sets $A, B \subseteq X$ are *generically equal* (or *equal mod meager*), written $A =^* B$, if A and B are equal mod $\mathrm{MGR}(X)$.

Definition 9.1. Let X be a topological space. A set $A \subseteq X$ is *Baire measurable* (or has the *Baire property*) if $A =^* U$ for some open set $U \subseteq X$.

Recall that a *σ-algebra on a set X* is a collection of subsets of X containing $\emptyset$ and closed under complements and countable unions (and thus under countable intersections). For a topological space X, let $\mathrm{BMEAS}(X)$ denote the collection of all Baire measurable subsets of X.

Proposition 9.2. $\mathrm{BMEAS}(X)$ *is a σ-algebra on X. In fact, it is the smallest σ-algebra containing all open sets and all meager sets.*

Proof. The second assertion follows from the first and the fact that any set $A \in \mathrm{BMEAS}(X)$ can be written as $A = U \Delta M$, where U is open and M is meager.

For the first assertion, we start by noting that if U is open, then $\overline{U} \setminus U$ is closed and nowhere dense, so $U =^* \overline{U}$. Taking complements, we see that if F is closed, $F \setminus \mathrm{Int}(F)$ is closed nowhere dense, so $F =^* \mathrm{Int}(F)$, and hence closed sets are Baire measurable. This implies that $\mathrm{BMEAS}(X)$ is closed under complements because if A is Baire measurable, then $A =^* U$ for some open U, and thus $A^c =^* U^c =^* \mathrm{Int}(U^c)$, so A^c is Baire measurable. Finally, if each A_n is Baire measurable, say $A_n =^* U_n$ with U_n open, then $\bigcup_n A_n =^* \bigcup_n U_n$, so $\bigcup_n A_n$ is Baire measurable. □

In particular, all open, closed, F_σ, G_δ, and more generally, Borel sets are Baire measurable.

Proposition 9.3. *Let X be a topological space and $A \subseteq X$. Then the following are equivalent:*
(1) *A is Baire measurable;*
(2) *$A = G \cup M$, where G is G_δ and M is meager;*
(3) *$A = F \setminus M$, where F is F_σ and M is meager.*

Proof. Follows from the fact that every meager set is contained in a meager F_σ set (see Proposition 6.5). □

Corollary 9.4. *For a nonempty perfect Polish space X, any nonmeager set $A \in \mathrm{BMEAS}(X)$ contains a nonempty perfect set.*

Proof. By the previous proposition, $A = G \cup M$, where G is G_δ and M is meager. Thus, G is nonmeager and hence is uncountable. So, G is an uncountable Polish space and therefore contains a copy of the Cantor space, by the Cantor–Bendixson theorem. □

This corollary in particular shows that AC implies that not all sets are Baire measurable. For example, we claim that any Bernstein set $B \subseteq \mathcal{N}$ (see Example 7.3) is not Baire measurable: indeed, otherwise, both of B, B^c would be Baire measurable, and at least one of them is nonmeager, so it must contain a nonempty perfect subset, contradicting the definition of a Bernstein set.

Definition 9.5. For topological spaces X, Y, a function $f : X \to Y$ is called *Baire measurable* if the preimage of every open set is Baire measurable.

Proposition 9.6. *Let X, Y be topological spaces and suppose Y is second countable. Then any Baire measurable function $f : X \to Y$ is continuous on a comeager set, i.e. there is a comeager set $D \subseteq X$ such that $f|_D : D \to Y$ is continuous.*

Proof. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis for Y. Because f is Baire measurable, $f^{-1}(V_n) =^* U_n$ for some open set $U_n \subseteq X$. Put $M_n = f^{-1}(V_n) \Delta U_n$ and let $D = X \setminus \bigcup_n M_n$. Now to show that $f|_D$ is continuous, it is enough to check that for each n, $(f|_D)^{-1}(V_n) = U_n \cap D$. For this, just note that $(f|_D)^{-1}(V_n) = f^{-1}(V_n) \cap D$, and since $M_n \cap D = \emptyset$, we have $f^{-1}(V_n) \cap D = U_n \cap D$. □

9.B. **Localization.** Recall that nonempty open subsets of Baire spaces are Baire themselves in the relative topology and all of the notions of category are absolute when relativizing to an open subset. This allows localizing the notions of category to open sets.

Definition 9.7. Let X be a topological space and $U \subseteq X$ an open set. We say that A *is meager in* U if $A \cap U$ is meager in X[9] and A is comeager in U if $U \setminus A$ is meager. If A is comeager in U, we say that A *holds generically in* U or that U *forces* A, in symbols $U \Vdash A$.

Thus, A is comeager iff $X \Vdash A$.

Note that if $A \subseteq B$, $U \subseteq V$ and $V \Vdash A$, then $U \Vdash B$. Also, $A =^* U \Rightarrow U \Vdash A$.

We now have the following simple fact that will be used over and over in our arguments below.

Proposition 9.8 (Baire alternative)**.** *Let A be a Baire measurable set in a topological space X. Then either A is meager or it is comeager in some nonempty open set*[10]*. If X is a Baire space, exactly one of these alternatives holds.*

Proof. By Baire measurability, $A =^* U$ for some open U. If $U = \emptyset$, then A is meager; otherwise, $U \neq \emptyset$ and $U \Vdash A$. □

We can now derive the following formulas concerning the forcing relation $U \Vdash A$. A *weak basis* for a topological space X is a collection $\mathcal{V}$ of nonempty open sets such that every nonempty open set $U \subseteq X$ contains at least one $V \in \mathcal{V}$.

Proposition 9.9. *Let X be a topological space.*

(a) *If $A_n \subseteq X$, then for any open $U \subseteq X$,*

$$U \Vdash \bigcap_n A_n \iff \forall n (U \Vdash A_n).$$

(b) *If X is a Baire space, A is Baire measurable and $U \subseteq X$ is nonempty open, then*

$$U \Vdash A^c \iff \forall V \subseteq U (V \nVdash A),$$

where V varies over a weak basis for X.

(c) *If X is a Baire space, the sets $A_n \subseteq X$ are Baire measurable, and U is nonempty open, then*

$$U \Vdash \bigcup_n A_n \iff \forall V \subseteq U \exists W \subseteq V \exists n (W \Vdash A_n).$$

where V, W vary over a weak basis for X.

Proof. Left as an exercise. □

9.C. **The Banach category theorem and a selector for $=^*$.** The following lemma gives an example of a case when an uncountable union of meager sets is still meager.

Lemma 9.10. *Let X be a topological space and let $(A_i)_{i \in I}$ be a family of nowhere dense (resp. meager) subsets of X (I may be uncountable). If there is a disjoint family $(U_i)_{i \in I}$ of open sets such that $A_i \subseteq U_i$, then $A = \bigcup_{i \in I} A_i$ is nowhere dense (resp. meager).*

[9] This is equivalent to $A \cap U$ being meager relative to U.

[10] Both alternatives can hold if the space X is not Baire.

Proof. The assertion with "meager" follows from that with "nowhere dense". For the assertion with "nowhere dense", let U be open and assume that $U \cap A \neq \emptyset$. We need to show that there is a nonempty open $V \subseteq U$ disjoint from A. Because $(U_i)_{i \in I}$ covers A, there is $i \in I$ with $U \cap U_i \neq \emptyset$. Since A_i is nowhere dense, there is nonempty open $V \subseteq U \cap U_i$ disjoint from A_i. But this V is also disjoint from A_j for $j \neq i$ simply because A_j is disjoint from $U_i \supseteq V$. Thus, V is disjoint from A, and we are done. □

Theorem 9.11 (Banach Category). *Let X be an arbitrary topological space and $A \subseteq X$. If A is locally meager, then it is meager; more precisely, if there is a (possibly uncountable) open cover $\mathcal{U}$ of A such that for each $U \in \mathcal{U}$, $A \cap U$ is meager, then A is meager. In particular,* ***arbitrary*** *unions of open meager sets are meager.*

Proof. The second assertion follows immediately from the first, so we only prove the first. Let $\mathcal{U}$ be the collection of *all* open sets $U \subseteq X$ such that $A \cap U$ is meager; by the hypothesis, $\mathcal{U}$ covers A. Using Zorn's lemma, take a maximal disjoint subfamily $\{V_i\}_{i \in I}$ of $\mathcal{U}$. Let $U = \bigcup \mathcal{U}$ and $V = \bigcup_{i \in I} V_i$.

Claim. V is dense in U, i.e. $U \subseteq \overline{V}$.

Proof of Claim. Otherwise, there is nonempty open $W \subseteq U$ disjoint from V. Although this W itself may not be a member of $\mathcal{U}$, there is $U' \in \mathcal{U}$ such that $V' = W \cap U' \neq \emptyset$. But then, by the definition of $\mathcal{U}$, $V' \in \mathcal{U}$ and is disjoint from V, contradicting the maximality of $\{V_i\}_{i \in I}$. ⊠

Thus, $A \setminus V \subseteq U \setminus V \subseteq \overline{V} \setminus V$, so $A \setminus V$ is nowhere dense. Therefore, it is enough to show that $A \cap V$ is meager, but this follows from the lemma above applied to $A_i = A \cap V_i$. □

We now draw a number of immediate corollaries.

Corollary 9.12. *Let X be a topological space and $A \subseteq X$. Put*

$$U(A) = \bigcup \{U \text{ open } : U \Vdash A\}.$$

Then $U(A) \Vdash A$, i.e. $U(A) \setminus A$ is meager. In particular, A is Baire measurable if and only if $A \setminus U(A)$ is meager if and only if $A =^* U(A)$.

Proof. Note that for every open $U \Vdash A$, $U \cap A^c$ is meager, so by the Banach category theorem, $U(A) \setminus A = U(A) \cap A^c$ is meager.

Now if A is Baire measurable, then $A =^* U$, for some open U. In particular, $U \Vdash A$ and hence $U \subseteq U(A)$. But then $A \setminus U(A) \subseteq A \setminus U$ is meager. □

A set U in a topological space X is called *regular open* if $U = \mathrm{Int}(\overline{U})$. Dually, a set F is *regular closed* if F^c is regular open (equivalently, $F = \overline{\mathrm{Int}(F)}$).

Proposition 9.13 (Canonical representatives for $=^*$-classes). *Let X be a Baire space. If $A \subseteq X$ is Baire measurable, then $U(A)$ is the unique regular open set U with $A =^* U$. Thus, $U(A) =^* A$ and $A =^* B \Rightarrow U(A) = U(B)$, i.e. the map $A \mapsto U(A)$ is a selector for the equivalence relation $=^*$ on* BMEAS(X).

Proof. Outlined in homework exercises. □

Letting RO(X) denote the class of regular open subsets of a Baire space X, what this proposition says is that we can canonically identify BMEAS(X)/MGR(X) with RO(X).

For a topological space X and $A \subseteq X$, recall the Baire alternative (Proposition 9.8): if A is Baire measurable, then A is meager or A is comeager in some nonempty open set. The converse is clearly false because for example we can take A to be a disjoint union of some nonempty open set and a Baire nonmeasurable set. Then the second alternative will hold, but A won't be Baire measurable. Not to mention that if X itself is meager (in particular, isn't a Baire space) then the second alternative vacuously holds for every set A. However, the following proposition shows that if X is Baire and the Baire alternative holds for $A \setminus U$, for every open set U, then A is Baire measurable.

Proposition 9.14. *Let X be a Baire space and $A \subseteq X$. The following are equivalent:*
(1) *A is Baire measurable;*
(2) *For every open set $U \subseteq X$, either $A \setminus U$ is meager or $A \setminus U$ is comeager in some nonempty open set $V \subseteq X$;*
(3) *Either $A \setminus U(A)$ is meager or $A \setminus U(A)$ is comeager in some nonempty open set $V \subseteq X$;*
(4) *$A \setminus U(A)$ is meager.*

Proof. (1)⇒(2) is just the statement of the Baire alternative, (2)⇒(3) is trivial, and (4)⇒(1) is stated in the previous corollary. For (3)⇒(4), assume for contradiction that $V \Vdash A \setminus U(A)$ for some nonempty open $V \subseteq X$; in particular, $V \Vdash A$ and hence $V \subseteq U(A)$, so V is disjoint from $A \setminus U(A)$. But then $V = V \setminus (A \setminus U(A))$ is meager, contradicting X being Baire. □

9.D. **The Banach–Mazur game.** In this subsection we define a game associated with the Baire alternative.

Notation 9.15. For sets A, B in a topological space X, write $A \subseteq_c B$ or $B \supseteq_c A$ if $\overline{A} \subseteq B$.

Let X be a nonempty Polish space and d a complete compatible metric on X. Also let $\mathcal{W}$ be a countable weak basis for X and let $A \subseteq X$. We define the *Banach–Mazur game* (or the *$**$-game*) $G^{**}(A)$ as follows:

$$\begin{array}{llllll} \text{I} & U_0 & & U_1 & & \\ & & & & & \dots \\ \text{II} & & V_0 & & V_1 & \end{array}$$

$U_n, V_n \in \mathcal{W}$, $\operatorname{diam}(U_n), \operatorname{diam}(V_n) < 1/n$, $U_0 \supseteq_c V_0 \supseteq_c U_1 \supseteq_c V_1 \dots$. Let x be such that $\{x\} = \bigcap_n \overline{U_n} (= \bigcap_n \overline{V_n})$. Then I wins iff $x \in A$.

Theorem 9.16 (Banach–Mazur, Oxtoby)**.** *Let X be a nonempty Polish space. Then*
(a) *A is meager iff II has a winning strategy in $G^{**}(A)$.*
(b) *A is comeager in a nonempty open set iff I has a winning strategy in $G^{**}(A)$.*

Proof. (a) ⇒: If A is meager, it is contained in $\bigcup_n F_n$, where each F_n is closed nowhere dense. Thus, at the n^{th} round, when Player I plays U_n, we let Player II respond by a legal $V_n \subseteq_c U_n \setminus F_n$. This is indeed a winning strategy for Player II since $\bigcap_n \overline{V_n} \subseteq \bigcap_n F_n^c \subseteq A^c$.

⇐: Now let σ be a winning strategy for Player II. For $x \in X$, call a position

$$p = (U_0, V_0, \dots, U_n, V_n)$$

good for x if it is played according to σ (i.e. $p \in \sigma$) and $x \in V_n$. By convention, the empty position $\emptyset$ is good for x. If $x \in A$ and every good for x position p has a proper extension that is also good for x, then there is a run of the game according to σ, which produces

$x \in A$, and hence Player I wins, giving a contradiction. Thus for every $x \in A$, there is a maximal good position for x, so $A \subseteq \bigcup_{p \in \sigma} M_p$, where M_p is the set of all $x \in X$, for which p is maximal good ($M_p = \emptyset$ if $|p|$ is odd). For p as above, note that

$$M_p = \{x \in V_n : \text{ for any legal Player I move } U_{n+1}, \text{ if } V_{n+1} \text{ is played by II according to } \sigma \\ \text{then } x \notin V_{n+1}\},$$

and hence M_p is nowhere dense since otherwise there would be a nonempty open set $U \subseteq V_n$ in which M_p is dense, so letting Player I play a legal $U_{n+1} \subseteq U$ and Player II respond by V_{n+1} according to σ, V_{n+1} should be disjoint from M_p by the very definition of the latter, contradicting M_p being dense in V_{n+1}. Thus, what we have shown is that A is meager.

(b) $\Rightarrow$: Let A be comeager in some nonempty open U. Then we let Player I play a legal $U_0 \subseteq U$ as his first move, and the rest is similar to $\Rightarrow$ of part (A) with U_0 instead of X, $U_0 \setminus A$ instead of A, and the roles of the players switched.

$\Leftarrow$: Suppose now that σ is a winning strategy for Player I, and let U_0 be his first move (in particular, $U_0 \neq \emptyset$). We can now run the same proof as for $\Leftarrow$ of (A) with U_0 instead of X, $U_0 \setminus A$ instead of A, and the roles of the players switched, to show that $U_0 \setminus A$ is meager, and hence A is comeager in U_0. □

Corollary 9.17. *Let X be a nonempty Polish space and let $A \subseteq X$. Then A is Baire measurable if and only if $G^{**}(A \setminus U(A))$ is determined.*

Proof. Follows immediately from Proposition 9.14 and the previous theorem. □

The $**$-game is played on a countable set $\mathcal{W}$, and thus, AD implies that it is determined for all $A \subseteq X$, so we have:

Corollary 9.18 (AD)**.** *All subsets of a Polish space are Baire measurable.*

9.E. **The Kuratowski–Ulam theorem.** In this subsection we prove an analog of Fubini's theorem for Baire category. We start by fixing convenient notation.

Let X be a topological space. For a set $A \subseteq X$ and $x \in X$, we put

$$A(x) \iff x \in A,$$

viewing A as a property of elements of X and writing $A(x)$ to mean that x has this property. We also use the following notation:

$$\forall^* x A(x) \iff A \text{ is comeager,}$$
$$\exists^* x A(x) \iff A \text{ is nonmeager.}$$

We read $\forall^*$ as "for comeager many" x, and $\exists^*$ as "for nonmeager many" x.

Similarly, for $U \subseteq X$ open, we write

$$\forall^* x \in U A(x) \iff A \text{ is comeager in } U,$$
$$\exists^* x \in U A(x) \iff A \text{ is nonmeager in } U.$$

Thus, denoting the negation by $\neg$, we have:

$$\neg \forall^* x \in U A(x) \iff \exists^* x \in U A^c(x).$$

With this notation, the content of Proposition 9.9 can be written as follows:

(a) $(\forall^* x)\,(\forall n)\,A_n(x) \iff (\forall n)\,(\forall^* x)\,A_n(x)$;
(b) $(\forall^* x \in U)\,A(x) \iff (\forall V \subseteq U)\,(\exists^* x \in V)\,A(x)$;
(c) $(\forall^* x \in U)\,(\exists n)\,A_n(x) \iff (\forall V \subseteq U)\,(\exists W \subseteq V)\,(\exists n)\,(\forall^* x \in W)\,A_n(x)$.

Recall that for arbitrary topological spaces X, Y, the projection function $\mathrm{proj}_1 : X \times Y \to X$ defined by $(x,y) \mapsto x$ is continuous and open (images of open sets are open). Conversely, for every $y \in Y$, the function $X \to X \times Y$ defined by $x \mapsto (x,y)$ is an embedding, i.e. a homeomorphism with its image.

Theorem 9.19 (Kuratowski–Ulam). *Let X, Y be second countable topological spaces. Let $A \subseteq X \times Y$ be Baire measurable, and denote $A_x = \{y \in Y : A(x,y)\}$, $A^y = \{x \in X : A(x,y)\}$.*

(i) *$\forall^* x(A_x$ is Baire measurable in $Y)$. Similarly, $\forall^* y(A^y$ is Baire measurable in $X)$.*
(ii) *A is meager $\iff \forall^* x(A_x$ is meager$) \iff \forall^* y(A^y$ is meager$)$.*
(iii) *A is comeager $\iff \forall^* x(A_x$ is comeager$) \iff \forall^* y(A^y$ is comeager$)$. In symbols:*

$$\forall^*(x,y)A(x,y) \iff \forall^* x \forall^* y A(x,y) \iff \forall^* y \forall^* x A(x,y).$$

Proof. First we need the following:

Claim. If $F \subseteq X \times Y$ is nowhere dense, then $\forall^* x(F_x$ is nowhere dense).

Proof of Claim. We may assume $Y \neq \emptyset$ and F is closed. Put $G = F^c$, and since G_x is open for every $x \in X$, it is enough to prove that $\forall^* x(G_x$ is dense). Fix a countable basis $\{V_n\}_{n \in \mathbb{N}}$ of nonempty open sets in Y and we need to show

$$\forall^* x \forall n(G_x \cap V_n \neq \emptyset),$$

which is equivalent to

$$\forall n \forall^* x(G_x \cap V_n \neq \emptyset).$$

Thus we fix n and show that the set $U_n = \{x \in X : G_x \cap V_n \neq \emptyset\}$ is open dense (and hence comeager). Note that $U_n = \mathrm{proj}_1(G \cap (X \times V_n))$ and hence is open. We claim that it is also dense: indeed, if $U \subseteq X$ is nonempty open then because G is dense in $X \times Y$, $G \cap (U \times V_n) \neq \emptyset$. But $U_n \cap U = \mathrm{proj}_1(G \cap (U \times V_n))$ and thus $U_n \cap U \neq \emptyset$. ⊠

This claim implies that if $M \subseteq X \times Y$ is meager, then $\forall^* x(M_x$ is meager), so we have shown $\Rightarrow$ of (ii).

For (i), let $A \subseteq X \times Y$ be Baire measurable, so $A = U \Delta M$ for some open U and meager M. Then for every $x \in X$, $A_x = U_x \Delta M_x$ and U_x is open. Since also $\forall^* x(M_x$ is meager), it follows that $\forall^* x(A_x$ is Baire measurable).

Since clearly (ii) implies (iii) by taking complements, it remains to prove $\Leftarrow$ of (ii).

Claim. Let $P \subseteq X, Q \subseteq Y$. $D = P \times Q$ is meager iff at least one of P, Q is meager.

Proof of Claim. $\Rightarrow$: By above we have $\forall^* x(D_x$ is meager). Thus either P is meager, or there is $x \in P$ such that D_x is meager in Y. But $D_x = Q$ for $x \in P$, so Q is meager.
$\Leftarrow$: It is enough to show that if P is nowhere dense, then so is $P \times Q$. Let $G \subseteq X \times Y$ be nonempty open. Then there is $U \times V \subseteq G$ with U, V nonempty open sets in X, Y, respectively. Because P is nowhere dense, there is nonempty open $U' \subseteq U$ with $P \cap U' = \emptyset$. Thus, $G' := U' \times V \subseteq G$ is nonempty open and $(P \times Q) \cap G' = \emptyset$. ⊠

We are now ready to prove $\Leftarrow$ of (ii). Let $A \subseteq X \times Y$ be Baire measurable and be such that $\forall^* x(A_x$ is meager). Because A is Baire measurable, $A = G \Delta M$ for some open G and meager

M in $X \times Y$. By $\Rightarrow$ of (ii), we have $\forall^* x(M_x$ is meager) and thus, since $G_x = A_x \Delta M_x$ for every $x \in X$, our assumption gives

$$\forall^* x(G_x \text{ is meager}).$$

Now suppose for contradiction that A is nonmeager, and hence G must also be nonmeager. Because X and Y are both second countable, G is a countable union of basic open sets of the form $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ open. Because G is nonmeager, one of these basic open sets $U \times V \subseteq G$ must be nonmeager. Thus, by the previous claim, both U and V are nonmeager, and hence there is $x \in U$ such that G_x is meager. But for this x, $G_x \supseteq V$, contradicting V being nonmeager. □

The Kuratowski–Ulam theorem fails if A is not Baire measurable. For example, using AC, one can construct a nonmeager set $A \subseteq \mathbb{R}^2$ so that no three points of A are on a straight line.

9.F. **Applications.**

Corollary 9.20. *A finite product of second-countable Baire spaces is Baire.*

Proof. Left as an exercise. □

Given a sequence $(X_n)_{n \in \mathbb{N}}$ of sets, let $X = \prod_n X_n$ and define an equivalence relation $\mathbb{E}_0^X$ on X as follows: for $x, y \in X$,

$$x \mathbb{E}_0^X y \iff \forall^\infty n \in \mathbb{N}\ x(n) = y(n).$$

A subset $A \subseteq X$ is called a *tail set* if it is $\mathbb{E}_0^X$-invariant, i.e. $x \in A$ and $y \mathbb{E}_0^X x$ implies that $y \in A$.

Theorem 9.21 (Second topological $0-1$ law)**.** *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of second countable Baire spaces and let $A \subseteq X := \prod_n X_n$ be Baire measurable. If A is a tail set, then A is either meager or comeager.*

Proof. Suppose that A is nonmeager. Because it is Baire measurable, it must be comeager in some nonempty basic open set $U \times Z \subseteq X$, where U is nonempty open in $Y := \prod_{i<n} X_i$ and $Z = \prod_{i \geqslant n} X_i$, for some n. By the corollary above, Y is Baire and hence U is nonmeager in Y. Note that A being comeager in $U \times Z$ simply means

$$\forall^*(y, z) \in U \times Z\ A(y, z),$$

so Kuratowski–Ulam gives

$$\forall^* z \in Z \forall^* y \in U\ A(y, z).$$

Because A is a tail set, (y, z) being in A depends only on z, so for each $z \in Z$, if there is $y \in Y$ such that $(y, z) \in A$, then actually $\forall y \in Y A(y, z)$. But $\forall^* z \in Z$ there is such a y in U because U is nonmeager. Thus,

$$\forall^* z \in Z \forall y \in Y A(y, z),$$

and hence, by Kuratowski–Ulam again, A is comeager. □

Theorem 9.22. *Let X be a nonempty perfect Polish space. No wellordering $<$ of X is Baire measurable (as a subset of X^2).*

Proof. Suppose for contradiction that $<$ is a Baire measurable wellordering of X. Call a set $A \subseteq X$ an *initial segment* if it is closed downward, i.e. for every $x \in A$, $<^x \subseteq A$.

Claim. Let $A \subseteq X$ be a nonmeager Baire measurable initial segment. Then $<|_A := < \cap A^2$ is nonmeager.

Proof of Claim. Suppose $<|_A$ is meager. Then Kuratowski–Ulam implies

$$\forall^* x \in X \ \Big((<|_A)^x \text{ and } (<|_A)_x \text{ are meager} \Big).$$

Thus, since A is nonmeager, there is $x \in A$ with $(<|_A)^x$ and $(<|_A)_x$ meager. But because also $\{x\}$ is nowhere dense (X is perfect), we get that

$$A = (<|_A)^x \cup \{x\} \cup (<|_A)_x$$

is meager, a contradiction. ⊠

Applying this claim to $A = X$, we get that $<$ must be nonmeager. Thus, by Kuratowski–Ulam,

$$\exists^* x \in X \ \Big(<^x \text{ is nonmeager and Baire measurable} \Big).$$

In particular, there exists $x \in X$ with $<^x$ being nonmeager and Baire measurable. Let x_0 be the $<$-least such and put $A = <^{x_0}$. By the claim, $<|_A$ is nonmeager. Thus, by Kuratowski–Ulam again,

$$\exists^* x \in X \ \Big((<|_A)^x \text{ is nonmeager and Baire measurable} \Big),$$

which is the same as

$$\exists^* x \in A \ \Big(<^x \text{ is nonmeager and Baire measurable} \Big).$$

Since A is nonmeager, there must exist such an $x \in A$, i.e. $x < x_0$ with $<^x$ being nonmeager and Baire measurable, contradicting the minimality of x_0. □

10. Measurability

Measures are one of the powerful useful tools in descriptive set theory and the measurability of sets plays a central role among the regularity properties. However, it takes work to properly develop the theory of measures, so in this section, we will only give some brief and noninformative definitions, as well as make some vague remarks about how measurability is tied with infinite games and determinacy.

10.A. **Definitions and examples.**

Let X be a Polish space and let $\mathcal{B}(X)$ denote the Borel σ-algebra of X, i.e. the smallest σ-algebra that contains all open sets.

Definition 10.1. A Borel measure on X is a function $\mu : \mathcal{B}(X) \to \mathbb{R}^+ \cup \{\infty\}$ that takes $\emptyset$ to 0 and that is countably additive, i.e. for pairwise disjoint Borel sets $A_n, n \in \mathbb{N}$, we have

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n).$$

Examples 10.2.

(a) The Lebesgue measure on $\mathbb{R}^n$ defined first on rectangles as the product of their side lengths, and then extended to all Borel sets using Caratheodory's extension theorem. On $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^3$, this measure corresponds to our intuition of what length, area, and volume of sets should be.

(b) The natural measure on the unit circle S^1 defined by pushing forward the measure from $[0,1]$ to S^1 via the map $x \mapsto e^{2\pi x i}$.

(c) The Cantor space can be equipped with the so-called *coin flip measure*, which is given by $\mu(N_s) = 2^{-|s|}$, thus $\mu(\mathcal{C}) = 1$.

(d) In general, it is a theorem of Haar that every locally compact Hausdorff topological group admits a unique, up to a constant multiple, nontrivial regular[11] left-invariant measure that is finite on compact sets; it is called *Haar measure*. This generalizes all of the above examples, including the coin flip measure on the Cantor space since we can identify $\mathcal{C} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.

(e) On any set X, one can define the so-called *counting measure* μ_c by giving each singleton measure 1. Similarly, when X is say $\mathbb{N}^+$, one can also assign measure $1/2^n$ to $\{n\}$, for $n \in \mathbb{N}^+$ and obtain a probability measure.

A Borel measure μ on X is called *continuous* (or *nonatomic*) if every singleton has measure zero. For example, the measures in all but the last example above are continuous, whereas in the last example it is purely atomic.

Furthermore, call a Borel measure μ on X *finite* if $\mu(X) < \infty$, and it is called *σ-finite* if X can be written as $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$. In case $\mu(X) = 1$, we call μ a *probability measure*. For example, the measures on S^1 and $\mathcal{C}$ defined above are probability measures, the Lebesgue measure on $\mathbb{R}^n$ is σ-finite, and the counting measure (Example 10.2(e)) on any uncountable set X is not σ-finite. In analysis and descriptive set theory, one usually deals with σ-finite measures, and even more often with probability measures.

10.B. **The null ideal and measurability.**

Definition 10.3. The null ideal of μ, noted NULL_μ, is the family of all subsets of Borel sets of measure 0.

Because of countable additivity, NULL_μ is a σ-ideal, and the sets in it are called *μ-null* (or just *null*) sets.

We now define *measurability of sets* analogously to Baire measurability, using NULL_μ instead of MGR. For two sets $A, B \subseteq X$, we write $A =_\mu B$ if $A \Delta B \in \mathrm{NULL}_\mu$. This is clearly an equivalence relation.

Definition 10.4. For a Borel measure μ, a set $A \subseteq X$ is called *μ-measurable* if $A =_\mu B$ for some Borel set B. In this case, we will define $\mu(A) := \mu(B)$, and extend μ to be defined on all measurable sets.

Clearly, μ-measurable subsets of X form a σ-algebra, which we denote by $\mathrm{MEAS}_\mu(X)$.

Definition 10.5. A subset A of a Polish space X is called *universally measurable* if it is μ-measurable for every σ-finite Borel measure μ.

Again, it is clear that universally measurable subsets of X form a σ-algebra and we denote it by $\mathrm{MEAS}(X)$.

In this definition, due to σ-finiteness of μ, the set A is μ-measurable if and only if $A \cap B$ is μ-measurable for every μ-measurable subset $B \subseteq X$ of finite μ-measure. This shows that we can replace "σ-finite" by "probability" in the definition of universal measurability. In fact,

[11] A Borel measure μ on a topological space X is called *regular* if for every measurable set $A \subseteq X$, $\mu(A) = \inf\{\mu(U) : U \supseteq A, U \text{ open}\} = \sup\{\mu(F) : F \subseteq A, F \text{ closed}\}$.

it is enough to consider continuous probability measures since probability measures can have at most countably many atoms and countable sets are clearly universally measurable.

This, together with the following theorem (which we won't prove in these notes), shows that the notion of universally measurable is very robust and it doesn't depend on the underlying Polish space:

Theorem 10.6 (Isomorphism of measures). *Let X be a Polish space and let μ be a continuous Borel probability measure on X. Then the measure space (X, μ) is Borel isomorphic to $([0,1], \lambda)$, where λ is the Lebesgue measure on $[0,1]$; more precisely, there is a Borel isomorphism*[12] *$f : X \to [0,1]$ such that the pushforward measure $f\mu$ is equal to λ.*

10.C. **Nonmeasurable sets.** Using AC, one can easily construct nonmeasurable subsets of $\mathbb{R}$. The most common example is the following.

Example 10.7. Let $\mathbb{E}_v$ be the equivalence relation on $\mathbb{R}$ defined as follows: for $x, y \in \mathbb{R}$, $x \mathbb{E}_v y$ iff $x - y \in \mathbb{Q}$. In other words it is the orbit equivalence relation of the translation action of $\mathbb{Q}$ on $\mathbb{R}$. This is known as the *Vitali equivalence relation*. A *transversal* for an equivalence relation is a set that meets every equivalence class at exactly one point. Let A be a transversal for $\mathbb{E}_v|_{[0,1]}$. We will show that it is not Lebesgue measurable.

Indeed, let $(q_n)_{n \in \mathbb{N}}$ enumerate (without repetitions) all rationals in $[-1,1]$. Note that $q_n + A \cap q_m + A = \emptyset$ for $n \neq m$ and that

$$[0,1] \subseteq \bigcup_n (q_n + A) \subseteq [-1,2].$$

If A is measurable, so is $q_n + A$, and thus we have

$$1 \leqslant \lambda(\bigcup_n q_n + A) \leqslant 3,$$

where λ denotes the Lebesgue measure. But because $q_n + A$ are pairwise disjoint and have equal measure (the Lebesgue measure is translation invariant),

$$\lambda(\bigcup_n q_n + A) = \sum_n \lambda(A),$$

and hence,

$$1 \leqslant \sum_n \lambda(A) \leqslant 3.$$

The second inequality implies that $\lambda(a) = 0$, but the first implies the opposite, a contradiction.

As with the PSP and Baire measurability, it is expected that "definable" sets are measurable. Borel sets for example, are measurable by definition. It can be shown that the so-called *analytic* sets (projections of Borel) are measurable, and thus so are their complements. But the measurability of definable sets beyond what's mentioned turns out to already be independent from ZFC.

As the measure isomorphism theorem above shows, when considering measurability of subsets of Polish spaces, we can restrict our attention to $X = [0,1]$ with the Lebesgue measure. Just like with the PSP and Baire measurability, there are infinite games associated

[12] Borel isomorphism is a bijection such that it and its inverse map Borel sets to Borel sets.

to measurability. One such game is the Banach–Mazur game for the so-called *Lebesgue density topology* on $[0,1]$, which we will discuss in the sequel.

10.D. **The Lebesgue density topology on $\mathbb{R}$.** Let λ denote the Lebesgue measure on $\mathbb{R}$. We now recall the notion of *density* and the related theorem from analysis.

Definition 10.8. For a measurable set $A \subseteq \mathbb{R}$, define the *density function* $d_A : \mathbb{R} \to [0,1]$ by letting I vary over bounded open intervals and setting

$$d_A(x) := \lim_{I \ni x, |I| \to 0} \frac{\lambda(A \cap I)}{\lambda(I)}$$

if this limit exists, and leaving it undefined otherwise.

Theorem 10.9 (Lebesgue density)**.** *For a measurable set $A \subseteq \mathbb{R}$, $d_A = \mathbb{1}_A$ a.e.*

Proof. Left as an exercise. □

For a measurable set $A \subseteq \mathbb{R}$, put $D(A) := \{x \in \mathbb{R} : d_A(x) = 1\}$, so by the Lebesgue density theorem, $A =_\lambda D(A)$. Note that for $A, B \subseteq \mathbb{R}$, if $A =_\lambda B$ then $d_A = d_B$. The converse is also true by the Lebesgue density theorem. Thus, $A =_\lambda B$ if and only if $D(A) = D(B)$; in other words, $D(A)$ is a canonical representative for the $=_\lambda$-equivalence class of A and the map $A \mapsto D(A)$ is a selector for the equivalence relation $=_\lambda$. This is the analogue of $U(A)$ in Baire category theory.

Definition 10.10. The *density topology* on $\mathbb{R}$ is defined by declaring a set $A \subseteq \mathbb{R}$ open if it is measurable and $A \subseteq D(A)$. That this is indeed a topology is outlined in exercises.

It is immediate that the density topology is finer than the usual topology on $\mathbb{R}$. Moreover, it is so fine that it is no longer second-countable: for any countable family $\mathcal{U}$ of nonempty open sets in this topology, we can choose a point x_U from each $U \in \mathcal{U}$ and have $V := \mathbb{R} \setminus \{x_U\}_{U \in \mathcal{U}}$ open (in the density topology), yet it is not a union of sets in $\mathcal{U}$. In particular, the density topology is not Polish. However, it still has some of the crucial properties of Polish spaces: namely, it is *regular*[13] and, more importantly, it has the property called *strong Choquet*, which ensures that certain decreasing sequences of open sets have nonempty intersection.

The main reason for defining the density topology is the following fact.

Proposition 10.11. *For a set $A \subseteq \mathbb{R}$, the following are equivalent:*

(1) *A is nowhere dense in the density topology;*
(2) *A is meager in the density topology;*
(3) *A is λ-null.*

Proof. Left as an exercise. □

Corollary 10.12. *A set $A \subseteq \mathbb{R}$ is Lebesgue measurable if and only if it is Baire measurable in the density topology.*

Proof. Follows immediately from the previous proposition and the Lebesgue density theorem. □

[13] A topology X is called *regular* if for any point $x \in X$ and closed set $C \subseteq X$ with $x \notin C$, there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $C \subseteq V$

The fact that the density topology is strong Choquet[14] implies that it is Baire. Moreover, because it contains a Polish topology (the usual topology on $\mathbb{R}$), Theorem 9.16 about Banach–Mazur games still holds for this topology. Thus, determinacy of Banach–Mazur games played on $\mathbb{R}$ with the density topology will imply Baire measurability for subsets of $\mathbb{R}$ in the density topology, and hence Lebesgue measurability.

[14] In fact, just Choquet is enough.

Part 3. Definable subsets of Polish spaces

In this part, we discuss important classes of the so-called *definable* subsets of Polish spaces, i.e. subsets that are defined explicitly from the very basic sets (the open sets) using simple set-theoretic operations such as complementation, countable unions and projections.

We will mainly study the Borel sets and the so-called *analytic* sets (projections of Borel). We will also mention the co-analytic sets (complements of analytic) without going deeply into their theory. We will see that the mentioned classes of sets enjoy most of the regularity properties. Nevertheless, the questions of whether the definable sets beyond co-analytic have the familiar regularity properties (such as the PSP, Baire measurability, measurability) turn out to be independent from ZFC. The latter fact, however, is beyond the realm of this course.

11. Borel sets

11.A. **σ-algebras and measurable spaces.** Recall that an *algebra $\mathcal{A}$ on a set X* is a family of subsets of X containing $\emptyset$ and closed under complements and finite unions (hence also finite intersections). An algebra $\mathcal{A}$ on X is called a *σ-algebra* if it is closed under countable unions (hence also countable intersections). For a family $\mathcal{E}$ of subsets of X, let $\sigma(\mathcal{E})$ denote the smallest σ-algebra containing $\mathcal{E}$. We say that $\mathcal{E}$ *generates the given σ-algebra $\mathcal{A}$* or that $\mathcal{E}$ is a *generating set* for $\mathcal{A}$ if $\sigma(\mathcal{E}) = \mathcal{A}$.

For a collection $\mathcal{E}$ of subsets of X, put $\neg\mathcal{E} := \{X \setminus A : A \in \mathcal{E}\}$ and refer to this as the *dual* of $\mathcal{E}$.

Proposition 11.1. *Let X be a set and $\emptyset \in \mathcal{E} \subseteq \mathscr{P}(X)$. Then $\sigma(\mathcal{E})$ is the smallest collection S of sets that contains $\mathcal{E}, \neg\mathcal{E}$, and is closed under countable unions and countable intersections.*

Proof. Put $S' = \{A \in S : A, A^c \in S\}$. Clearly, $S' \supseteq \mathcal{E}$ and it is trivially closed under complements. Because complement of a union is the intersection of complements, S' is also closed under countable unions, and thus is a σ-algebra. Hence, $\sigma(\mathcal{E}) \subseteq S' \subseteq S \subseteq \sigma(\mathcal{E})$. □

Definition 11.2. A measurable space is a pair $(X, \mathcal{S})$ where X is a set and $\mathcal{S}$ is a σ-algebra on X. For measurable spaces $(X, \mathcal{S}), (Y, \mathcal{A})$, a map $f : X \to Y$ is called *measurable* if $f^{-1}(A) \in \mathcal{S}$ for each $A \in \mathcal{A}$.

Recall that for a topological space Y, $\mathcal{B}(Y)$ denotes the σ-algebra generated by all open sets and it is called the *Borel σ-algebra of Y*. For a measurable space $(X, \mathcal{S})$, a map $f : X \to Y$ is called *measurable* if it is measurable as a map from $(X, \mathcal{S})$ to $(Y, \mathcal{B}(Y))$, i.e. the preimage of a Borel set is in $\mathcal{S}$.

For topological spaces X, Y, recall that a map $f : X \to Y$ is said to be *Baire measurable* if it is measurable as a map from $(X, \mathrm{BMEAS}(X))$ to Y, i.e. the preimages of Borel sets are Baire measurable in X. Furthermore, f is called *Borel* (or *Borel measurable*) if it is measurable as a map from $(X, \mathcal{B}(X))$ to Y, i.e. the preimages of Borel sets are Borel.

Proposition 11.3. *Let $(X, \mathcal{S}), (Y, \mathcal{A})$ be measurable spaces and let $\mathcal{F}$ be a generating set for $\mathcal{A}$. Then, a map $f : X \to Y$ is measurable if $f^{-1}(A) \in \mathcal{S}$ for every $A \in \mathcal{F}$. In particular, if Y is a topological space and $\mathcal{A} = \mathcal{B}(Y)$, then f is measurable if the preimage of every open set is in $\mathcal{S}$.*

Proof. It is easy to check that $\mathcal{A}' := \left\{A \in \mathcal{A} : f^{-1}(A) \in \mathcal{S}\right\}$ is a σ-algebra and contains $\mathcal{F}$. Thus, $\mathcal{A}' = \mathcal{A}$, and hence, $f^{-1}(A) \in \mathcal{S}$ for every $A \in \mathcal{A}$. □

This proposition in particular implies that continuous functions are Borel.

11.B. **The stratification of Borel sets into a hierarchy.** Let X be a topological space. We will now define the *hierarchy of the Borel subsets of X*, i.e. the recursive construction of Borel sets level-by-level, starting from the open sets.

Let ω_1 denote the first uncountable ordinal, and for $1 \leqslant \xi < \omega_1$, define by transfinite recursion the classes $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ of subsets of X as follows:

$$\begin{aligned}
\mathbf{\Sigma}^0_1(X) &:= \{U \subseteq X : U \text{ is open}\}\\
\mathbf{\Pi}^0_\xi(X) &:= \neg\mathbf{\Sigma}^0_\xi(X)\\
\mathbf{\Sigma}^0_\xi(X) &:= \left\{\bigcup_n A_n : A_n \in \mathbf{\Pi}^0_{\xi_n}(X), \xi_n < \xi, n \in \mathbb{N}\right\}, \text{ if } \xi > 1.
\end{aligned}$$

In addition, we define the so-called *ambiguous classes* $\mathbf{\Delta}^0_\xi(X)$ by

$$\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X).$$

Traditionally, one denotes by $G(X)$ the class of open subsets of X, and by $F(X)$ the class of closed subsets of X. For any collection $\mathcal{E}$ of subsets of subsets of X, let

$$\begin{aligned}
\mathcal{E}_\sigma &= \left\{\bigcup_n A_n : A_n \in \mathcal{E}, n \in \mathbb{N}\right\}\\
\mathcal{E}_\delta &= \left\{\bigcap_n A_n : A_n \in \mathcal{E}, n \in \mathbb{N}\right\}.
\end{aligned}$$

Then we have $\mathbf{\Sigma}^0_1 = G(X), \mathbf{\Pi}^0_1(X) = F(X), \mathbf{\Sigma}^0_2(X) = F_\sigma(X), \mathbf{\Pi}^0_2(X) = G_\delta(X), \mathbf{\Sigma}^0_3(X) = G_{\delta\sigma}(X), \mathbf{\Pi}^0_3(X) = F_{\sigma\delta}(X)$, etc. Also, note that $\mathbf{\Delta}^0_1(X) = \{A \subseteq X : A \text{ is clopen}\}$.

Proposition 11.4 (Closure properties)**.** *For a topological space X and for each $\xi \geqslant 1$, the classes $\mathbf{\Sigma}^0_\xi(X), \mathbf{\Pi}^0_\xi(X)$ and $\mathbf{\Delta}^0_\xi(X)$ are closed under finite intersections and finite unions. Moreover, $\mathbf{\Sigma}^0_\xi$ is closed under countable unions, $\mathbf{\Pi}^0_\xi$ under countable intersections, and $\mathbf{\Delta}^0_\xi$ under complements.*

Proof. The only statement worth checking is the closedness of the classes $\mathbf{\Sigma}^0_\xi$ under finite intersections, but it easily follows by induction on ξ using the fact that

$$\bigcup_n A_n \cap \bigcup_n B_n = \bigcup_{n,m} (A_n \cap B_m).$$

The statements about $\mathbf{\Pi}^0_\xi$ follows from those about $\mathbf{\Sigma}^0_\xi$ by taking complements. □

Proposition 11.5. *Let X be a metrizable space.*

(a) $\mathbf{\Sigma}^0_\xi(X) \cup \mathbf{\Pi}^0_\xi(X) \subseteq \mathbf{\Delta}^0_{\xi+1}(X)$.

(b) $\mathcal{B}(X) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi(X) = \bigcup_{\xi<\omega_1} \mathbf{\Delta}^0_\xi(X) = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi(X)$.

Proof. For part (a), by taking complements, it is enough to show that $\boldsymbol{\Sigma}^0_\xi(X) \subseteq \boldsymbol{\Delta}^0_{\xi+1}(X)$. By the definition of $\boldsymbol{\Pi}^0_{\xi+1}(X)$, $\boldsymbol{\Sigma}^0_\xi(X) \subseteq \boldsymbol{\Pi}^0_{\xi+1}(X)$, so it remains to show that $\boldsymbol{\Sigma}^0_\xi(X) \subseteq \boldsymbol{\Sigma}^0_{\xi+1}(X)$, which we do by induction on ξ. For $\xi = 0$ this is just the fact that open sets are F_σ in metrizable spaces. For the successor case, assume that $\boldsymbol{\Sigma}^0_\xi(X) \subseteq \boldsymbol{\Sigma}^0_{\xi+1}(X)$, and hence $\boldsymbol{\Pi}^0_\xi(X) \subseteq \boldsymbol{\Pi}^0_{\xi+1}(X)$, and show that $\boldsymbol{\Sigma}^0_{\xi+1}(X) \subseteq \boldsymbol{\Sigma}^0_{\xi+2}(X)$. Every set in $\boldsymbol{\Sigma}^0_{\xi+1}(X)$ is a countable union of sets in $\boldsymbol{\Pi}^0_\xi(X)$. But $\boldsymbol{\Sigma}^0_{\xi+2}(X) \supseteq \boldsymbol{\Pi}^0_{\xi+1}(X) \supseteq \boldsymbol{\Pi}^0_\xi(X)$ and $\boldsymbol{\Sigma}^0_{\xi+2}(X)$ is closed under countable unions, so $\boldsymbol{\Sigma}^0_{\xi+2}(X) \supseteq \boldsymbol{\Sigma}^0_{\xi+1}(X)$.

Finally, in case ξ is a limit, we don't even need the inductive assumption; simply note that $\forall \eta < \xi$, $\boldsymbol{\Pi}^0_\eta(X) \subseteq \boldsymbol{\Sigma}^0_{\eta+1}(X) \subseteq \boldsymbol{\Pi}^0_\xi(X) \subseteq \boldsymbol{\Sigma}^0_{\xi+1}(X)$ and $\boldsymbol{\Sigma}^0_{\xi+1}(X)$ is closed under countable unions, so it must contain $\boldsymbol{\Sigma}^0_\xi(X)$.

From part (a), we immediately get $\bigcup_{\xi<\omega_1} \boldsymbol{\Sigma}^0_\xi(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Delta}^0_\xi(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Pi}^0_\xi(X)$. For the first equality in (b), the inclusion $\supseteq$ follows by an easy induction on ξ, while $\subseteq$ follows from the claim that $\bigcup_{\xi<\omega_1} \boldsymbol{\Sigma}^0_\xi(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Delta}^0_\xi(X) = \bigcup_{\xi<\omega_1} \boldsymbol{\Pi}^0_\xi(X)$ is a σ-algebra containing all open sets. To verify that the latter set is closed under complements, look at $\bigcup_{\xi<\omega_1} \boldsymbol{\Delta}^0_\xi(X)$, while the closure under countable unions follows from the fact that ω_1 is regular (using AC), i.e. any countable sequence of ordinals below ω_1 is bounded from above by a countable ordinal. □

Thus, we have the following picture:

$$\underbrace{\begin{array}{ccccccccccccccc}
 & & \boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & & & \boldsymbol{\Sigma}^0_\xi & & & \\
 & \subseteq & & \supseteq & & \subseteq & & \supseteq & & & & \subseteq & & \supseteq & & \\
\boldsymbol{\Delta}^0_1 & & & & \boldsymbol{\Delta}^0_2 & & & & \boldsymbol{\Delta}^0_3 & \dots & \boldsymbol{\Delta}^0_\xi & & & & \boldsymbol{\Delta}^0_{\xi+1} & \dots \\
 & \subseteq & & \supseteq & & \subseteq & & \supseteq & & & & \subseteq & & \supseteq & & \\
 & & \boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & & & \boldsymbol{\Pi}^0_\xi & & &
\end{array}}_{\mathcal{B}}$$

Note that if X is second countable, then $|\boldsymbol{\Sigma}^0_1(X)| \leqslant 2^{\aleph_0}$ and hence, by induction on $\xi < \omega_1$, $|\boldsymbol{\Sigma}^0_\xi(X)| \leqslant |(2^{\aleph_0})^{\aleph_0}| = |2^{\aleph_0 \times \aleph_0}| = 2^{\aleph_0}$ and $|\boldsymbol{\Pi}^0_\xi(X)| \leqslant 2^{\aleph_0}$. Thus, it follows from (b) of the previous proposition that $|\mathcal{B}(X)| \leqslant |\omega_1 \times 2^{\aleph_0}|$, and by AC, $|\omega_1 \times 2^{\aleph_0}| = 2^{\aleph_0}$, so there are at most continuum many Borel sets.

Example 11.6. Let C^1 be the set of all continuously differentiable functions in $C([0,1])$ (at the endpoints we take one-sided derivatives). We will show that C^1 is $\boldsymbol{\Pi}^0_3$ and hence Borel.

It is not hard to check that for $f \in C([0,1])$, $f \in C^1$ iff for all $\varepsilon \in \mathbb{Q}^+$ there exist rational open intervals $I_0, \dots, I_{n-1}$ covering $[0,1]$ such that for all $j < n$:

$$\forall a,b,c,d \in I_j \cap [0,1] \text{ with } a \neq b, c \neq d \left(\left| \frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(d)}{c-d} \right| \leqslant \varepsilon \right).$$

So if for an open interval J and $\varepsilon > 0$, we put

$$A_{J,\varepsilon} := \left\{ f \in C([0,1]) : \forall a,b,c,d \in J \cap [0,1] \text{ with } a \neq b, c \neq d, \left| \frac{f(a)-f(b)}{a-b} - \frac{f(c)-f(d)}{c-d} \right| \leqslant \varepsilon \right\},$$

we have that $A_{J,\varepsilon}$ is closed in $C([0,1])$ and for $f \in C([0,1])$,

$$f \in C^1 \iff \forall \varepsilon \in \mathbb{Q}^+ \ \exists n \in \mathbb{N} \ \exists (I_0, \dots, I_{n-1}) \ \forall j < n \ f \in A_{I_j,\varepsilon},$$

where $(I_0, \dots, I_{n-1})$ varies over all n-tuples of rational open intervals with $\bigcup_{i<n} I_i \supseteq [0,1]$. Because $\forall$ corresponds to intersection and $\exists$ to union, we get

$$C^1 = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_n \bigcup_{(I_0,\dots,I_{n-1})} \bigcap_{j<n} A_{I_j,\varepsilon},$$

so C^1 is $\mathbf{\Pi}^0_3$. In practice, one does not need to write the set in terms of unions, intersections, and complements to understand its complexity. Instead, we write the definition of the set (or its equivalent) in terms of quantifiers and specify the classes as follows:

$$f \in C^1 \iff \underbrace{\forall \varepsilon \in \mathbb{Q}^+ \ \underbrace{\exists n \in \mathbb{N} \ \exists (I_0, \dots, I_{n-1}) \ \underbrace{\forall j < n \ \underbrace{f \in A_{I_j,\varepsilon}}_{\mathbf{\Pi}^0_1}}_{\mathbf{\Pi}^0_1}}_{\mathbf{\Sigma}^0_2}}_{\mathbf{\Pi}^0_3}.$$

11.C. **The classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.** Let $\mathcal{B}$, $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Delta}^0_\xi$ denote the classes of the corresponding types of sets in metrizable spaces, for example $\mathbf{\Sigma}^0_\xi$ is the union of $\mathbf{\Sigma}^0_\xi(X)$, where X varies over all metrizable spaces.

Proposition 11.7. *For each $\xi \geqslant 1$, if Γ is of the classes $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$, and $\mathbf{\Delta}^0_\xi$, then it is closed under the following operations.*

(11.7.a) *Continuous preimages, i.e. if X, Y are Polish spaces, $f : X \to Y$ is continuous and $A \subseteq \Gamma(Y)$, then $f^{-1}(A) \in \Gamma(X)$.*

(11.7.b) *Putting countably many sets as fibers over $\mathbb{N}$, i.e. if X is a Polish space and sets $A_n \subseteq \Gamma(X)$, then the set $B \subseteq \mathbb{N} \times X$ defined by $B_n := A_n$ is in $\Gamma(\mathbb{N} \times X)$, where B_n is the fiber of B over $n \in \mathbb{N}$.*

Proof. Easy induction on ξ. □

Let $\mathbf{\Gamma}$ be a class of sets in various spaces (such as $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$, $\mathbf{\Delta}^0_\xi$, $\mathcal{B}$, etc.). We denote by $\mathbf{\Gamma}(X)$ the collection of subsets of X that are in $\mathbf{\Gamma}$. We also denote by $\neg\mathbf{\Gamma}(X) := \{X \setminus A : A \in \mathbf{\Gamma}(X)\}$ the *dual* of $\mathbf{\Gamma}(X)$, and we let $\mathbf{\Delta}$ denote its ambiguous part: $\mathbf{\Delta}(X) := \mathbf{\Gamma}(X) \cap \neg\mathbf{\Gamma}(X)$. Furthermore, for any Polish space Y put

$$\exists^Y \mathbf{\Gamma} := \{\mathrm{proj}_1(A) : X \text{ is Polish}, A \in \mathbf{\Gamma}(X \times Y)\},$$
$$\forall^Y \mathbf{\Gamma} := \neg\exists^Y \mathbf{\Gamma}.$$

With this notation, we have the following:

Proposition 11.8. *For all $1 \leqslant \xi < \omega_1$, $\mathbf{\Sigma}^0_{\xi+1} = \exists^{\mathbb{N}} \mathbf{\Pi}^0_\xi$. Thus also $\mathbf{\Pi}^0_{\xi+1} = \forall^{\mathbb{N}} \mathbf{\Sigma}^0_\xi$.*

Proof. Follows from (11.7.b) and the fact that the operation $\exists^{\mathbb{N}}$ is the same as taking the union of the fibers. □

Thus, $\bigcup_{1 \leqslant n < \omega} \mathbf{\Sigma}^0_\xi$ are all the sets that can be obtained from open sets using operations $\neg$, $\exists^{\mathbb{N}}$ (and also the binary operation $\cup$). These set-theoretic operations obviously correspond to the logical operations $\neg$, $\exists$, and $\vee$, where $\exists$ varies over $\mathbb{N}$ (arithmetical definability). The superscript 0 in the notation $\mathbf{\Sigma}^0_n, \mathbf{\Pi}^0_n$ corresponds to the order of quantification: it is 0 in our case as the quantification $\exists^{\mathbb{N}}$ is done over $\mathbb{N}$. We will later define classes $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$, where the quantification is done over $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, i.e. functions from $\mathbb{N}$ to $\mathbb{N}$. One could also define the classes $\mathbf{\Sigma}^2_n, \mathbf{\Pi}^2_n$ using quantification over $\mathbb{N}^{\mathcal{N}}$, i.e. functions from $\mathcal{N}$ to $\mathbb{N}$, and so on.

11.D. **Universal sets for $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.** The classes $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ provide for each Polish space X a hierarchy for $\mathcal{B}(X)$ of at most ω_1 levels. We will next show that this is indeed a proper hierarchy, i.e., all these classes are distinct, when X is uncountable. We will use the usual diagonalization technique due to Cantor:

Lemma 11.9 (Diagonalization)**.** *For a set X and $R \subseteq X^2$, put* $\operatorname{AntiDiag}(R) = \{x \in X : \neg R(x,x)\}$. *Then* $\operatorname{AntiDiag}(R) \neq R_x$ *for any $x \in X$.*

Proof. Assume for contradiction that $\operatorname{AntiDiag}(R) = R_x$, for some $x \in X$. Then we get a contradiction because

$$\neg R(x,x) \iff x \in \operatorname{AntiDiag}(R) \iff x \in R_x \iff R(x,x). \qquad \square$$

Thus, we first need to construct, for each ξ, a set that parameterizes $\mathbf{\Sigma}^0_\xi$. The following definition makes this precise:

Definition 11.10. Let $\mathbf{\Gamma}$ be a class of sets in topological spaces (such as $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$, $\mathbf{\Delta}^0_\xi$, $\mathcal{B}$, etc.) and let X, Y be topological spaces. We say that a set $U \subseteq Y \times X$ *parameterizes* $\mathbf{\Gamma}(X)$ if

$$\left\{U_y : y \in Y\right\} = \mathbf{\Gamma}(X).$$

If, moreover, U itself is in $\mathbf{\Gamma}$ (i.e. $U \in \mathbf{\Gamma}(Y \times X)$), we say that U is *Y-universal for* $\mathbf{\Gamma}(X)$.

The following observation justifies the definition of a universal set for $\mathbf{\Gamma}(X)$ and we use it below without mention.

Observation 11.11 (Closure under fibers)**.** *Let $\mathbf{\Gamma}$ be a class of sets in topological spaces that is closed under continuous preimages and let X, Y be topological spaces. For any $A \in \mathbf{\Gamma}(X \times Y)$ and each $x \in X, y \in Y$, the fibers A_x, A^y are in $\mathbf{\Gamma}(Y)$.*

Theorem 11.12. *Let X be a separable metrizable space. Then for each $\xi \geqslant 1$, there is a $\mathcal{C}$-universal set for $\mathbf{\Sigma}^0_\xi(X)$, and similarly for $\mathbf{\Pi}^0_\xi(X)$.*

Proof. We prove by induction on ξ. Let $(V_n)_{n \in \mathbb{N}}$ be an open basis for X. Because every $\mathbf{\Sigma}^0_1$(=open) set is a union of some subsequence of these V_n, we define $U \subseteq \mathcal{C} \times X$ as follows: for $y \in \mathcal{C}$, put

$$U_y := \bigcup_{n, y(n)=1} V_n.$$

It is clear U parameterizes $\mathbf{\Sigma}^0_1(X)$. Moreover, U is open because for $(y,x) \in \mathcal{C} \times X$,

$$(y,x) \in U \iff x \in \bigcup_{n, y(n)=1} V_n \iff \exists n \in \mathbb{N}\ (y(n) = 1 \wedge x \in V_n).$$

Thus indeed, U is $\mathcal{C}$-universal for $\mathbf{\Sigma}^0_1$.

Note next that if $U \subseteq \mathcal{C} \times X$ is $\mathcal{C}$-universal for $\mathbf{\Gamma}(X)$, then U^c is $\mathcal{C}$-universal for the dual class $\neg\mathbf{\Gamma}(X)$. In particular, if there is a $\mathcal{C}$-universal set for $\mathbf{\Sigma}^0_\xi(X)$, there is also one for $\mathbf{\Pi}^0_\xi(X)$.

Assume now that we have already defined $\mathcal{C}$-universal sets V^η for $\mathbf{\Pi}^0_\eta$ for all $\eta < \xi$. Let $\eta_n < \xi$, $n \in \mathbb{N}$, be such that $\eta_n \leqslant \eta_{n+1}$ and $\sup\{\eta_n + 1 : n \in \mathbb{N}\} = \xi$. Because $\mathcal{C}$ is homeomorphic to $\mathcal{C}^{\mathbb{N}}$, it is enough to construct a $\mathcal{C}^{\mathbb{N}}$-universal set for $\mathbf{\Sigma}^0_\xi$. Just like for the open sets, define $U \subseteq \mathcal{C}^{\mathbb{N}} \times X$ as follows: for $y \in \mathcal{C}^{\mathbb{N}}$, put

$$U_y := \bigcup_n V^{\eta_n}_{y(n)}.$$

By definition, U parameterizes $\mathbf{\Sigma}^0_\xi(X)$. To see why U itself is in $\mathbf{\Sigma}^0_\xi$, note that for $(y, x) \in \mathcal{C}^{\mathbb{N}} \times X$,

$$(y,x) \in U \iff x \in \bigcup_n V^{\eta_n}_{y(n)} \iff \exists n \in \mathbb{N}\ V^{\eta_n}(y(n), x).$$

The latter condition defines a set in $\mathbf{\Sigma}^0_\xi(X)$ because V^{η_n} is in $\mathbf{\Sigma}^0_\xi(\mathcal{C} \times X)$, the projection function $y \mapsto y(n)$ is continuous for each $n \in \mathbb{N}$, and $\mathbf{\Sigma}^0_\xi$ is closed under continuous preimages. □

Lemma 11.13 (Relativization to subsets)**.** *Let X be a topological space, $Y \subseteq X$, and let ξ be an ordinal with $1 \leqslant \xi < \omega_1$.*

(a) *If $\mathbf{\Gamma}$ is one of $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathcal{B}$, then $\mathbf{\Gamma}(Y) = \mathbf{\Gamma}(X)|_Y := \{A \cap Y : A \in \mathbf{\Gamma}(X)\}$.*

(b) *We also always have $\mathbf{\Delta}^0_\xi(Y) \supseteq \mathbf{\Delta}^0_\xi(X)|_Y$. If moreover, $Y \in \mathbf{\Delta}^0_\xi(X)$, then we also have $\mathbf{\Delta}^0_\xi(Y) \subseteq \mathbf{\Delta}^0_\xi(X)|_Y$. However, the last inclusion is in general false for arbitrary Y.*

Proof. Left as an exercise. □

Corollary 11.14. *Let X be separable metrizable and Y be uncountable Polish. For any $1 \leqslant \xi < \omega_1$, there is a Y-universal set for $\mathbf{\Sigma}^0_\xi(X)$, and similarly for $\mathbf{\Pi}^0_\xi$.*

Proof. The existence of a Y-universal set for $\mathbf{\Sigma}^0_\xi(X)$ follows from that for $\mathbf{\Pi}^0_\xi(X)$, so it is enough to construct a Y-universal set for $\mathbf{\Pi}^0_\xi(X)$.

By the perfect set property for Polish spaces, there is a homeomorphic copy $C \subseteq Y$ of the Cantor space. By the above theorem, there is a C-universal set $U_\xi \in \mathbf{\Pi}^0_\xi(C \times X)$ for $\mathbf{\Pi}^0_\xi(X)$; in particular, U parameterizes $\mathbf{\Pi}^0_\xi(X)$. By the previous lemma, $\mathbf{\Pi}^0_\xi(C \times X) = \mathbf{\Pi}^0_\xi(Y \times X)|_{C \times X}$, and since $C \times X$ is closed in $Y \times X$ (i.e. is in $\mathbf{\Pi}^0_1(Y \times X)$) and $\mathbf{\Pi}^0_\xi(Y \times X)$ is closed under finite (in fact, countable) intersections, $\mathbf{\Pi}^0_\xi(Y \times X)|_{C \times X} \subseteq \mathbf{\Pi}^0_\xi(Y \times X)$, so U_ξ is still $\mathbf{\Pi}^0_\xi$ as a subset of $Y \times X$, i.e. $U_\xi \in \mathbf{\Pi}^0_\xi(Y \times X)$. (This wouldn't be true for $\mathbf{\Sigma}^0_1$ and that's why we chose to construct a universal set for $\mathbf{\Pi}^0_\xi$ instead of $\mathbf{\Sigma}^0_\xi$.) Thus, U_ξ is Y-universal for $\mathbf{\Pi}^0_\xi(X)$. □

Corollary 11.15. *For every uncountable Polish space X and every $1 \leqslant \xi < \omega_1$, $\mathbf{\Sigma}^0_\xi(X) \neq \mathbf{\Pi}^0_\xi(X)$. In particular, $\mathbf{\Delta}^0_\xi(X) \subsetneq \mathbf{\Sigma}^0_\xi(X) \subsetneq \mathbf{\Delta}^0_{\xi+1}(X)$, and the same holds for $\mathbf{\Pi}^0_\xi$.*

Proof. Let $U \subseteq X \times X$ be an X-universal set for $\mathbf{\Sigma}^0_\xi(X)$ and take $A = \mathrm{AntiDiag}(U)$. Since $A = \delta^{-1}(U^c)$, where $\delta : X \to X^2$ by $x \mapsto (x,x)$, $A \in \mathbf{\Pi}^0_\xi(X)$. However, by the Diagonalization lemma, $A \neq U_x$ for any $x \in X$, and thus $A \notin \mathbf{\Sigma}^0_\xi(X)$. □

11.E. **Turning Borel sets into clopen sets.** The following theorem is truly one of the most useful facts about Borel sets. Recall that a Polish space X is formally a set X with a topology $\mathcal{T}$ on it (i.e. the collection of the open sets), so it is really a pair $(X,\mathcal{T})$. We denote the Borel subsets of X by $\mathcal{B}(X,\mathcal{T})$ or just $\mathcal{B}(\mathcal{T})$, when we want to emphasize the topology with respect to which the Borel sets are taken.

Theorem 11.16. *Let $(X,\mathcal{T})$ be a Polish space. For any Borel set $A \subseteq X$, there is a finer Polish topology $\mathcal{T}_A \supseteq \mathcal{T}$ with respect to which A is clopen, yet $\mathcal{B}(\mathcal{T}_A) = \mathcal{B}(\mathcal{T})$.*

We will prove this theorem after proving the following useful lemmas.

Lemma 11.17. *Let $(X,\mathcal{T})$ be a Polish space. For any closed set $F \subseteq X$, the topology $\mathcal{T}_F$ generated by $\mathcal{T} \cup \{F\}$ is Polish. Moreover, F is clopen in $\mathcal{T}_F$ and $\mathcal{B}(\mathcal{T}_F) = \mathcal{B}(\mathcal{T})$.*

Proof. The assertions in the second sentence of the statement are obvious. To see that $\mathcal{T}_F$ is Polish, note that $(X,\mathcal{T}_F)$ is a direct sum of the topological spaces $(F,\mathcal{T}|_F)$ and $(F^c,\mathcal{T}|_{F^c})$, where $\mathcal{T}|_F$ and $\mathcal{T}|_{F^c}$ are the relative topologies on F and F^c as subspaces of $(X,\mathcal{T})$. But then $(X,\mathcal{T}_F)$ is Polish being a direct sum of two Polish spaces. □

Lemma 11.18. *Let $(X,\mathcal{T})$ be a Polish space and let $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ be a sequence of Polish topologies on X with $\mathcal{T} \subseteq \mathcal{T}_n$, for all $n \in \mathbb{N}$. Then the topology $\mathcal{T}_\infty$ generated by $\bigcup_n \mathcal{T}_n$ is Polish. Moreover, if $\mathcal{T}_n \subseteq \mathcal{B}(\mathcal{T})$, for all $n \in \mathbb{N}$, then $\mathcal{B}(\mathcal{T}_\infty) = \mathcal{B}(\mathcal{T})$.* (We will actually see later that $\mathcal{B}(\mathcal{T}_\infty) = \mathcal{B}(\mathcal{T})$ is already implied by $\forall n \in \mathbb{N}(\mathcal{T} \subseteq \mathcal{T}_n)$).

Proof. [15] Let $(X_n,\mathcal{T}_n)$ be the topological space with the underlying set $X_n = X$ and topology $\mathcal{T}_n$. Then the product space $Y := \prod_n (X_n,\mathcal{T}_n)$ is Polish, and we let $\delta : (X,\mathcal{T}_\infty) \to Y$ be the diagonal map: $x \mapsto (x,x,\ldots)$. By definition, δ is a continuous bijection from $(X,\mathcal{T}_\infty)$ to $\delta(X)$ because each coordinate function $\mathrm{proj}_n \circ \delta : (X,\mathcal{T}_\infty) \to (X,\mathcal{T}_n)$ is continuous.

Claim. If a sequence $\big(\delta(x_m)\big)_{m\in\mathbb{N}}$ converges in Y, then $(x_m)_{m\in\mathbb{N}}$ converges in $(X,\mathcal{T}_\infty)$.

Proof of Claim. By the definition of product topology, for each $n \in \mathbb{N}$, the projections $\mathrm{proj}_n\big(\delta(x_m)\big)$ converge in $\mathcal{T}_n$, as $m \to \infty$, to some $y_n \in X_n = X$. But $\mathrm{proj}_n\big(\delta(x_m)\big) = x_m$, so $(x_m)_{m\in\mathbb{N}}$ converges to y_n in $\mathcal{T}_n$, and hence also in $\mathcal{T}$. Because $\mathcal{T}$ is Hausdorff, these y_n must all be equal and we denote the common value by y. Thus, $(x_m)_{m\in\mathbb{N}}$ converges to y in $\mathcal{T}_n$ for all $n \in \mathbb{N}$, and hence also in $\mathcal{T}_\infty$. ⊠

This claim implies that $\delta(X)$ is closed in Y and δ^{-1} is continuous, so δ is a homeomorphism between $(X,\mathcal{T}_\infty)$ and a closed subset of Y, and hence, $(X,\mathcal{T}_\infty)$ is Polish. □

Proof of Theorem 11.16. Let $\mathcal{S}$ be the collection of all sets $A \subseteq X$ for which there exists a Polish topology $\mathcal{T}_A \supseteq \mathcal{T} \cup \{A\}$ with $\mathcal{B}(\mathcal{T}_A) = \mathcal{B}(\mathcal{T})$. It is obvious that $\mathcal{S}$ is closed under complements and by Lemmas 11.17 and 11.18 it contains all closed sets and is closed under countable unions. Thus, $\mathcal{S}$ contains all Borel sets. □

[15] Thanks to Anton Bernshteyn for suggesting the exposition of this proof using sequences as opposed to open sets.

Theorem 11.16 is extensively used throughout Descriptive Set Theory. Before going into its applications, we mention a generalization of Lemma 11.18 in terms of fiber products.

Lemma 11.19. *Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a sequence of topological spaces, Y a topological space, and $f_i : X_i \to Y$ continuous maps. For $j \in I$, let $\mathcal{T}_j'$ denote the topology generated by $\mathcal{T}_j$ and the sets of the form $f_j^{-1} \circ f_i(U_i)$ for $i \in I \setminus \{j\}$ and $U_i \in \mathcal{T}_i$. If for all $i \in I \setminus \{j\}$, f_i is a bijection, then the fiber product $\prod_{i\in I} \big((X_i, \mathcal{T}_i), f_i\big)$ is homeomorphic to $(X_j, \mathcal{T}_j')$ via the projection map proj_j.*[16]

Proof. Straightforward once the definitions are unraveled. We leave it as an exercise. □

Here are several very useful applications of Theorem 11.16.

Corollary 11.20. *Borel subsets of Polish spaces have the* PSP.

Proof. Let B be an uncountable Borel subset of a Polish space $(X, \mathcal{T})$. By the previous theorem, there is a Polish topology $\mathcal{T}' \supseteq \mathcal{T}$ in which B is clopen and hence $(B, \mathcal{T}'|_B)$ is Polish, where $\mathcal{T}'|_B$ denotes the relative topology on B with respect to $\mathcal{T}'$. Now by the PSP for Polish spaces, there is an embedding $f : \mathcal{C} \hookrightarrow (B, \mathcal{T}'|_B)$. But then f is still continuous as a map from $\mathcal{C}$ into $(B, \mathcal{T}|_B)$ as $\mathcal{T}|_B$ has fewer open sets. Hence, because $\mathcal{C}$ is compact, f is still automatically an embedding from $\mathcal{C}$ into $(B, \mathcal{T}|_B)$. □

Corollary 11.21. *Let $(X, \mathcal{T})$ be a Polish space, Y be a second countable space, and $f : X \to Y$ be a Borel function. There is a Polish topology $\mathcal{T}_f \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_f) = \mathcal{B}(\mathcal{T})$ that makes f continuous.*

Proof. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis for Y and let $\mathcal{T}_n \supseteq \mathcal{T}$ be a Polish topology on X that makes $f^{-1}(V_n)$ open and has the same Borel sets as $\mathcal{T}$. By Lemma 11.18, the topology $\mathcal{T}_\infty$ generated by $\bigcup_n T_n$ is Polish, and clearly, $f : (X, \mathcal{T}_\infty) \to Y$ is continuous. □

Corollary 11.22. *Let $(X, \mathcal{T})$ be a Polish space and $B \subseteq X$ be Borel. There is a closed subset $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \to B$. In particular, if $B \neq \emptyset$, there is a continuous surjection $\bar{f} : \mathcal{N} \twoheadrightarrow B$.*

Proof. Let $\mathcal{T}' \supseteq \mathcal{T}$ be a Polish topology making B clopen. Hence $(B, \mathcal{T}'|_B)$ is Polish and and we apply Theorem 5.9. □

The proofs of the following two corollaries are left as exercises.

Corollary 11.23. *Any Borel action $\Gamma \curvearrowright (X, \mathcal{T})$ of a countable group Γ on a Polish space $(X, \mathcal{T})$ has a continuous realization, i.e. there is a finer topology $\mathcal{T}_\Gamma \supseteq \mathcal{T}$ such that the action $\Gamma \curvearrowright (X, \mathcal{T}_\Gamma)$ is continuous.*

Corollary 11.24. *For any Polish $(X, \mathcal{T})$, there is a zero-dimensional Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$.*

12. Analytic sets

It is clear that the class of Borel sets is closed under continuous preimages, but is it closed under continuous images?

Definition 12.1. A subset A of a Polish space X is called *analytic* if it is a continuous image of a Borel subset of some Polish space; more precisely, if there is a Polish space Y, a Borel set $B \subseteq Y$ and a continuous function $f : Y \to X$ such that $f(B) = A$.

[16] Many thanks to Ruiyuan (Ronnie) Chen for pointing this out.

Clearly, all Borel sets are analytic, but is the converse true? Historically, Lebesgue had a "proof" that continuous images of Borel sets are Borel, but several years later Souslin found a mistake in Lebesgue's proof; moreover, he constructed an example of a closed set whose projection was not Borel. Hence continuous images of Borel sets were new kinds of sets, which he and his advisor Luzin called analytic and systematically studied the properties thereof. This is often considered the birth of descriptive set theory.

12.A. **Basic facts and closure properties.** Before we exhibit an analytic set that is not Borel, we give the following equivalences to being analytic.

Proposition 12.2. *Let X be Polish and $\emptyset \neq A \subseteq X$. The following are equivalent:*

(1) *A is analytic.*
(2) *There is Polish Y and continuous $f : Y \to X$ with $A = f(Y)$.*
(3) *There is continuous $f : \mathcal{N} \to X$ with $A = f(\mathcal{N})$.*
(4) *There is closed $F \subseteq X \times \mathcal{N}$ with $A = \mathrm{proj}_1(F)$.*
(5) *There is Polish Y and Borel $B \subseteq X \times Y$ with $A = \mathrm{proj}_1(B)$.*
(6) *There is Polish Y, Borel $B \subseteq Y$ and Borel $f : Y \to X$ with $A = f(B)$.*

Proof. (4)⇒(5)⇒(1) are trivial, (1)⇒(2) is immediate from Theorem 11.16, (2)⇒(3) follows from Theorem 5.9, and (3)⇒(4) follows from the fact that graphs of continuous functions are closed and $f(Y) = \mathrm{proj}_1(\mathrm{Graph}(f))$.

Finally, the implication (1)⇒(6) is trivial and the reverse implication follows from Corollary 11.21. Alternatively, one could deduce (6)⇒(5) from the fact that if $f : Y \to X$ and $B \subseteq Y$ are Borel, then $\mathrm{Graph}(f|_B)$ is Borel (see Corollary 12.8) and $f(B) = \mathrm{proj}_1(\mathrm{Graph}(f|_B))$. □

Let $\mathbf{\Sigma}^1_1$ denote the class of all analytic subsets of Polish spaces, so for a Polish space X, $\mathbf{\Sigma}^1_1(X)$ is the set of all analytic subsets of X. Let $\mathbf{\Pi}^1_1 := \neg\mathbf{\Sigma}^1_1$ denote the dual class, and we call the elements of $\mathbf{\Pi}^1_1$ *co-analytic*. By (4) of the above proposition, we have

$$\mathbf{\Sigma}^1_1 = \exists^{\mathcal{N}}\mathcal{B} = \exists^{\mathcal{N}}\mathbf{\Pi}^0_1,$$

and consequently,

$$\mathbf{\Pi}^1_1 = \forall^{\mathcal{N}}\mathcal{B} = \forall^{\mathcal{N}}\mathbf{\Sigma}^0_1.$$

Furthermore, put $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. It is clear that $\mathcal{B} \subseteq \mathbf{\Delta}^1_1$, and we will see later that they are actually equal. For now, we will just list some closure properties of $\mathbf{\Sigma}^1_1$:

Proposition 12.3. *The class $\mathbf{\Sigma}^1_1$ is closed under*

(i) *continuous images and preimages;*
(ii) *(in fact) Borel images and preimages;*
(iii) *countable intersections and unions.*

Proof. We only prove the closure under countable intersections and leave the rest as an exercise. Let A_n be analytic subsets of a Polish space X. By (4) of Proposition 12.2, there are closed sets $C_n \subseteq X \times \mathcal{N}$ such that $A_n = \mathrm{proj}_1(C_n)$. Let $Y = X \times \mathcal{N}^{\mathbb{N}}$ and consider the set $C \subseteq Y$ defined by

$$(x, (y_n)_{n\in\mathbb{N}}) \in C :\Leftrightarrow \forall n \in \mathbb{N}(x, y_n) \in C_n.$$

Clearly, C is Borel (in fact it is closed) and $\bigcap_n A_n = \mathrm{proj}_1(C)$. □

12.B. **A universal set for Σ_1^1.** We now focus on showing that $\mathbf{\Sigma}_1^1 \neq \mathbf{\Pi}_1^1$ and hence there are analytic sets that are not Borel. As with the Borel hierarchy, we start with a universal analytic set:

Theorem 12.4 (Souslin)**.** *For any uncountable Polish Y and Polish X, there is a Y-universal set $U \subseteq Y \times X$ for $\mathbf{\Sigma}_1^1(X)$. The same holds for $\mathbf{\Pi}_1^1(X)$.*

Proof. The idea is to use (4) of Proposition 12.2, so we start with a Y-universal set $F \subseteq Y \times (X \times \mathcal{N})$ for $\mathbf{\Pi}_1^0(X \times \mathcal{N})$, which exists by Corollary 11.14. Put $U := \mathrm{proj}_{1,2}(F) := \{(y,x) \in Y \times X : \exists z \in \mathcal{N}(y,x,z) \in F\}$ and note that U is analytic being a projection of a closed set. We claim that U is universal for $\mathbf{\Sigma}_1^1(X)$. Indeed, let $A \subseteq X$ be analytic, so by (4) of Proposition 12.2, there is a closed set $C \subseteq X \times \mathcal{N}$ with $A = \mathrm{proj}_1(C)$. Then there is $y \in Y$ with $F_y = C$ and hence $A = \mathrm{proj}_1(C) = \mathrm{proj}_1(F_y) = (\mathrm{proj}_{1,2}(F))_y = U_y$ and we are done. $\square$

Corollary 12.5 (Souslin)**.** *For any uncountable Polish space X, $\mathbf{\Sigma}_1^1(X) \neq \mathbf{\Pi}_1^1(X)$. In particular, $\mathcal{B}(X) \subseteq \mathbf{\Delta}_1^1(X) \neq \mathbf{\Sigma}_1^1(X)$, and same for $\mathbf{\Pi}_1^1(X)$.*

Proof. Take an X-universal set $U \subseteq X \times X$ for $\mathbf{\Sigma}_1^1(X)$ and put

$$A := \mathrm{AntiDiag}(U) = \{x \in X : (x,x) \notin U\}.$$

Let $\delta : X \to X \times X$ by $x \mapsto (x,x)$ and note that it is continuous. Because $A = \delta^{-1}(U^c)$ and U^c is co-analytic, A is also co-analytic. However, it is not analytic since otherwise A would have to be equal to a fiber U_x of U, for some $x \in X$, contradicting the diagonalization lemma.

In particular, A is not Borel, so A^c is analytic but not Borel. $\square$

12.C. **Analytic separation and Borel = $\mathbf{\Delta}_1^1$.**

Theorem 12.6 (Luzin)**.** *Let X be a Polish space and let $A, B \subseteq X$ be disjoint analytic sets. There is a Borel set $D \subseteq X$ that separates A and B, i.e. $D \supseteq A$ and $D^c \supseteq B$.*

Proof. Call disjoint sets $P, Q \subseteq X$ *Borel-separable* if there is a Borel set $R \subseteq X$ with $R \supseteq P$ and $R \cap Q = \emptyset$. Note that the collection of sets that are Borel-separable from a given set Q forms a σ-ideal: indeed, if $P = \bigcup_n P_n$ and each P_n is separable from Q by R_n, then the set $\bigcup_n R_n$ separates P from Q. Thus we have:

Claim. If $P = \bigcup_n P_n$ and $Q = \bigcup_m Q_m$, and P_n, Q_m are Borel-separable for any $n, m \in \mathbb{N}$, then P, Q are Borel-separable.

Proof of Claim. First fix n and note that P_n is separable from Q since it is separable from each Q_m. But then Q is separable from P. ⊠

An obvious example of disjoint sets that are Borel-separable are distinct singletons $\{x\}, \{y\}$ (because X is Hausdorff). Iterating the above claim, we will show that if A, B are not Borel-separable, then it should boil down to two singletons not being Borel-separable, which would be a contradiction.

Let $f : \mathcal{N} \to A$ and $g : \mathcal{N} \to B$ be continuous surjections, which exist by (3) of Proposition 12.2. Put $A_s := f(N_s)$ and $B_s := g(N_s)$ for each $s \in \mathbb{N}^{<\mathbb{N}}$. Using the claim, we can follow the non-Borel-separable branch of $\mathbb{N}^{<\mathbb{N}}$ and recursively define $x, y \in \mathcal{N}$ such that for every $n \in \mathbb{N}$, $A_{x|_n}$ and $B_{y|_n}$ are not Borel-separable. Put $a := f(x)$ and $b := g(y)$. Because A and B are disjoint, $a \neq b$, so there are disjoint open neighborhoods $U \ni a$ and $V \ni b$. By the continuity of f and g, there is n such that $A_{x|_n} = f(N_{x|_n}) \subseteq U$ and $B_{y|_n} = g(N_{y|_n}) \subseteq V$. So $A_{x|_n}$ and $B_{y|_n}$ are Borel-separable, contradicting the choice of x and y. ☐

Corollary 12.7 (Souslin). *Let X be Polish and $A \subseteq X$. If A and A^c are both analytic, then A is Borel. In other words, $\mathcal{B}(X) = \mathbf{\Delta}^1_1(X)$.*

Proof. Take a Borel set B separating A and A^c and note that B has to be equal to A. ☐

Corollary 12.8. *Let X, Y be Polish and $f : X \to Y$. The following are equivalent:*

(1) *f is Borel;*
(2) *The graph of f is Borel;*
(3) *The graph of f is analytic.*

Proof. (1)⇒(2): Fix a countable basis $\{V_n\}_{n\in\mathbb{N}}$ for Y and note that for $(x,y) \in X \times Y$, we have

$$f(x) = y \iff \forall n(y \in V_n \to x \in f^{-1}(V_n)),$$

which makes it clear that Graph(f) is Borel, but for extra clarity, we write it set-theoretically one last time:

$$\mathrm{Graph}(f) = \bigcap_n \Big(\mathrm{proj}_2^{-1}(V_n^c) \cup \mathrm{proj}_1^{-1}(f^{-1}(V_n))\Big).$$

(3)⇒(1): Assume (3) and let $U \subseteq Y$ be open; we need to show that $f^{-1}(U)$ is Borel. But for $x \in X$, we have

$$\begin{aligned} x \in f^{-1}(U) &\iff \exists y \in Y(f(x) = y \text{ and } y \in U) \\ &\iff \forall y \in Y(f(x) = y \to y \in U), \end{aligned}$$

so $f^{-1}(U)$ is both analytic and co-analytic, and hence is Borel by Souslin's theorem. ☐

Corollary 12.9. *Let X be Polish and let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint family of analytic subsets of X. Then there is a disjoint family $\{B_n\}_{n\in\mathbb{N}}$ of Borel sets with $B_n \supseteq A_n$.*

12.D. **Souslin operation $\mathcal{A}$.** In this subsection, we will define an important operation on schemes of sets and we will give yet another characterization of analytic sets in terms of this operation.

For a set X and a pruned tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we refer to a sequence of subsets $(P_s)_{s\in T}$ of X as a *Souslin scheme on X*. Call this scheme *monotone* if

(i) $P_t \subseteq P_s$ for all $t \supseteq s$, $s, t \in T$,

and call it *proper*[17] if it is monotone and also

(ii) $P_{s^\frown i} \cap P_{s^\frown j} = \emptyset$, for all $s \in T$, $i \neq j$, $i, j \in T(s)$.

Definition 12.10. We define the *Souslin operation $\mathcal{A}$ applied to an arbitrary Souslin scheme* $(P_s)_{s\in T}$ as follows:

$$\mathcal{A}(P_s)_{s\in T} := \bigcup_{y\in[T]} \bigcap_{n\in\mathbb{N}} P_{y|_n}.$$

Note that by taking some of the sets to be empty, we can always assume that $T = \mathbb{N}^{<\mathbb{N}}$. Also note that we don't require $(P_s)_{s\in T}$ to be proper (not even monotone). In fact, the following lemma shows that when it is proper, this operation trivializes in the sense that the uncountable union is replaced by a countable union:

[17] This generalizes our earlier definition of a Luzin scheme.

Lemma 12.11. *If $(P_s)_{s\in T}$ is a proper Souslin scheme, then $\mathcal{A}(P_s)_{s\in T} = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in T, |s|=n} P_s$.*

Proof. The inclusion $\subseteq$ follows easily by taking $s = y|_n$ for each n. For $\supseteq$, take

$$x \in \bigcap_{n\in\mathbb{N}} \bigcup_{s\in T, |s|=n} P_s,$$

so for each n, there is $s_n \in T$ of length n such that $x \in P_{s_n}$. The sequence $(s_n)_n$ must be coherent (i.e. increasing) because otherwise, if $n < m$ and $s_n \not\subseteq s_m$, then by monotonicity, $x \in P_t$ with $t = s_m|_n$, so $x \in P_t \cap P_{s_n} \neq \emptyset$, contradicting properness. □

For a class $\mathbf{\Gamma}$ of subsets in topological spaces, let $\mathcal{A}\mathbf{\Gamma}$ denote the class of all sets of the form $\mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, where each $P_s \in \mathbf{\Gamma}(X)$, for a fixed topological space X.

The following observation shows that the operation $\mathcal{A}$ can be implemented via projection.

Observation 12.12. *For a Souslin scheme $(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ on a set X, $\mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}} = \mathrm{proj}_1(P)$, where $P \subseteq X \times \mathcal{N}$ is defined as follows: for $(x,y) \in X \times \mathcal{N}$,*

$$\begin{aligned}(x,y) \in P :&\iff \forall n \in \mathbb{N}\ \exists s \in \mathbb{N}^{<\mathbb{N}}\ y|_n = s \wedge x \in P_s \\ &\iff \forall n \in \mathbb{N}\ \forall s \in \mathbb{N}^{<\mathbb{N}}\ y|_n \neq s \vee x \in P_s.\end{aligned}$$

In particular, if $\mathbf{\Gamma}$ is a class of subsets in topological spaces that contains clopen sets and is closed under finite unions, countable intersections, and continuous preimages, then $\mathcal{A}\mathbf{\Gamma} \subseteq \exists^{\mathcal{N}}\mathbf{\Gamma}$.

The next lemma shows that the converse also holds for $\mathbf{\Gamma} = \mathbf{\Sigma}^1_1$.

Lemma 12.13. *Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a pruned tree, X a Polish space and $f : [T] \to X$ continuous. Then $P_s := f([T_s])$ is analytic for each $s \in T$, and the Souslin scheme $(P_s)_{s\in T}$ satisfies the following:*

(i) *for each $s \in T$, $P_s \neq \emptyset$;*
(ii) *for each $s \in T$, $P_s = \bigcup_{i\in T(s)} P_{s^\frown i}$ (in particular, it is monotone);*
(iii) *for each $y \in [T]$ and $U \subseteq X$ open, if $P_y := \bigcap_n P_{y|_n} \subseteq U$, then $P_{y|_n} \subseteq U$ for some $n \in \mathbb{N}$;*
(iv) *it is of vanishing diameter, i.e. for all $y \in [T]$, $\mathrm{diam}(P_{y|_n}) \to 0$ as $n \to \infty$.*

Moreover, $f([T]) = \mathcal{A}(P_s)_{s\in T} = \mathcal{A}(\overline{P_s})_{s\in T}$.

Proof. The properties (i)-(ii) and the equality $f([T]) = \mathcal{A}(P_s)_{s\in T}$ are immediate from the definition of $(P_s)_{s\in T}$, and (iii)-(iv) follow from the continuity of f. The equality $\mathcal{A}(P_s)_{s\in T} = \mathcal{A}(\overline{P_s})_{s\in T}$ also follows from the continuity of f as follows: it is enough to show that for fixed $y \in [T]$, $\bigcap_n \overline{P_{y|_n}} \subseteq \bigcap_n P_{y|_n}$, so take $x \in \bigcap_n \overline{P_{y|_n}}$, and we claim that $f(y) = x$. Otherwise, there are disjoint open sets $U, V \subseteq X$ such that $U \ni f(y)$ and $V \ni x$, so $x \notin \overline{U}$. But by (iii), there is $n \in \mathbb{N}$ such that $P_{y|_n} \subseteq U$, so $x \notin \overline{P_{y|_n}}$, a contradiction. □

Proposition 12.14 (Characterization of analytic via operation $\mathcal{A}$)**.** *Let X be Polish and $\emptyset \neq A \subseteq X$. The following are equivalent:*

(1) *A is analytic.*
(2) *$A = \mathcal{A}(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, where each F_s is nonempty closed, and the Souslin scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ is monotone and of vanishing diameter (for any compatible metric on X).*
(3) *$A = \mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, where each P_s is analytic.*

In particular, $\mathbf{\Sigma}^1_1 = \mathcal{A}\mathbf{\Sigma}^1_1 = \mathcal{A}\mathbf{\Pi}^0_1$.

Proof. (1)⇒(2): Suppose A is analytic, so, by Proposition 12.2, $A = f(\mathcal{N})$ for some continuous function $f : \mathcal{N} \to X$. Putting $P_s = f(N_s)$, (2) follows from 12.13 by taking $F_s = \overline{P_s}$.
(2)⇒(3): Trivial.
(3)⇒(1): Follows immediately from Observation 12.12. □

13. More on Borel sets

13.A. **Closure under small-to-one images.** In general, images of Borel sets under Borel functions are analytic and may not be Borel. However, the situation may be different if the preimage of every point is "small". In this subsection, we will state and prove some results with various notions of "small", starting from the cases where the domain space itself if "small".

Below we use the terms *σ-compact* or K_σ for subsets of topological spaces that are countable unions of compact sets.

Proposition 13.1.

(a) *Continuous functions map compact sets to compact sets.*
(b) *Continuous functions map K_σ sets to K_σ sets.*
(c) **Tube lemma.** *For topological spaces X, Y with Y compact,* proj_1 *maps closed subsets of $X \times Y$ to closed subsets of X.*

Proof. (a) is just by unraveling the definitions and it immediately implies (b). For (c), let $F \subseteq X \times Y$ be closed, $x \notin \mathrm{proj}_1(F)$ and consider the open cover $(V_y)_{y \in Y}$ of Y where $V_y \ni y$ is open and is such that for some nonempty open neighborhood $U_y \subseteq X$ of x, $U_y \times V_y$ is disjoint from F. □

Examples 13.2.

(a) $\mathrm{proj}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ does not, in general, map closed sets to closed sets: e.g., take F to be the graph of $1/x$ with domain $(0, 1]$; then F is closed, but its projection is $(0, 1]$.

(b) However, because $\mathbb{R}$ is σ-compact (hence $K_\sigma = F_\sigma$) and Hausdorff (hence compact sets are closed), it follows from (b) of Proposition 13.1 that $\mathrm{proj}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ maps F_σ sets (in particular, closed sets) to F_σ sets.

The following is one of the most used results in descriptive set theory.

Theorem 13.3 (Luzin–Souslin). *Let X, Y be Polish spaces and $f : X \to Y$ be Borel. If $A \subseteq X$ is Borel and $f|_A$ is injective, then $f(A)$ is Borel.*

Proof (Ruiyuan Chen).[18] Turning A into a clopen set, we may replace X by A, so it is enough to show that $f(X)$ is Borel. Let $\mathcal{U}$ be a countable basis for the topology $\mathcal{T}_X$ of X. For each $U \in \mathcal{U}$, since f is injective, the sets $f(U)$ and $f(X \setminus U)$ are disjoint, so by the Luzin separation theorem 12.6, there is a Borel set $B_U \subseteq Y$ with $U \subseteq B_U \subseteq Y \setminus f(X \setminus U)$. Let $\mathcal{T}_Y'$ be a finer Polish topology $\mathcal{T}_Y$ on Y making each B_U clopen but still having the same Borel sets as Y. Note that $\mathcal{T}_X' := \mathcal{T}_X \cup \left\{ f^{-1}(V) : V \in \mathcal{T}_Y' \right\}$ is a topology on X and $f : (X, \mathcal{T}_X') \to (Y, \mathcal{T}_Y')$ is an embedding. Moreover, by Lemma 11.19, $(X, \mathcal{T}_X')$ is homeomorphic to the fiber product $\left((X, \mathcal{T}_X), f\right) \times_Y \left((Y, \mathcal{T}_Y'), \mathrm{id}_Y\right)$, so it is Polish, and hence $f(X)$ is G_δ in $(Y, \mathcal{T}_Y')$. □

[18] Thanks to Ruiyuan (Ronnie) Chen for permission to include his proof here.

Proof (classical). By Corollary 11.21 (or by replacing X with $X \times Y$, A with $\mathrm{Graph}(f|_A)$, and f with proj_2), we may assume that f is continuous. Moreover, by Corollary 11.22, we may assume that $X = \mathcal{N}$ and $A \subseteq \mathcal{N}$ is closed. Thus, $A = [T]$ for some pruned tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$. For each $s \in T$, put $P_s = f([T_s])$ and hence each P_s is analytic and, by Lemma 12.13, we have

$$f([T]) = \mathcal{A}(P_s)_{s\in T} = \mathcal{A}(\overline{P_s})_{s\in T}.$$

Note that by injectivity of $f|_{[T]}$, the scheme $(P_s)_{s\in T}$ is proper, so

$$f([T]) = \mathcal{A}(P_s)_{s\in T} = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in T, |s|=n} P_s,$$

but this still doesn't imply that $f([T])$ is Borel since each P_s may not be Borel. On the other hand, with $(\overline{P_s})_{s\in T}$ it is the opposite: each $\overline{P_s}$ is Borel, but the scheme (although monotone) may not be proper. We fix this by approximating P_s from outside by a Borel set B_s while staying within $\overline{P_s}$. This is done using iterative applications of the analytic separation theorem as follows: for each n, recursively apply Corollary 12.9 to the collection $\{P_s : s \in T, |s| = n\}$ to get pairwise disjoint sequence $(B_s)_{s\in T}$ of Borel sets with $P_s \subseteq B_s$. By taking intersections, we may assume that $B_s \subseteq \overline{P_s}$, as well as $B_s \subseteq B_t$ for every $t \subseteq s$. Thus, $(B_s)_{s\in T}$ is a proper Souslin scheme, as desired.

Because $P_s \subseteq B_s \subseteq \overline{P_s}$, we have

$$\mathcal{A}(P_s)_{s\in T} \subseteq \mathcal{A}(B_s)_{s\in T} \subseteq \mathcal{A}(\overline{P_s})_{s\in T},$$

so all these inclusions are actually equalities. Finally, because $(B_s)_{s\in T}$ is proper, we have

$$f([T]) = \mathcal{A}(P_s)_{s\in T} = \mathcal{A}(B_s)_{s\in T} = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in T, |s|=n} B_s,$$

and hence $f([T])$ is Borel. □

Corollary 13.4. *Let X, Y be Polish and $f : X \to Y$ be Borel. If f is injective, then it is a Borel embedding, i.e. f maps Borel sets to Borel sets.*

The Luzin–Souslin theorem together with Corollary 11.22 gives the following characterization of Borel sets:

Corollary 13.5. *A subset B of a Polish space X is Borel iff it is an injective continuous image of a closed subset of $\mathcal{N}$.*

This shows the contrast between Borel and analytic as the latter sets are just continuous images of closed subsets of $\mathcal{N}$.

Now, how big can the "small" be so that the Borel sets are still closed under "small"-to-one images? It turns out that for small being σ-compact, this is still true and this is a deep theorem of Arsenin and Kunugui [Kec95, 18.18]. Here we will only state a very important special case of this, which will be enough for our purposes.

For topological spaces X, Y, call a set $A \subseteq X \times Y$ a *function graph* if $A = \mathrm{Graph}(\gamma)$ for some partial function $\gamma : X \rightharpoonup Y$; equivalently, for every $x \in X$, the fiber $A_x := \{y \in Y : (x, y) \in A\}$ has at most one element. Note that if X, Y are Polish and the function graph $\mathrm{Graph}(\gamma)$ is Borel, then $\mathrm{dom}(\gamma)$ is Borel by the Luzin–Souslin theorem and the function $\gamma : \mathrm{dom}(\gamma) \to Y$ is Borel by Corollary 12.8.

Theorem 13.6 (Luzin–Novikov). *Let X, Y be Polish spaces and $R \subseteq X \times Y$ be a Borel set all of whose X-fibers are countable, i.e. for every $x \in X$, R_x is countable. Then R can be partitioned into countably many disjoint Borel function graphs $R = \bigsqcup_n \mathrm{Graph}(\gamma_n)$.*

In Subsection 22.B, we will deduce the last theorem from another big theorem about graph colorings.

Corollary 13.7. *The class of Borel subsets of Polish spaces is closed under countable-to-one Borel images.*

Proof. Let Z, X be Polish spaces, $f : Z \to X$ be a countable-to-one Borel function, $B \subseteq Z$ a Borel set, and we show that $f(B)$ is Borel. By replacing Z with $X \times Z$ and B with $\mathrm{Graph}^{-}(f|_B) := \{(x,z) : z \in B \text{ and } f(z) = x\}$, we may assume that $Z = X \times Y$, for some Polish space Y, $B \subseteq X \times Y$ is a Borel set with countable X-fibers, and $f = \mathrm{proj}_1$. By the Luzin–Novikov theorem, $B = \bigsqcup_n B_n$, where each B_n is a Borel function graph. For each n, $f(B_n)$ is Borel by the Luzin–Souslin theorem, and thus, so is $f(B) = \bigcup_n f(B_n)$. □

The next corollary says, in particular, that given a Borel set $B \subseteq X \times Y$ with countable X-fibers, for each $x \in \mathrm{proj}_1(B)$, we can choose *in a Borel way* ("uniformly") a witness $y \in Y$ with $(x,y) \in B$.

Corollary 13.8 (Countable uniformization). *For Polish spaces Z, X, any countable-to-one Borel function $f : Z \to X$ admits a Borel right inverse $g : f(Z) \to Z$.*

Proof. Just like in the proof of Corollary 13.7, we may assume that $Z = X \times Y$ and $f = \mathrm{proj}_1$. By the Luzin–Novikov theorem, $B = \bigsqcup_n B_n$, where each B_n is a Borel graph. Define $k : X \to \mathbb{N}$ by $x \mapsto$ the least $n \in \mathbb{N}$ with $x \in \mathrm{proj}_1(B_n)$, and finally define $g : X \to X \times Y$ by $x \mapsto (x,y)$, where $y \in Y$ is the unique element with $(x,y) \in B_{k(x)}$. It is straightforward to check that the function k, and hence also g, is Borel. □

13.B. **The Borel isomorphism theorem.** Using that the Borel sets are closed under one-to-one Borel images, we show in this subsection that any two uncountable Polish spaces are Borel isomorphic.

Corollary 13.9 (Borel Schröder–Bernstein). *Let X, Y be Polish and $f : X \hookrightarrow Y$, $g : Y \hookrightarrow X$ be Borel injections. Then X and Y are Borel isomorphic.*

Proof. Run the same proof as for the regular Schröder–Bernstein theorem and note that all the sets involved are images of Borel sets under f or g, and hence are themselves Borel. Thus, the resulting bijection is a Borel isomorphism. □

The following theorem shows how robust the framework of Polish spaces is when studying Borel sets and beyond.

Theorem 13.10 (Borel Isomorphism). *Any two Polish spaces of the same cardinality are Borel isomorphic. In particular, any two uncountable Polish spaces are Borel isomorphic.*

Proof. The statement for countable Polish spaces is obvious since their Borel σ-algebra is all of their powerset. For uncountable Polish space, it is enough to show that if X is uncountable, then it is Borel isomorphic to $\mathcal{C}$. By the Borel Schröder–Bernstein, it is enough to show that there are Borel injections $X \hookrightarrow \mathcal{C}$ and $\mathcal{C} \hookrightarrow X$. The first is by Proposition 1.9 and the second is by the Cantor–Bendixson theorem and the perfect set theorem. □

13.C. **Standard Borel spaces.** As the Borel Isomorphism Theorem shows, it really does not matter which Polish space to consider when working in the Borel context. The following definition makes abstracting from the topology but keeping the Borel structure precise.

Definition 13.11. A measurable space $(X, \mathcal{S})$ is called a *standard Borel space* if there is a Polish topology $\mathcal{T}$ on X such that $\mathcal{B}(\mathcal{T}) = \mathcal{S}$. In this case, we call $\mathcal{T}$ a *compatible Polish topology* and refer to the sets in $\mathcal{S}$ as *Borel sets*. Similarly, we call a subset $A \subseteq X$ *analytic* (resp. *co-analytic*) if for some (equivalently any) compatible Polish topology $\mathcal{T}$, A is analytic (resp. co-analytic) as a subset of $(X, \mathcal{T})$.

In the definition above, the notion of an analytic set is well-defined, i.e. it does not depend on which compatible Polish topology one picks; indeed, if $\mathcal{T}, \mathcal{T}'$ are compatible Polish topologies on X, then the identity map from $(X, \mathcal{T})$ to $(X, \mathcal{T}')$ is a Borel isomorphism, and hence A is analytic in $(X, \mathcal{T})$ iff it is analytic in $(X, \mathcal{T}')$.

Examples 13.12.

(a) An obvious example of a standard Borel space is a Polish space with its Borel σ-algebra: $(X, \mathcal{B}(X))$.

(b) A less immediate example is a Borel subset A of a Polish space X with the relative Borel σ-algebra: $(A, \mathcal{B}(X)|_A)$, where $\mathcal{B}(X)|_A = \{B \cap A : B \in \mathcal{B}(X)\} = \{B \in \mathcal{B}(X) : B \subseteq A\}$. This is because there is a Polish topology on X making A clopen and hence Polish in the relative topology.

13.D. **The Effros Borel space.** We now consider an interesting and important example of a standard Borel space. For a topological space X, let $\mathcal{F}(X)$ denote the collection of the closed subsets of X. We endow $\mathcal{F}(X)$ with the σ-algebra $\mathcal{E}$ generated by the sets of the form

$$[U]_{\mathcal{F}(X)} := \{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\},$$

for U open in X. If X has a countable basis $\{U_n\}_{n \in \mathbb{N}}$, it is of course enough to consider U in that basis. The measurable space $(\mathcal{F}(X), \mathcal{E})$ is called the *Effros Borel space of X*.

Theorem 13.13. *For any Polish space X, the Effros Borel space of X is standard.*

Proof. Let $(U_n)_{n \in \mathbb{N}}$ be a countable basis for X and consider the map $c : \mathcal{F}(X) \to \mathcal{C}$ by $F \mapsto$ the characteristic function of $\{n \in \mathbb{N} : F \cap U_n \neq \emptyset\}$. It is clear that c is measurable since the preimage of a pre-basic open set $\{x \in \mathcal{C} : x(n) = i\}$, for $i \in \{0, 1\}$, is $[U_n]_{\mathcal{F}(X)}$ or its complement, depending on whether $i = 1$ or 0. It is also clear that c is injective and, letting $Y := c(\mathcal{F}(X))$,

$$c\Big([U_n]_{\mathcal{F}(X)}\Big) = \{x \in \mathcal{C} : x(n) = 1\} \cap Y,$$

so $c^{-1} : Y \to \mathcal{F}(X)$ is also measurable. This makes c an isomorphism between measurable spaces $(\mathcal{F}(X), \mathcal{E})$ and $(Y, \mathcal{B}(\mathcal{C})|_Y)$. Hence, if Y is Borel, then these measurable spaces are standard Borel.

We in fact show that Y is a G_δ subset of $\mathcal{C}$. Indeed, fix a complete compatible metric on X. Then one can verify (left as a homework exercise) that for $x \in \mathcal{C}$,

$$
\begin{aligned}
x \in Y \iff & \forall U_n \subseteq U_m [x(n) = 1 \to x(m) = 1] \\
& \text{and} \\
& \forall U_n \forall \varepsilon \in \mathbb{Q}^+ \Big[x(n) = 1 \to \exists \overline{U_m} \subseteq U_n \text{ with } \operatorname{diam}(U_m) < \varepsilon \text{ such that } x(m) = 1) \Big].
\end{aligned}
$$

Thus, Y is clearly G_δ. □

As the following example shows, the fact that the Effros space is standard Borel allows considering spaces of seemingly third order objects, such as Polish spaces themselves, in the context of Polish spaces.

Example 13.14. Theorem 3.7 states that we can think of $\mathcal{F}(\mathbb{R}^{\mathbb{N}})$ as the space of all Polish spaces, and by Theorem 13.13, it is a standard Borel space. This allows us, for example, to talk about the homeomorphism of Polish spaces as an equivalence relation on $\mathcal{F}(\mathbb{R}^{\mathbb{N}})$.

Lastly, we will discuss the possibility of choosing a "canonical" point from every nonempty closed subset of a Polish space.

Definition 13.15. Let X be a Polish space. A function $s : \mathcal{F}(X) \to X$ is called a *selector* if $s(F) \in F$ for every nonempty $F \in \mathcal{F}(X)$.

Intuitively, one can recognize when such a selector is canonical; for example, choosing the leftmost branch of T_C for a given nonempty closed subset $C \subseteq \mathcal{N}$. Another example is for a nonempty closed subset $C \subseteq \mathbb{R}$, let $M \in \mathbb{N}$ be the least such that $C \cap [-M, M] \neq \emptyset$ and choose the point $c = \min(C \cap [-M, M])$ from C. For a general Polish space X, the Effros structure on $\mathcal{F}(X)$ makes the notion of canonical precise: simply require the selector function to be Borel! The following shows that such a function always exists.

Theorem 13.16. *For every Polish space X, the Effros Borel space $\mathcal{F}(X)$ admits a Borel selector.*

Proof. Outlined in a homework exercise. □

13.E. **Borel determinacy.** We have already proven that Borel sets have the PSP and they are also Baire measurable because the latter sets form a σ-algebra containing all open sets. Borel sets are also obviously measurable under any Borel measure. In this subsection, we discuss the determinacy of Borel sets, and we start with open/closed sets. For the rest of the section let A be a discrete set and let $T \subseteq A^{<\mathbb{N}}$ be a pruned tree on which the games will be played (so T is a game with rules).

Theorem 13.17 (Gale–Stewart)**.** *Any open or closed subset $D \subseteq [T]$ is determined, i.e. the game $G(T, D)$ is determined.*

Proof. Suppose D is open or closed, so the payoff set O for one of the players is open and for the other one the payoff set C is closed (of course, $\{O, C\} = \{D, D^c\}$). We refer the former as Player **O** and to the latter as Player **C**. Call a position $p \in T$ *winning for Player* **O** if he has a winning strategy starting from p. Clearly we have the following:

Claim. Let $p \in T$ be not winning for Player **O**. If it is Player **C**'s turn to play, then there is a legal move $a \in A$ that Player **C** can make so that the position $p^\frown a \in T$ is still not winning for Player **O**. If it is Player **O**'s turn to play, then no matter what legal $a \in A$ he plays, the new position $p^\frown a \in T$ will still be not winning for Player **O**.

Now suppose Player **O** does not have a winning strategy; in other words, $\emptyset \in T$ is not winning for Player **O**. We inductively construct a winning strategy for Player **C** as follows: assuming that the game is at position $p \in T$ that is not winning for Player **O** and it is Player **C**'s turn to play, then Player **C** chooses an extension of p that is still not winning for Player **O**. If Player **C** plays according to this strategy, then the run of the game $x \in [T]$ is such that for every $n \in \mathbb{N}$, $x|_n$ is not winning for Player **O**. Thus, x must be in the closed payoff set C since otherwise, if $x \in O$, then there would be $n \in \mathbb{N}$ with $[T_{x|_n}] \subseteq O$ and hence $x|_n$ would be winning for Player **O**, a contradiction. $\square$

Although this theorem only proves determinacy for open/closed sets, we will use it in proving regularity properties of analytic sets. We can do this mainly because analytic sets are projections of closed subsets of $X \times \mathcal{N}$, so we will somehow construct equivalent games on these closed subsets and use their determinacy to conclude determinacy for the original games for analytic sets.

The following is one of the most important and grandiose results in descriptive set theory:

Theorem 13.18 (Borel Determinacy, Martin 1975)**.** *For any set A (possibly uncountable) and any tree $T \subseteq A^{<\mathbb{N}}$, all Borel sets $B \subseteq [T]$ are determined.*

We won't give the proof of this theorem here, but we will describe its general flow. By definition, every Borel set in $[T]$ can be (transfinitely) "unraveled" up to clopen sets. Similarly, the main idea of the proof is to "unravel" every Borel game to a clopen game in such a way that the determinacy of the latter (which we know holds) implies that of the former. The following makes this precise:

Definition 13.19. Let $T \subseteq A^{<\mathbb{N}}$ be a tree. A *covering* of T is a triple $(\tilde{T}, \pi, \varphi)$ where

(i) $\tilde{T}$ is a pruned tree on some set $\tilde{A}$.

(ii) $\pi : \tilde{T} \to T$ is a length-preserving (i.e. $|\pi(s)| = |s|$ for $s \in \tilde{T}$) monotone map (think of it as a projection). Thus, π gives rise to a continuous function $\pi^* : [\tilde{T}] \to [T]$. We will abuse the notation and still write π for π^*.

(iii) φ maps strategies for Player I (resp. II) in $\tilde{T}$ (i.e. certain pruned subtrees) to strategies for Player I (resp. II) in T in such a way that for a strategy $\tilde{\sigma}$ in $\tilde{T}$, $\varphi(\tilde{\sigma})$ restricted to positions of length $\leqslant n$ depends only on $\tilde{\sigma}$ restricted to positions of length $\leqslant n$.

(iv) If $\tilde{\sigma}$ is a strategy in $\tilde{T}$, then $\varphi([\tilde{\sigma}]) \supseteq [\varphi(\tilde{\sigma})]$. In other words, if a run $x \in [T]$ is played according to $\varphi(\tilde{\sigma})$, then there is a run $\tilde{x} \in [\tilde{T}]$ played according to $\tilde{\sigma}$ such that $\pi(\tilde{x}) = x$.

It should be clear from the definition that for $D \subseteq [T]$, the game $G(T, D)$ can be simulated by the game $G(\tilde{T}, \tilde{D})$, where $\tilde{D} = \pi^{-1}(D)$. More precisely, if $\tilde{\sigma}$ is a winning strategy for Player I (resp. II) in $G(\tilde{T}, \tilde{D})$, then $\varphi(\tilde{\sigma})$ a winning strategy for the same player in $G(T, D)$.

Definition 13.20. We say that a covering $(\tilde{T}, \pi, \varphi)$ of T *unravels* $D \subseteq [T]$ if $\pi^{-1}(D) \subseteq [\tilde{T}]$ is clopen.

Thus, if $(\tilde{T}, \pi, \varphi)$ unravels $D \subseteq [T]$, then, by the Gale–Stewart theorem, $G(\tilde{T}, \pi^{-1}(D))$ is determined, and hence so is $G(T, D)$. So to prove Borel determinacy, it is enough to prove that for every Borel set $D \subseteq [T]$ there is a covering $(\tilde{T}, \pi, \varphi)$ of T that unravels D. The proof is done by transfinite induction on the construction of Borel sets[19]. The hardest part (the heart of the proof) is the base case, i.e. showing that open sets can be unraveled. As for the inductive step, first note that if $D \subseteq [T]$ can be unraveled, then the same covering also unravels $[T] \setminus D$. So we only need to show that if A_n are unraveled by $(\tilde{T}_n, \pi_n, \varphi_n)$, then $A = \bigcup_n A_n$ can be unraveled as well. By taking an inverse limit (in some appropriate sense) of the coverings[20] $(\tilde{T}_n, \pi_n, \varphi_n)$, we get a covering that unravels all A_n simultaneously, so the preimage of A is open and hence it can be unraveled further using the base of the induction.

14. Regularity properties of analytic sets

In this section we prove that analytic sets enjoy the PSP and the Baire measurability. The determinacy of analytic sets is already independent of ZFC (under some large cardinal hypothesis[21]).

We have already considered games that are associated to these properties, namely the $*$-game $G^*(D)$ and the $**$-game $G^{**}(D)$, for $D \subseteq X$, where X is a perfect Polish space. Note that either of these games is played on a certain countable pruned tree T of legal positions. In the case of the $*$-game, the moves of the players are from the set $A = \{0,1\} \cup \mathcal{W}^2$, where $\mathcal{W}$ is a weak basis for X, and in the case of the $**$-game $A = \mathcal{W}$. Let $g : [T] \to X$ be the function that associates an element $x \in X$ with a given run $a \in [T]$ in either of these games: for the $*$-game, $\{x\} = \bigcap_n U^{(n)}_{i_n}$, and for the $**$-game, $\{x\} = \bigcap_n \overline{U_n}$. In either game, it is clear that this function is continuous. In particular, if $D \subseteq X$ is closed (resp. Borel, analytic, etc), then so is $g^{-1}(D) \subseteq [T]$. Thus $\boldsymbol{\Sigma}^1_1$-determinacy automatically implies that analytic sets have the PSP and are Baire measurable.

However, as mentioned above, $\boldsymbol{\Sigma}^1_1$-determinacy is independent from ZFC, so we can't use it to prove that all analytic sets have the PSP and are Baire measurable (in ZFC). Instead, based on the fact that analytic subsets of X are projections of closed subsets of $X \times \mathcal{N}$, we will use the so-called *unfolding* technique to reduce the determinacy of the $*$- and the $**$-games for analytic sets to that for closed sets, whose determinacy we already know (the Gale–Stewart theorem).

14.A. The perfect set property.

We start with defining the unfolded $*$-game. Suppose X is a perfect Polish space and let $F \subseteq X \times \mathcal{N}$. The *unfolded $*$-game $G^*_u(F)$ for F* is the following:

$$\begin{array}{lccccc}
\text{I} & (U^{(0)}_0, U^{(0)}_1), y_0 & & (U^{(1)}_0, U^{(1)}_1), y_1 & & \\
 & & & & & \cdots \\
\text{II} & & i_0 & & i_1 &
\end{array}$$

where Players I and II play as they do in the $*$-game, but additionally Player I plays $y_n \in \mathbb{N}$ is his n^{th} move. If $\{x\} = \bigcap_n U^{(n)}_{i_n}$ and $y = (y_n)_{n \in \mathbb{N}}$, then Player I wins iff $(x, y) \in F$.

Theorem 14.1. *Let X be a perfect Polish space, $F \subseteq X \times \mathcal{N}$, and $A = \mathrm{proj}_1(F)$. Then*

(a) *Player I has a winning strategy in $G^*_u(F) \Rightarrow A$ contains a Cantor set.*

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[19] For the induction to go through, one actually has to construct so-called *k-coverings* instead of coverings, but we will keep this technicality out of our exposition.

[20] One has to actually take a coherent sequence of coverings $(\tilde{T}_n, \pi_n, \varphi_n)$, which we assume exists by induction.

[21] It is actually equivalent to the existence of sharps.

(b) *Player II has a winning strategy in $G^*_u(F) \Rightarrow A$ is countable.*

Proof. (a) If Player I has a winning strategy in $G^*_u(F)$, then ignoring the $y(n)$-s, we get a winning strategy for Player I in the original game $G^*(A)$, so A contains a Cantor set.
(b) Assume Player II has a winning strategy σ in $G^*_u(F)$. For each $x \in A$, we choose a witness $y \in \mathcal{N}$, so that $(x,y) \in F$. Call a position

$$p = (((U_0^{(0)}, U_1^{(0)}), y_0,), i_0, \dots, ((U_0^{(n)}, U_1^{(n)}), y_n), i_n)$$

good for (x,y) if it is played according to σ (i.e. $p \in \sigma$), $y \supseteq (y_i)_{i \leqslant n}$ and $x \in U_{i_n}$. For a position p as above, and for $b \in \mathbb{N}$, let $A_{p,b}$ denote the set of $x \in X$ such that p doesn't have a good extension for (x,y), for any $y \in \mathcal{N}$ that extends $(y_i)_{i \leqslant n} {}^\frown b$, i.e.

$$A_{p,b} := \{x \in X : \forall \text{ legal moves } ((U_0^{(n+1)}, U_1^{(n+1)}), b), \text{ if } i_{n+1} \text{ is played by II according to } \sigma \text{ then } x \notin U_{i_{n+1}}^{(n+1)}\}.$$

As before, for every $(x,y) \in F$, there must exist a maximal good position $p \in \sigma$ since otherwise $(x,y) \in [\sigma]$, contradicting σ being a winning strategy for Player II. Thus, for such p and $b = y(n+1)$, $x \in A_{p,b}$, and hence

$$A \subseteq \bigcup_{p \in \sigma, b \in \mathbb{N}} A_{p,b}.$$

To conclude that A is countable, it remains to note that each $A_{p,b}$ can have at most one element; indeed, if $x, z \in A_{p,b}$ and $x \neq z$, then by Hausdorffness, there is a legal move $((U_0^{(n+1)}, U_1^{(n+1)}), b)$ of Player I such that $x \in U_0^{(n+1)}$ and $z \in U_1^{(n+1)}$, so regardless of what Player II's response i_{n+1} is, $U_{i_{n+1}}^{(n+1)}$ contains either x or z, contradicting $x, z \in A_{p,b}$. □

Corollary 14.2. *Analytic subsets of Polish spaces have the* PSP.

Proof. Let X be a Polish space and let $A \subseteq X$ be analytic. By taking $X =$ its perfect kernel, we may assume that X is perfect. Now let $F \subseteq X \times \mathcal{N}$ be a closed set such that $A = \text{proj}_1(F)$. If $g : [T] \to X \times \mathcal{N}$ is the function associating an element $(x,y) \in X \times \mathcal{N}$ to every run of $G^*_u(F)$, then $g^{-1}(F)$ is closed and hence the game $G^*_u(F)$ is determined (by the Gale–Stewart theorem). Thus, by the previous theorem, A is either countable or contains a Cantor set. □

We now give a completely different (probabilistic in spirit) proof of the PSP for analytic sets using the notion of Baire category in the hyperspace $\mathcal{K}(X)$ of compact sets, for a Polish space X. Here we use that analytic sets are continuous images of Polish spaces, i.e. $A \subseteq Y$ is analytic if there is Polish X and continuous $f : X \to Y$ such that $f(X) = A$.

Theorem 14.3. *Let X, Y be Polish spaces and $f : X \to Y$ be a continuous function such that $A = f(X)$ is uncountable. Then there is a Cantor set $C \subseteq X$ such that $f|_C$ is injective. In particular, $f(C)$ is a Cantor set in A, and hence any analytic set has the* PSP.

Proof. By restricting to the perfect kernel of X, we may assume without loss of generality that X is perfect. In particular, by Corollary 4.5, the set $\mathcal{K}_p(X)$ of all perfect compact sets is dense G_δ in $\mathcal{K}(X)$. Thus, to prove the theorem, it is enough to show that the set

$$\mathcal{K}_f(X) = \{K \in \mathcal{K}(X) : f|_K \text{ is injective}\}$$

is a dense G_δ subset of $\mathcal{K}(X)$ since then $\mathcal{K}_p(X) \cap \mathcal{K}_f(X) \neq \emptyset$, and hence there is a compact perfect set $K \in \mathcal{K}_f(X)$, which of course contains a Cantor set, concluding the proof.

To ensure the density of $\mathcal{K}_f(X)$, we assume without loss of generality that for every nonempty open $U \subseteq X$, $f(U)$ is uncountable. This can be achieved as follows: fix a countable basis $\{U_n\}_{n\in\mathbb{N}}$ and subtract from X all the U_n for which $f(U_n)$ is countable. By doing so, we have thrown away all open sets U for which $f(U)$ is countable. The remaining set X' is nonempty since $f(X')$ is still uncountable, and X' is perfect since for every open set $U \subseteq X'$, $f(U)$ is uncountable. Thus, we assume that $X = X'$ to start with.

Claim. $\mathcal{K}_f(X)$ is dense in $\mathcal{K}(X)$.

Proof of Claim. Let $\langle U_0; U_1, ..., U_n\rangle_{\mathcal{K}}$ be a Vietoris basic open set in $\mathcal{K}(X)$ with $U_i \subseteq U_0$ for all $i \leqslant n$. Since each $f(U_i)$ is uncountable, we can recursively define a sequence $(x_i)_{i=1}^n$ such that $x_i \in U_i$ and $f|_K$ is injective, where $K := \{x_i\}_{i=1}^n$. Thus, $K \in \mathcal{K}_f(X) \cap \langle U_0; U_1, ..., U_n\rangle_{\mathcal{K}}$. ⊠

It remains to show that $\mathcal{K}_f(X)$ is a G_δ subset of $\mathcal{K}(X)$. To this end, fix a countable basis $\mathcal{U}$ for X and note that for $K \in \mathcal{K}(X)$,

$$K \in \mathcal{K}_f(X) \iff \forall U_1, U_2 \in \mathcal{U} \text{ with } \overline{U_1} \cap \overline{U_2} = \emptyset [f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset].$$

To finish the proof, it is enough to show that for fixed $U_1, U_2 \in \mathcal{U}$ with $\overline{U_1} \cap \overline{U_2} = \emptyset$ the set

$$\mathcal{V} = \left\{K \in \mathcal{K}(X) : f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset\right\}$$

is open in $\mathcal{K}(X)$, which we leave as an exercise. □

14.B. **Baire measurability and measurability.** We now define the unfolded $**$-game. Suppose X is a Polish space and let $F \subseteq X \times \mathcal{N}$. The *unfolded $**$-game $G_u^{**}(F)$ for F* is the following:

$$\begin{array}{lccccc} \mathrm{I} & U_0, y_0 & & U_1, y_1 & & \\ & & & & & \cdots \\ \mathrm{II} & & V_0 & & V_1 & \end{array}$$

where Players I and II play as they do in the $**$-game, but additionally Player I plays $y_n \in \mathbb{N}$ is his n^{th} move. If $\{x\} = \bigcap_n \overline{U_n}$ and $y = (y_n)_{n\in\mathbb{N}}$, then Player I wins iff $(x, y) \in F$.

Theorem 14.4. *Let X be a Polish space, $F \subseteq X \times \mathcal{N}$ and $A = \mathrm{proj}_1(F)$.*

(a) *Player II has a winning strategy in $G_u^{**}(F) \Rightarrow A$ is meager.*
(b) *Player I has a winning strategy in $G_u^{**}(F) \Rightarrow A$ is comeager in some nonempty open set.*

Proof. (a) Modify the proof of (a) of Theorem 9.16 just like we modified the proof of (b) of Theorem 8.2 in the proof of (b) of Theorem 14.1.
(b) If Player I has a winning strategy in $G_u^{**}(F)$, then forgetting the $y(n)$-s gives a winning strategy in the original game $G^{**}(A)$, so A is comeager in some nonempty open set. □

Corollary 14.5 (Luzin–Sierpiński). *Analytic subsets of Polish spaces are Baire measurable.*

Proof. Let X be a Polish space and $A \subseteq X$ be analytic. By (3) of Proposition 9.14, it is enough to show that the Baire alternative holds for $A \setminus U(A)$. But $A \setminus U(A)$ is still analytic and hence there is a closed set $F \subseteq X \times \mathcal{N}$ such that $A \setminus U(A) = \mathrm{proj}_1(F)$. Then, if $g : [T] \to X \times \mathcal{N}$ is the function associating an element $(x, y) \in X \times \mathcal{N}$ to every run of $G_u^{**}(F)$, then $g^{-1}(F)$ is closed and hence the game $G_u^{**}(F)$ is determined. Thus, by the previous theorem, $A \setminus U(A)$ satisfies the Baire alternative. □

The same proof applied to the density topology on $\mathbb{R}$ shows that analytic sets are universally measurable. However, since we didn't prove the Banach–Mazur theorem for the density topology, we will give an alternative proof of measurability of analytic sets (as well as, Baire measurability) in the next subsection.

14.C. **Closure of** BMEAS **and** MEAS **under the operation** $\mathcal{A}$**.** We now isolate a property of σ-algebras (satisfied by BMEAS and MEAS_μ), which ensures closure under the operation $\mathcal{A}$. First note that both BMEAS and MEAS_μ come with corresponding σ-ideals MGR and NULL_μ. The following definition extracts this ideal for a given σ-algebra.

Definition 14.6. Let $\mathcal{S}$ be a σ-algebra on a set X. We denote by $\mathrm{Ideal}_\sigma(\mathcal{S})$ the collection of sets $A \subseteq X$ with the property that for every $B \subseteq A$, $B \in \mathcal{S}$.

It is straightforward to check that $\mathrm{Ideal}_\sigma(\mathcal{S})$ is indeed a σ-ideal, and it is immediate from the definitions that $\mathrm{MGR} \subseteq \mathrm{Ideal}_\sigma(\mathrm{BMEAS})$ and $\mathrm{NULL}_\mu \subseteq \mathrm{Ideal}_\sigma(\mathrm{MEAS}_\mu)$. Although we will not use it below, both of the latter inclusions are actually equalities because one can show using AC that every nonmeager (resp. μ-positive) set contains a set which is Baire nonmeasurable (resp. μ-nonmeasurable).

Definition 14.7. Let $(X, \mathcal{S})$ be a measurable space. We call a set $B \in \mathcal{S}$ an $\mathcal{S}$-*envelope* for a set $A \subseteq X$ if $A \subseteq B$ and any subset $C \subseteq B \setminus A$ is either in $\mathrm{Ideal}_\sigma(\mathcal{S})$ or is not in $\mathcal{S}$ (i.e. it is either "small" or "nonmeasurable"). We say that $\mathcal{S}$ (or $(X, \mathcal{S})$) *admits envelopes* if every subset $A \subseteq X$ has an $\mathcal{S}$-envelope.

It follows from this definition that if E_1, E_2 are $\mathcal{S}$-envelopes for $A \subseteq X$, then $E_1 \Delta E_2 \in \mathrm{Ideal}_\sigma(\mathcal{S})$. Moreover, for any measurable space $(X, \mathcal{S})$, because $\mathrm{Ideal}_\sigma(\mathcal{S})$ is a σ-ideal, the collection $E(X, \mathcal{S})$ of subsets of X that admit $\mathcal{S}$-envelopes is closed under countable unions.

Examples 14.8.

(a) *For any σ-finite measure space $(X, \mathcal{S}, \mu)$,* $\mathrm{MEAS}_\mu(X)$ *admits envelopes*[22]. Indeed, first note that because μ is σ-finite and $E(X, \mathcal{S})$ is a σ-algebra, we may assume that μ is a probability measure. Now let $A \subseteq X$ and put

$$\mu^*(A) = \inf\left\{\mu(B) : B \supseteq A \text{ and } B \in \mathrm{MEAS}_\mu(X)\right\}.$$

Note that this infimum is actually achieved because if $(B_n)_n$ is a sequence of μ-measurable sets with $B_n \supseteq A$ and $\lim_{n\to\infty} \mu(B_n) = \mu^*(A)$, then $B := \bigcap_n B_n$ is μ-measurable, contains A and $\mu(B) \leqslant \mu(B_n)$ for all n, so $\mu(B) = \mu^*(A)$. It easily follows now that B is a MEAS_μ-envelope for A.

(b) *For any topological space X,* $\mathrm{BMEAS}(X)$ *admits envelopes.* Indeed, one can easily construct envelopes using Corollary 9.12 (or the Banach category theorem) and we leave it as an exercise.

Theorem 14.9 (Szpilrajn–Marczewski)**.** *Let $(X, \mathcal{S})$ be a measurable space. If $\mathcal{S}$ admits envelopes then it is closed under the operation $\mathcal{A}$, i.e. if $(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ is a Souslin scheme of sets in $\mathcal{S}$, then $\mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}} \in \mathcal{S}$.*

[22] μ is only defined on $\mathcal{S}$, but as usual, we take the completion $\overline{\mu}$ and denote by $\mathrm{MEAS}_\mu(X)$ the set of $\overline{\mu}$-measurable sets.

Proof. Let $(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ be a Souslin scheme of sets in $\mathcal{S}$ and assume, as we may, that it is monotone. For each $s \in \mathbb{N}^{<\mathbb{N}}$, put $\tilde{P}_s = \mathcal{A}(P_t)_{s\subseteq t\in\mathbb{N}^{<\mathbb{N}}}$, so

$$P_s \supseteq \tilde{P}_s = \bigcup_{i\in\mathbb{N}} \tilde{P}_{s^\frown i}.$$

For each $s \in \mathbb{N}^{<\mathbb{N}}$, let E_s denote an $\mathcal{S}$-envelope for $\tilde{P}_s$, and we may assume $E_s \subseteq P_s$ by replacing E_s with $P_s \cap E_s$. Because $\tilde{P}_s = \bigcup_{i\in\mathbb{N}} \tilde{P}_{s^\frown i}$, the set $\bigcup_{i\in\mathbb{N}} E_{s^\frown i}$ is also an $\mathcal{S}$-envelope for $\tilde{P}_s$, so

$$Q_s := E_s \setminus \bigcup_{i\in\mathbb{N}} E_{s^\frown i} \in \mathrm{Ideal}_\sigma(\mathcal{S}),$$

and hence $Q := \bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} Q_s \in \mathrm{Ideal}_\sigma(\mathcal{S})$.

Claim. $E_\emptyset \setminus Q \subseteq \mathcal{A}(E_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$.

Proof of Claim. This is true for any Souslin scheme in general. Indeed, let $x \in E_\emptyset \setminus Q$ and recursively construct an $\subseteq$-increasing sequence $(s_n)_n \subseteq \mathbb{N}^{<\mathbb{N}}$ with $|s_n| = n$ such that $x \in E_{s_n}$ for all $n \in \mathbb{N}$ as follows: if $x \in E_{s_n}$ then because $x \notin Q_{s_n}$, it must be that $x \in \bigcup_{i\in\mathbb{N}} E_{s_n{}^\frown i}$, so there is $i \in \mathbb{N}$ with $x \in E_{s_n{}^\frown i}$ and we let $s_{n+1} := s_n{}^\frown i$. Putting $y = \Uparrow_n s_n$, we get that $x \in \bigcap_n E_{y|_n} \subseteq \mathcal{A}(E_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$. ⊠

Because $E_s \subseteq P_s$ for each $s \in \mathbb{N}^{<\mathbb{N}}$, we get $E_\emptyset \setminus Q \subseteq \mathcal{A}(P_s)_{s\in\mathbb{N}^{<\mathbb{N}}} = \tilde{P}_\emptyset$, or equivalently, $E_\emptyset \setminus \tilde{P}_\emptyset \subseteq Q$, which implies that $E_\emptyset \setminus \tilde{P}_\emptyset \in \mathcal{S}$ and hence $\tilde{P}_\emptyset \in \mathcal{S}$. □

Corollary 14.10. *Analytic subsets of Polish spaces are Baire measurable and universally measurable.*

Proof. This is immediate from the previous theorem recalling that for a Polish space X, $\Sigma^1_1(X) = \mathcal{A}\mathcal{B}(X)$, as well as $\mathcal{B}(X) \subseteq \mathrm{BMEAS}(X)$ and $\mathcal{B}(X) \subseteq \mathrm{MEAS}_\mu(X)$ for any Borel σ-finite measure μ on X. □

15. The projective hierarchy

We now define the hierarchy of all subsets of Polish spaces that are definable from open sets using operations $\exists^{\mathcal{N}}$, $\exists^{\mathbb{N}}$, $\neg$ and $\vee$. To indicate that we are allowing quantification over $\mathcal{N}$, the superscript in the notation below is 1.

For each $n \geqslant 1$, we define the *projective classes* $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ of subsets of Polish spaces as follows: let $\mathbf{\Sigma}^1_1$ be the class of analytic sets, and let

$$\begin{aligned}
\mathbf{\Pi}^1_n &:= \neg\mathbf{\Sigma}^1_n \\
\mathbf{\Sigma}^1_{n+1} &:= \exists^{\mathcal{N}}\mathbf{\Pi}^1_n \\
\mathbf{\Delta}^1_n &:= \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n.
\end{aligned}$$

An easy induction shows that $\mathbf{\Sigma}^1_n \subseteq \mathbf{\Sigma}^1_{n+1}$ and similarly for $\mathbf{\Pi}^1_n$. Thus we have that $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$. Put

$$\mathbf{P} = \bigcup_n \mathbf{\Sigma}^1_n = \bigcup_n \mathbf{\Pi}^1_n = \bigcup_n \mathbf{\Delta}^1_n,$$

and call the sets in $\mathbf{P}$ *projective*. Thus, we have the following picture of the *projective hierarchy*:

$$\underbrace{\begin{array}{cccccccccccccc}
 & & \mathbf{\Sigma}^1_1 & & & & \mathbf{\Sigma}^1_2 & & & & & \mathbf{\Sigma}^1_n & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\mathbf{\Delta}^1_1 & & & & \mathbf{\Delta}^1_2 & & & & \mathbf{\Delta}^1_3 \ \ldots & \mathbf{\Delta}^1_n & & & & \mathbf{\Delta}^1_{n+1} \ \ldots \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \mathbf{\Pi}^1_1 & & & & \mathbf{\Pi}^1_2 & & & & & \mathbf{\Pi}^1_n & &
\end{array}}_{\mathbf{P}}$$

Proposition 15.1.

(a) *The classes $\mathbf{\Sigma}^1_n$ are closed under Borel preimages, countable intersections and unions, and Borel images, i.e. if $A \subseteq X$ is $\mathbf{\Sigma}^1_n$ and $f : X \to Y$ is Borel (where X, Y are Polish spaces), then $f(A)$ is $\mathbf{\Sigma}^1_n$.*

(b) *The classes $\mathbf{\Pi}^1_n$ are closed under Borel preimages, countable intersections and unions, and co-projections (i.e. universal quantification over Polish spaces).*

(c) *The classes $\mathbf{\Delta}^1_n$ are closed under Borel preimages, complements and countable unions. In particular, each $\mathbf{\Delta}^1_n$ is a σ-algebra.*

Proof. Part (c) follows immediately from (a) and (b).

Borel preimages: We only show this for Σ^1_1 since it would then follow for $\mathbf{\Pi}^1_1$ by taking complements. Let $A \in \Sigma^1_1(Y)$, so $A = \mathrm{proj}_1(B)$, where $B \in \mathcal{B}(Y \times \mathcal{N})$. Let $f : X \to Y$ be a Borel function. We need to show that $\tilde{A} = f^{-1}(A)$ is Σ^1_1. We lift f to a function $\tilde{f} : X \times \mathcal{N} \to Y \times \mathcal{N}$ by $(x, z) \to (f(x), z)$. Note that the diagram commutes, i.e. $\mathrm{proj}_1 \circ \tilde{f} = f \circ \mathrm{proj}_1$. Also, $\tilde{B} = \tilde{f}^{-1}(B)$ is Borel, so $\tilde{A} = \mathrm{proj}_1(\tilde{B})$ is Σ^1_1. ‘ **Countable unions and intersections:** We will only prove this for Σ^1_n since it would then follow for $\mathbf{\Pi}^1_n$ by taking complements. For each $i \in \mathbb{N}$, let $A_i := \mathrm{proj}_1(B_i)$ with $B_i \in \mathbf{\Gamma}(X \times \mathcal{N})$, where $\mathbf{\Gamma} = \mathcal{B}$ if $n = 1$ and $\mathbf{\Gamma} = \mathbf{\Pi}^1_{n-1}$, otherwise. The closure under countable unions is easily proven by induction on n because $\bigcup_i A_i = \mathrm{proj}_1(\bigcup_i B_i)$. To prove the closure under intersections, we use a coding trick: for x to be in A_i, there has to be a witness $y_i \in \mathcal{N}$ such that $(x, y_i) \in B_i$; now we code these witnesses y_i into one witness y as follows:

$$\begin{aligned} x \in \bigcap_i A_i &\iff \forall i \in \mathbb{N}\ \exists y_i \in \mathcal{N}\ (x, y_i) \in B_i \\ &\iff \exists y \in \mathcal{N}^{\mathbb{N}}\ \forall i \in \mathbb{N}\ (x, y(i)) \in B_i. \end{aligned}$$

Thus, we are done since $\mathcal{N}^{\mathbb{N}}$ is homeomorphic to $\mathcal{N}$ by $\mathcal{N}^{\mathbb{N}} = (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \simeq_h \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \simeq_h \mathbb{N}^{\mathbb{N}} = \mathcal{N}$, where $\simeq_h$ denotes the relation of being homeomorphic.

Borel images: Again the proof is by induction on n. For $n = 1$ it follows from the definition of Σ^1_1, so suppose it is true for Σ^1_n, and let X, Y be Polish spaces, $A \in \Sigma^1_{n+1}(X)$, and $f : X \to Y$ Borel. We need to show that $f(A)$ is Σ^1_{n+1}. Take $B \in \mathbf{\Pi}^1_n(X \times \mathcal{N})$ such that $A = \mathrm{proj}_1(B)$. By

Theorem 5.9, let $g : \mathcal{N} \to X \times \mathcal{N}$ be a continuous surjection. Then we have

$$\begin{aligned} y \in f(A) &\iff \exists x \in X \left(x \in A \text{ and } f(x) = y \right) \\ &\iff \exists x \in X \, \exists z \in \mathcal{N} \left((x,z) \in B \text{ and } f(x) = y \right) \\ &\iff \exists w \in \mathcal{N} \left(g(w) \in B \text{ and } f(\mathrm{proj}_1(g(w))) = y \right). \end{aligned}$$

The latter condition defines a set in $\exists^{\mathcal{N}} \mathbf{\Pi}^1_n := \mathbf{\Sigma}^1_{n+1}$, so $f(A) \in \mathbf{\Sigma}^1_{n+1}$. $\square$

Above we proved various properties of Borel and analytic sets. Using infinite games, we showed that analytic sets enjoy the PSP and Baire measurability. Similarly, one could also show that they are universally measurable. This implies that $\mathbf{\Pi}^1_1$ sets are also Baire measurable and universally measurable. However, whether or not all $\mathbf{\Pi}^1_1$ sets satisfy the PSP is already independent from ZFC[23]. The same is true for $\mathbf{\Sigma}^1_2$ sets regarding all of the regularity properties mentioned, i.e. whether or not $\mathbf{\Sigma}^1_2$ sets have the PSP, are Baire measurable, or universally measurable (i.e. measurable with respect to any σ-finite Borel measure), is independent from ZFC.

Proposition 15.2. *For any uncountable Polish space Y, $\mathbf{\Sigma}^1_{n+1} = \exists^Y \mathbf{\Pi}^1_n$.*

Proof. Any such Y is Borel isomorphic to $\mathcal{N}$ and $\mathbf{\Pi}^1_n$ is closed under Borel preimages. $\square$

Proposition 15.3. *For any uncountable Polish space X and Polish Y, there is an X-universal set $U \subseteq X \times Y$ for $\mathbf{\Sigma}^1_n(Y)$. Same for $\mathbf{\Pi}^1_n$.*

Proof. We have already constructed this for $n = 1$, and note that if U is X-universal $U \subseteq X \times Y$ for $\mathbf{\Sigma}^1_n(Y)$, then U^c is X-universal $U \subseteq X \times Y$ for $\mathbf{\Pi}^1_n(Y)$. We prove by induction on n. Suppose that $F \subseteq X \times Y \times \mathcal{N}$ is an X-universal set for $\mathbf{\Pi}^1_n(Y \times \mathcal{N})$. Then clearly $U = \mathrm{proj}_{1,2}(F)$ is $\mathbf{\Sigma}^1_{n+1}(Y)$, and it is obvious that U parameterizes $\mathbf{\Sigma}^1_{n+1}(Y)$. $\square$

From this, using the usual trick of taking the antidiagonal, we get:

Corollary 15.4. *The projective hierarchy is strict for any uncountable Polish space X, i.e. $\mathbf{\Sigma}^1_n(X) \subsetneq \mathbf{\Delta}^1_{n+1}(X) \subsetneq \mathbf{\Sigma}^1_{n+1}(X)$. Same for $\mathbf{\Pi}^1_n$.*

16. $\mathbf{\Gamma}$-complete sets

The following definition gives a notion of relative complexity of sets in topological spaces.

Definition 16.1. Let X, Y be sets and $A \subseteq X, B \subseteq Y$. A map $f : X \to Y$ is called a *reduction of A to B* if $f^{-1}(B) = A$, in other words, $x \in A \Leftrightarrow f(x) \in B$. If X, Y are topological spaces, then we say that *A is Wadge reducible to B*, and write $A \leqslant_W B$, if there is a continuous reduction of A to B.

So if $A \leqslant_W B$ then A is simpler than B; more precisely, the question of membership in B is at least as hard to answer as that for A. Now let $\mathbf{\Gamma}$ be a class of certain subsets of Polish space, e.g. $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Sigma}^1_1, \mathbf{\Delta}^1_3$, etc. One may wonder whether there is a most complicated

[23]Strictly speaking, a large cardinal hypothesis (existence of an inaccessible cardinal) is needed to show that it is consistent with ZFC that all $\mathbf{\Pi}^1_1$ sets satisfy the PSP. However, the consistency of the failure of this statement can be shown in ZFC.

set in this class $\mathbf{\Gamma}$ with respect to Wadge reducibility, and the following definition makes this precise.

Definition 16.2. Let Y be Polish. A set $C \subseteq Y$ is called $\mathbf{\Gamma}$-*hard* if for any zero-dimensional Polish X and any set $S \in \mathbf{\Gamma}(X)$, $S \leqslant_W C$. Moreover, if C itself is in $\mathbf{\Gamma}$, then we say that C is $\mathbf{\Gamma}$-*complete*.

The requirement of X to be zero-dimensional is enforced to get rid of topological obstructions that may appear in trying to make the reduction continuous (e.g. if X is connected/compact but Y isn't). Doing so allows measuring the purely descriptive complexity of the sets disregarding the topological properties of the ambient space.

Once we have found a $\mathbf{\Gamma}$-hard (resp. complete) set C, then a common method for showing that some other set D is $\mathbf{\Gamma}$-hard (resp. complete) is showing that C is Wadge reducible to D.

16.A. **$\mathbf{\Sigma}^0_\xi$- and $\mathbf{\Pi}^0_\xi$-complete sets.** Note that if a set $A \subseteq Y$ is $\mathbf{\Gamma}$-hard (resp. complete), then A^c is $\neg\mathbf{\Gamma}$-hard (resp. complete). On the other hand, if $\mathbf{\Gamma}$ is not self-dual (i.e. $\mathbf{\Gamma} \neq \neg\mathbf{\Gamma}$) on zero-dimensional Polish spaces and is closed under continuous preimages, then no $\mathbf{\Gamma}$-hard set is in $\neg\mathbf{\Gamma}$. In particular, if A is, say, $\mathbf{\Sigma}^0_\xi$-complete, then $A \notin \mathbf{\Pi}^0_\xi$. The following theorem shows that the converse also holds for a zero-dimensional Y:

Theorem 16.3 (Wadge)**.** *Let Y be a zero-dimensional Polish space. A Borel set $B \subseteq Y$ is $\mathbf{\Sigma}^0_\xi$-hard iff it is not in $\mathbf{\Pi}^0_\xi$. In particular, B is $\mathbf{\Sigma}^0_\xi$-complete iff B is in $\mathbf{\Sigma}^0_\xi \setminus \mathbf{\Pi}^0_\xi$. The same statements are true with the roles of $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Sigma}^0_\xi$ swapped.*

To prove this theorem, we first need the following amusing yet important lemma:

Lemma 16.4 (Wadge's Lemma)**.** *Let X, Y be zero-dimensional Polish spaces and let $A \subseteq X$ and $B \subseteq Y$ be Borel sets. Then either $A \leqslant_W B$ or $B \leqslant_W A^c$.*

Proof. By Theorem 5.8, we may assume that X, Y are closed subsets of $\mathcal{N}$. Thus $X = [S]$ and $Y = [T]$ for some pruned trees S, T on $\mathbb{N}$.

Consider the Wadge game $G_W(A, B)$:

$$\begin{array}{llllll} \mathrm{I} & x_0 & & x_1 & & \\ & & & & & \dots \\ \mathrm{II} & & y_0 & & y_1 & \end{array}$$

where $x_n, y_n \in \mathbb{N}$, $(x_i)_{i<n} \in S$ and $(y_i)_{i<n} \in T$ for all n. Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$. Player II wins iff $(x \in A \leftrightarrow y \in B)$.

Note that this game is a usual game with rules with a Borel payoff set since A, B are Borel. Thus, it is determined. Suppose Player II has a winning strategy. We can view this strategy as a monotone map $\varphi : S \to T$ such that $|\varphi(s)| = |s|$, for $s \in S$. By (a) of Proposition 2.9, φ gives rise to a continuous map $\varphi^* : [S] \to [T]$. Since φ is a winning strategy for Player II, $x \in A \leftrightarrow \varphi^*(x) \in B$; in other words, $A = (\varphi^*)^{-1}(B)$, so $A \leqslant_W B$.

Now suppose that Player I has a winning strategy. Note that I wins the above game if $(y \in B \leftrightarrow x \notin A)$. Thus, repeating the argument above with roles of the players switched, we get $B \leqslant_W A^c$. $\square$

Now we are ready to prove the above theorem.

Proof of Theorem 16.3. Because $\mathbf{\Sigma}^0_\xi \neq \mathbf{\Pi}^0_\xi$, it is clear that if B is $\mathbf{\Sigma}^0_\xi$-hard then B is not in $\mathbf{\Pi}^0_\xi$. For the converse, suppose $B \subseteq Y$ is a Borel set not in $\mathbf{\Pi}^0_\xi$ and let $A \in \mathbf{\Sigma}^0_\xi(X)$ for some zero-dimensional Polish X. By Wadge's lemma, either $A \leqslant_W B$ or $B \leqslant_W A^c$. The latter alternative cannot happen since it would imply that $B \in \mathbf{\Pi}^0_\xi$, which isn't the case. □

16.B. $\mathbf{\Sigma}^1_1$**-complete sets.** Every analytic set in $\mathcal{N}$ is a projection of a closed set in $\mathcal{N}^2$. Identify $\mathcal{N}^2$ with $(\mathbb{N}\times\mathbb{N})^{\mathbb{N}}$ and note that this turns the projection function $\mathrm{proj}_1 : \mathcal{N}^2 \to \mathcal{N}$ to $p : (\mathbb{N}\times\mathbb{N})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined by $(x_n, y_n)_{n\in\mathbb{N}} \to (x_n)_{n\in\mathbb{N}}$. Recall that closed sets in $(\mathbb{N}\times\mathbb{N})^{\mathbb{N}}$ are the sets of infinite branches through a tree T on $\mathbb{N}\times\mathbb{N}$. Thus, a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is analytic iff there is a tree T on $\mathbb{N}\times\mathbb{N}$ such that $A = p[T]$. (Here we abused the notation and wrote $p[T]$ instead of $p([T])$.)

For a tree T on $\mathbb{N}\times\mathbb{N}$ and $x \in \mathbb{N}^{\mathbb{N}}$, put

$$T_x = \left\{ s \in \mathbb{N}^{<\mathbb{N}} : (x|_{|s|}, s) \in T \right\}$$

and note that T_x is a tree on $\mathbb{N}$. Also note that

$$x \in p[T] \iff [T_x] \neq \emptyset.$$

This allows us to construct a $\mathbf{\Sigma}^1_1$-complete set.

Definition 16.5. Let T be a tree on a set A. Call T *well-founded* if the partial order $\supseteq$ on T is well-founded, i.e. there is no infinite chain $s_0 \subsetneq s_1 \subsetneq s_2 \subsetneq \ldots$ with $s_n \in T$. Otherwise, call T *ill-founded*.

Note that T is well-founded iff $[T] = \emptyset$.

We identify $\mathscr{P}(\mathbb{N}^{<\mathbb{N}})$ with $2^{\mathbb{N}^{<\mathbb{N}}}$ and thus a tree T on $\mathbb{N}$ is an element of $2^{\mathbb{N}^{<\mathbb{N}}}$. Let Tr denote the set of trees on $\mathbb{N}$ and note that it is a closed subset of $2^{\mathbb{N}^{<\mathbb{N}}}$ and hence Polish. Let IF denote the set of ill-founded trees and note that it is an analytic subset of Tr because for $T \in \mathrm{Tr}$,

$$T \in \mathrm{IF} \iff \exists (s_n)_{n\in\mathbb{N}} \in (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}} \forall n (s_n \in T \text{ and } s_n \subseteq s_{n+1} \text{ and } |s_n| = n).$$

Proposition 16.6. IF *is* $\mathbf{\Sigma}^1_1$*-complete.*

Proof. Let X be zero-dimensional Polish and $A \subseteq X$ be analytic. By Theorem 5.8, we can identify X with a closed subset of $\mathcal{N}$. Because closed subsets of $\mathcal{N}$ are analytic, A is still analytic when viewed as a subset of $\mathcal{N}$, so it is enough to show that $(A, \mathcal{N}) \leqslant_W (\mathrm{IF}, \mathrm{Tr})$.

Let T be a tree on $\mathbb{N}\times\mathbb{N}$ such that $A = p[T]$, so for $x \in \mathcal{N}$, we have

$$x \in A \iff T_x \in \mathrm{IF}.$$

Thus, the map $\mathcal{N} \to \mathrm{Tr}$ given by $x \mapsto T_x$ is a reduction of A to IF, and it is straightforward to check that this map is continuous, so we are done. □

Part 4. Definable equivalence relations, group actions, and graphs

In the past twenty five years, a major focus of descriptive set theory has been the study of equivalence relations on Polish spaces that are definable when viewed as sets of pairs (e.g. orbit equivalence relations of continuous actions of Polish groups are analytic). This study is motivated by foundational questions such as understanding the nature of classification of mathematical objects (measure-preserving transformations, unitary operators, Riemann surfaces, etc.) up to some notion of equivalence (isomorphism, conjugacy, conformal equivalence, etc.), and creating a mathematical framework for measuring the complexity of such classification problems. Due to its broad scope, it has natural interactions with other areas of mathematics, such as ergodic theory and topological dynamics, functional analysis and operator algebras, representation theory, topology, model theory, etc.

The following definition makes precise what it means for one classification problem to be easier (not harder) than another.

Definition. Let E and F be equivalence relations on Polish spaces X and Y, respectively. We say that *E is Borel reducible to F* and write $E \leqslant_B F$ if there is a Borel map $f : X \to Y$ such that for all $x_0, x_1 \in X$, $x_0 E x_1 \iff f(x_0) F f(x_1)$.

We call E *smooth* (or *concretely classifiable*) if it Borel reduces to the identity relation $\mathrm{Id}(X)$ on some (any) Polish space X (note that such E is automatically Borel). An example of such an equivalence relation is the similarity relation of matrices; indeed, if $J(A)$ denotes the Jordan canonical form of a matrix $A \in \mathbb{R}^{n^2}$, then for $A, B \in \mathbb{R}^{n^2}$, we have $A \sim B \iff J(A) = J(B)$. It is not hard to check that the computation of $J(A)$ is Borel, so $J : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is a Borel reduction of $\sim$ to $\mathrm{Id}(\mathbb{R}^{n^2})$, and hence $\sim$ is smooth. Another (much more involved) example is the isomorphism of Bernoulli shifts, which, by Ornstein's famous theorem, is reduced to the equality on $\mathbb{R}$ by the map assigning to each Bernoulli shift its entropy.

However, many equivalence relations that appear in mathematics are nonsmooth. For example, the Vitali equivalence relation $\mathbb{E}_v$ on $[0,1]$ defined by $x \mathbb{E}_v y \iff x - y \in \mathbb{Q}$ can be easily shown to be nonsmooth using measure-theoretic or Baire category arguments. The following theorem (known as the General Glimm–Effros dichotomy, see [HKL90]) shows that in fact containing $\mathbb{E}_v$ is the only obstruction to smoothness:

Theorem (Harrington–Kechris–Louveau '90)**.** *Let E be a Borel equivalence relation on a Polish space X. Then either E is smooth, or else $\mathbb{E}_v \sqsubseteq_B E$.*[24]

This was one of the first major victories of descriptive set theory in the study of equivalence relations. It in particular implies that $\mathbb{E}_v$ is the easiest among all nonsmooth Borel equivalence relations in the sense of Borel reducibility. Besides its foundational importance in the theory of Borel equivalence relations, it also generalized earlier important results of Glimm and Effros. By now, many other dichotomy theorems have been proved and general methods of placing a given equivalence relation among others in the Borel reducibility

[24]Here, $\sqsubseteq_B$ means that there is an injective Borel reduction.

hierarchy have been developed. However, there are still many fascinating open problems left and the Borel reducibility hierarchy is yet to be explored.

Another very active area of descriptive set theory is combinatorics of definable graphs, in particular, coloring problems of various classes of definable graphs on Polish spaces. There are dichotomy theorems known for graph colorings as well, and they are tightly connected to the dichotomy theorems for equivalence relations. Below, we will discuss the first dichotomy theorem for graphs by Kechris–Solecki–Todorčević [KST99].

17. Examples of equivalence relations and Polish group actions

17.A. **Equivalence relations.** Let X denote a Polish space. We start by listing some familiar examples of equivalence relations that appear in various areas of mathematics.

Examples 17.1.

(a) The *identity* (*equality*) *relation* $\mathrm{Id}(X)$ on X is a closed equivalence relation.

(b) The *Vitali equivalence relation* $\mathbb{E}_v$ on $[0,1]$, defined by $x\mathbb{E}_v y :\Leftrightarrow x-y \in \mathbb{Q}$, is clearly an F_σ equivalence relation.

(c) Define the equivalence relation $\mathbb{E}_0(X)$ on $X^{\mathbb{N}}$ of *eventual equality of sequences*, namely: for $x,y \in X^{\mathbb{N}}$, $x\mathbb{E}_0(X)y :\Leftrightarrow \forall^\infty n(x(n)=y(n))$. This is again an F_σ equivalence relation. Important special cases when $X=2$, i.e. $X^{\mathbb{N}} = \mathcal{C}$, and when $X = \mathcal{N}$. In the first case we simply write $\mathbb{E}_0 := \mathbb{E}_0(2)$ and in the second case we write $\mathbb{E}_1 := \mathbb{E}_0(\mathcal{N})$.

(d) The similarity relation $\sim$ of matrices on the space $M_n(\mathbb{C})$ of $n \times n$ matrices: for $A,B \in M_n(\mathbb{C})$, $A \sim B \Leftrightarrow \exists Q \in GL_n(\mathbb{C})\ QAQ^{-1} = B$. By definition, this is an analytic equivalence relation, but we will see below that it is actually Borel.

(e) Consider the following subgroups of $\mathbb{R}^{\mathbb{N}}$ under addition:
   - $\ell_p = \left\{x \in \mathbb{R}^{\mathbb{N}} : \sum_n |x(n)|^p < \infty\right\}$, for $1 \leqslant p < \infty$,
   - $\ell_\infty = \left\{x \in \mathbb{R}^{\mathbb{N}} : \sup_n |x(n)| < \infty\right\}$,
   - $c_0 = \left\{x \in \mathbb{R}^{\mathbb{N}} : \lim_n x(n) = 0\right\}$.

   The first two are F_σ subsets of $\mathbb{R}^{\mathbb{N}}$ and the last is $\mathbf{\Pi}^0_3$. Thus, if $\mathcal{I}$ is one of these subgroups, then the equivalence relation $E_{\mathcal{I}}$ on $\mathbb{R}^{\mathbb{N}}$, defined by

   $$xE_{\mathcal{I}}y :\Leftrightarrow x-y \in \mathcal{I},$$

   is F_σ for $\mathcal{I} = \ell_p$, $1 \leqslant p \leqslant \infty$, and is $\mathbf{\Pi}^0_3$ for $\mathcal{I} = c_0$.

(f) Fix a countable first-order relational language $\mathcal{L} = \{R_i\}_{i\in\mathbb{N}}$, where R_i is a relation symbol of arity n_i. The set of countable $\mathcal{L}$-structures can be turned into a Polish space by fixing their underlying set to be $\mathbb{N}$ and, for each i, identifying the interpretation of R_i (i.e. a relation on $\mathbb{N}^{n_i}$) with its characteristic function. Such a structure is simply an element of $X_{\mathcal{L}} := \prod_{i\in\mathbb{N}} 2^{\mathbb{N}^{n_i}}$. This allows talking about the Polish spaces of countable orderings and countable graphs, for example. Also, because any first-order language can be turned into a relational language by replacing function symbols with relation symbols for their graphs, we can also consider Polish spaces of countable groups, rings, fields, etc.

Thus, *isomorphism of countable $\mathcal{L}$-structures*, denoted by $\simeq_{\mathcal{L}}$, naturally falls into the framework of descriptive set theory as it is an analytic equivalence relation on $X_{\mathcal{L}}$; indeed, two structures are isomorphic if and only if there exists a certain bijection f from $\mathbb{N}$ to $\mathbb{N}$, i.e. a certain element $f \in \mathcal{N}$.

17.B. **Polish groups.** Many natural analytic equivalence relations arise as orbit equivalence relations of continuous (or Borel) actions of Polish groups.

Definition 17.2. A topological group is a group with a topology on it so that group multiplication $(x, y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Such a group is called *Polish* if its topology happens to be Polish.

Here are some important examples of Polish groups.

Examples 17.3.

(a) All countable groups with the discrete topology are Polish. In fact, it is an exercise to show that the only Polish topology on a countable group is the discrete topology.

(b) The unit circle $S^1 \subseteq \mathbb{C}$ is a Polish group under multiplication.

(c) $\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ are Polish groups under coordinatewise addition (note that the latter is just the Cantor space $\mathcal{C}$).

(d) The group S_∞ of permutations of $\mathbb{N}$ (i.e. bijections from $\mathbb{N}$ to $\mathbb{N}$) is a G_δ subset of $\mathcal{N}$, so is a Polish group with the relative topology.

(e) Let $(X, \mathcal{B}, \mu)$ be a standard probability space, i.e. $(X, \mathcal{B})$ is a standard Borel space and μ is a probability measure on $\mathcal{B}$; we will often simply write (X, μ). A *measure-preserving automorphism* of (X, μ) is a bimeasurable[25] bijection $T : X \to X$ such that for every measurable $A \subseteq X$, $\mu(T^{-1}(A)) = \mu(A)$. For example, take $X = [0, 1)$ with the Lebesgue measure and let $T_\alpha : X \to X$ be the translation modulo 1 by a real $\alpha \in (0, 1)$.

Let $\mathrm{Aut}(X, \mu)$ denote the set of all measure-preserving automorphisms up to a.e. equality. This is clearly a group under composition and we equip it with the so-called *weak topology* defined in terms of convergent sequences as follows:

$$T_n \to T :\Leftrightarrow \forall A \in \mathcal{B}\ \mu(T_n(A) \Delta T(A)) \to 0.$$

One can show that this is indeed a Polish topology, making $\mathrm{Aut}(X, \mu)$ a Polish group.

(f) Let $\mathcal{H}$ be a separable Hilbert space and let $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$, i.e. invertible linear operators $U : \mathcal{H} \to \mathcal{H}$ that preserve the inner product (equivalently, $U^* = U^{-1}$). This is a Polish group under the strong operator topology[26] defined in terms of convergent sequences as follows:

$$U_n \to U :\Leftrightarrow \forall h \in \mathcal{H}\ \|U_n(h) - U(h)\|_{\mathcal{H}} \to 0.$$

[25] Bimeasurable means both T and T^{-1} are measurable.

[26] The strong and weak operator topologies (defined on the space $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$) coincide on $U(\mathcal{H})$.

17.C. **Actions of Polish groups.**

Definition 17.4. Let G be a Polish group and X be a Polish space. An action $a : G \curvearrowright X$ of G on X is said to be *continuous* (resp. *Borel*) if the action function $a : G \times X \to X$ given by $(g, x) \mapsto g \cdot_a x$ is continuous (resp. Borel).

We denote by E_G (or sometimes by E_a) the orbit equivalence relation induced by such an action. Note that E_G is analytic because for $x, y \in X$,

$$xE_G y \iff \exists g \in G(g \cdot_a x = y).$$

Here we list some examples of continuous actions of Polish groups.

Examples 17.5.

(a) Any Polish group acts on itself by left multiplication, as well as by conjugation. It follows from the definition of topological groups that these actions are continuous.

(b) Let G be a Polish group and $H < G$ be a Polish (equivalently, closed) subgroup. The left multiplication action of H on G is clearly continuous and the induced orbit equivalence relation E_H is the relation of being in the same right H-coset, i.e. $xE_H y \Leftrightarrow Hx = Hy$. We refer to E_H as the *H-coset equivalence relation*.

(c) The Vitali equivalence relation $\mathbb{E}_v$ is exactly the orbit equivalence relation of the translation action of $\mathbb{Q}$ on $\mathbb{R}$.

(d) The relation $\mathbb{E}_0$ of eventual equality on $\mathcal{C}$ is induced by a continuous action of a countable group as follows: for each $n \in \mathbb{N}$, let $\sigma_n : \mathcal{C} \to \mathcal{C}$ be the map that flips the n^{th} bit:

$$\sigma_n(x)(i) := \begin{cases} 1 - x(i) & \text{if } i = n \\ x(i) & \text{otherwise,} \end{cases}$$

and let Γ be the group generated by $\{\sigma_n : n \in \mathbb{N}\}$, which is just the direct sum $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. It is clear that each σ_n is a homeomorphism of $\mathcal{C}$ and $E_\Gamma = \mathbb{E}_0$.

One can also show that, after throwing away two orbits (more precisely, restricting $\mathbb{E}_0$ to $\mathcal{C}' = \{x \in \mathcal{C} : x \text{ has infinitely many 0-s and 1-s}\}$), we can realize $\mathbb{E}_0$ by a continuous action of $\mathbb{Z}$ known as the *odometer* action. We leave this as an exercise.

(e) A rotation of S^1 is simply an action $\mathbb{Z} \curvearrowright S^1$, where $1 \in \mathbb{Z}$ acts as multiplication by $e^{\alpha\pi i}$, for a fixed $\alpha \in \mathbb{R}$. Clearly this action is continuous and we denote the orbit equivalence relation by $\mathbb{E}_\alpha$.

(f) The equivalence relations $\mathbb{E}_0, \mathbb{E}_v, \mathbb{E}_\alpha$ are examples of countable equivalence relations. In general, orbit equivalence relations induced by continuous or Borel actions of countable groups are examples of countable Borel (why?) equivalence relations. Curiously enough, these are all of the examples! More precisely, any countable Borel equivalence relation arises as the orbit equivalence relation of a Borel action of a countable group. This is a theorem by Feldman and Moore, and we will prove it below.

(g) The similarity relation $\sim$ of matrices in $M_n(\mathbb{R})$ is induced as the orbit equivalence relation of the conjugation action of $GL_n(\mathbb{R})$ on $M_n(\mathbb{R})$.

(h) For a first-order relational language $\mathcal{L}$, the group S_∞ admits a natural action on the Polish space $X_\mathcal{L}$ of countable $\mathcal{L}$-structures by permuting their underlying sets.

Clearly, the induced orbit equivalence relation is exactly the relation of isomorphism of $\mathcal{L}$-structures.

(i) For a standard probability space $(X, \mathcal{B}, \mu)$, let MALG denote the σ-algebra of μ-measurable sets modulo the σ-ideal of μ-null sets. The metric $d(A, B) := \mu(A \triangle B)$ on MALG is complete and the fact that $\mathcal{B}$ is countably generated implies that it is separable, so MALG is a Polish space and the natural action of $\mathrm{Aut}(X, \mu)$ on MALG is by isometries. It also follows that this action is continuous.

(j) Similarly, for a separable Hilbert space $\mathcal{H}$, the natural action of $U(\mathcal{H})$ on $\mathcal{H}$ is continuous.

18. Borel reducibility

Let E and F be equivalence relations on Polish spaces X and Y, respectively. The following defines the class of functions from X to Y that induce functions from X/E to Y/F.

Definition 18.1. A function $f : X \to Y$ is called a *homomorphism from E to F* if for all $x_0, x_1 \in X$,

$$x_0 E x_1 \Rightarrow f(x_0) F f(x_1).$$

$f : X \to Y$ is called a *reduction of E to F* if for all $x_0, x_1 \in X$,

$$x_0 E x_1 \Leftrightarrow f(x_0) F f(x_1).$$

Note that reductions induce injections $X/E \hookrightarrow Y/F$.

The following makes it precise what it means for a classification problem in mathematics to be easier (not harder) than another classification problem:

Definition 18.2. Let E and F be equivalence relations on Polish spaces X and Y, respectively. We say that *E is Borel reducible to F*, and write $E \leqslant_B F$, if there is a Borel reduction of E to F. Furthermore, we say that *E is strictly below F*, and write $E <_B F$, if $E \leqslant_B F$ but $F \nleqslant_B E$.

The choice of "Borel" as the regularity condition on the reduction is mainly because any two uncountable Polish spaces are Borel isomorphic, so the existence of Borel reductions does not depend on the particular choice of the underlying Polish spaces and it only depends on the inherent complexity of the equivalence relations, which is what we want to measure.

We replace the subscript B in $\leqslant_B$ by c if there is a continuous reduction, and we write $\sqsubseteq$ instead of $\leqslant$ if the reduction is injective.

It is clear that $\leqslant_B$ is a quasi-order[27] on the class of all equivalence relations on Polish spaces[28]. We call E and F *Borel bireducible*, and write $E \sim_B F$, if $E \leqslant_B F$ and $F \leqslant_B E$. Since Borel reductions induce Borel embeddings $X/E \hookrightarrow Y/E$, we refer to the bireducibility class of E as *the Borel cardinality of X/E*.

We also call E and F *Borel isomorphic*, and write $E \simeq_B F$, if there is a bijective Borel reduction (thus a Borel isomorphism from X to Y) of E to F.

[27]Quasi-order is a reflexive and transitive relation, not necessarily antisymmetric.

[28]This is actually a set if we fix a particular uncountable Polish space, which we can do as any two of them are Borel isomorphic.

Remark 18.3. In general, $E \sqsubseteq_B F$ and $F \sqsubseteq_B E$ does not imply $E \simeq_B F$; more precisely, the Schröder–Bernstein argument doesn't work unless both reductions are "locally surjective on classes". The latter means that the Borel reduction $f : X \to Y$ has the property that for every $x \in X$, $f([x]_E)$ is an entire F-class, i.e. $f([x]_E) = [f(x)]_F$. Indeed, imagine a situation of having injective Borel reductions $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$ of E to F and F to E, respectively, such that for some $x \in X$, $[x]_E \cap g(Y) = \emptyset$ and $f([x]_E) \subsetneq [f(x)]_F$. Then, the Schröder–Bernstein argument would map the elements of $[x]_E$ by f into $[f(x)]_F$ and the elements in $[f(x)]_F \setminus f([x]_E)$ by g into $[g(f(x))]_E$. But $[g(f(x))]_E \neq [x]_E$ because $[x]_E$ is disjoint from $[g(Y)]_F$, so the elements in $f([x]_E)$ would go to a different E-class (namely, $[x]_E$) than the elements in $[f(x)]_F \setminus f([x]_E)$, and hence, the resulting map will not be a reduction.

The systematic study of the Borel reducibility hierarchy of definable equivalence relations is sometimes referred to as *invariant descriptive set theory*. It was pioneered by Silver, Harrington, Kechris, Louveau, and others, in the late '80s and early '90s. The goal of invariant descriptive set theory is to understand the Borel reducibility hierarchy (and hence, the complexity of classification problems that appear in many areas of mathematics such as analysis, ergodic theory, operator algebras, model theory, recursion theory, etc.), and to develop methods for placing a given equivalence relation into its "correct" spot in this hierarchy.

19. Perfect set property for quotient spaces

Given a definable equivalence relation E on a Polish space X, first thing one would want to know about the quotient space X/E is its cardinality. A strengthening of this is the question of whether or not X/E has the perfect set property and the following definition makes it precise:

Definition 19.1. We say that E has *perfectly many classes* if $\mathrm{Id}(\mathcal{C}) \sqsubseteq_c E$.

Proposition 19.2. *For an equivalence relation E on a Polish space X,*

$$\mathrm{Id}(\mathcal{C}) \sqsubseteq_c E \iff \mathrm{Id}(\mathcal{C}) \sqsubseteq_B E \iff \mathrm{Id}(\mathcal{C}) \leqslant_B E.$$

Proof. Left as an exercise. □

Proposition 19.3. *Let E be an analytic or co-analytic equivalence relation on a Polish space X. If E has countably many equivalence classes, then E is Borel and hence $E \leqslant_B \mathrm{Id}(\mathbb{N})$.*

Proof. Say E is analytic (the proof is the same for co-analytic), and hence, so is each E-class (being a fiber of E). But the complement of each E-class C is a countable union of E-classes, so is analytic as well. Thus, C is Δ^1_1 and hence is Borel. Letting $\{x_n\}_{n<k}$, $k \leqslant \omega$, be a set of representatives of the E-classes (one from each), we see that for all $x, y \in X$,

$$xEy \Leftrightarrow \exists n < k \ (x, y \in [x_n]_E).$$

This shows that E is Borel since each $[x_n]_E$ is Borel. Moreover, the function that maps all elements of $[x_n]_E$ to n is a Borel map from X to k witnessing $E \leqslant_B \mathrm{Id}(k) \sqsubseteq_c \mathrm{Id}(\mathbb{N})$. □

In the light of last two propositions, the question of whether X/E has the perfect set property is the same as whether

$$\mathrm{Id}(\mathbb{N}) <_B E <_B \mathrm{Id}(\mathcal{C}).$$

We give some answers to this question in the next subsections.

19.A. **Co-analytic equivalence relations: Silver's dichotomy.** In 1980, Silver showed that the perfect set property holds for $\mathbf{\Pi}_1^1$ equivalence relations, namely:

Dichotomy 19.4 (Silver '80)**.** *Any co-analytic equivalence relation E on a Polish space X has either countably many or perfectly many classes. In other words, either $E \leqslant_B \mathrm{Id}(\mathbb{N})$, or $\mathrm{Id}(\mathcal{C}) \leqslant_B E$.*

Silver's original proof was quite complicated and used forcing. Later on Harrington reproved it using a finer topology (not Polish) on X that comes from recursion theory, the so-called *Gandy–Harrington topology*. Finally, in 2008, Ben Miller found a classical proof (using only Baire category arguments) of a dichotomy theorem by Kechris–Solecki–Todorčević about Borel colorings of analytic graphs, and from this he deduced Silver's dichotomy (and other dichotomies as well). We will give this proof later on in the notes.

19.B. **Analytic equivalence relations: Burgess' trichotomy.** The perfect set property does not hold for analytic equivalence relations! For example, let LO denote the Polish space of all linear orderings (this is a closed subset of $X_{\mathcal{L}}$, with $\mathcal{L} = \{<\}$), let $\mathrm{WO} \subseteq \mathrm{LO}$ denote the set of all wellorderings, and define an equivalence relation E on LO as follows: for $x, y \in \mathrm{LO}$,

$$xEy :\Leftrightarrow (x \notin \mathrm{WO} \wedge y \notin \mathrm{WO}) \vee (x \simeq y),$$

where $\simeq$ stands for isomorphism of orderings. E is clearly analytic since WO is co-analytic (it is an exercise to show that WO is actually $\mathbf{\Pi}_1^1$-complete) and $\simeq$ is analytic (isomorphism of structures is analytic because it holds when there *exists* a certain bijection from $\mathbb{N}$ to $\mathbb{N}$). All nonwellorderings in LO are E-equivalent (belong to one E-class) and there are precisely ω_1-many nonisomorphic wellorderings of $\mathbb{N}$. Thus, E has exactly ω_1-many classes and hence if the continuum hypothesis doesn't hold, E won't have the perfect set property.

So what are the possibilities for the cardinality of X/E for a given analytic E?

Trichotomy 19.5 (Burgess '78)**.** *Any analytic equivalence relation E on a Polish space X has either countably many, ω_1-many, or perfectly many classes.*

We won't prove this theorem in these notes, but the proof can be found in [Gao09].

The Vaught conjecture. Let $\mathcal{L}$ be a countable language and let T be first-order $\mathcal{L}$-theory, i.e. a set of $\mathcal{L}$-sentences. A straightforward induction on the length of formulas in T shows that the set $\mathrm{Mod}(T)$ of countable models of T is a Borel subset of $X_{\mathcal{L}}$.

Vaught conjecture. *Any countable first-order theory T has either countably many or perfectly many nonisomorphic countable models.*

As mentioned above, the isomorphism relation of countable structures is precisely the orbit equivalence relation E_{S_∞} induced by the natural action of S_∞ on $X_{\mathcal{L}}$. Thus, the Vaught conjecture is simply the statement that Silver's dichotomy holds for $E_{S_\infty}|_{\mathrm{Mod}(T)}$, and it has the following generalization (a present from model theory to descriptive set theory):

Topological Vaught conjecture. *Borel actions of Polish groups on Polish spaces have either countably many or perfectly many orbits.*

The best currently known result in this direction is a theorem of Becker stating that topological Vaught conjecture holds for the so-called *cli* groups, i.e. Polish groups that admit a complete left-invariant metric.

19.C. **Meager equivalence relations: Mycielski's theorem.** Many interesting equivalence relations have small equivalence classes and the following proposition shows that these are exactly the meager equivalence relations.

Proposition 19.7. *Let E be a Baire measurable equivalence relation on a Polish space X. Then E is meager if and only if each E-class is meager.*

Proof. Implication $\Leftarrow$ is by the Kuratowski–Ulam theorem. For $\Rightarrow$, the Kuratowski–Ulam theorem merely gives

$$\forall^* x \in X \; ([x]_E \text{ is meager}), \tag{$*$}$$

so a priori there may be an $x_0 \in X$ with $[x_0]_E$ nonmeager. But then, since for every $x \in [x_0]_E$, $[x]_E = [x_0]_E$, the set

$$A = \{x \in X : [x]_E \text{ is nonmeager}\}$$

contains $[x_0]_E$, so is nonmeager, contradicting $(*)$. □

Examples 19.8.

(a) All *countable*[29] Borel equivalence relations on nonempty perfect Polish spaces are meager. This includes $\mathbb{E}_0$, $\mathbb{E}_v$, the irrational rotation $\mathbb{E}_\alpha$, and in general, any orbit equivalence relation induced by a Borel action of a countable group.

(b) For any Polish space X with $|X| \geqslant 2$, the equivalence relation $\mathbb{E}_0(X)$ of eventual equality on $X^{\mathbb{N}}$ is meager; in particular, $\mathbb{E}_1 = \mathbb{E}_0(\mathcal{N})$ is meager. This is because for each $x \in X^{\mathbb{N}}$, $[x]_{\mathbb{E}_0(X)} = \bigcup_n A_n$, where $A_n := \{y \in Y : \forall k \geqslant n \; y(k) = x(k)\}$ is nowhere dense (has empty interior and is closed).

(c) The equivalence relations $E_{\mathcal{I}}$ for $\mathcal{I} = c_0$ or ℓ_p, $1 \leqslant p \leqslant \infty$. This is because every $E_{\mathcal{I}}$ equivalence class is homeomorphic to $\mathcal{I}$, $c_0 \subseteq \ell_p \subseteq \ell_\infty$, and ℓ_∞ is meager because $\ell_\infty = \bigcup_n A_n$, where $A_n := \left\{x \in \mathbb{R}^{\mathbb{N}} : \sup_n |x(n)| \leqslant n\right\}$ is nowhere dense (has empty interior and is closed).

Note that since each equivalence class of a meager equivalence relation is meager, there must be uncountably many classes. In fact, we have:

Theorem 19.9 (Mycielski)**.** *Any meager equivalence relation E on a Polish space X has perfectly many classes.*

Proof. Write $E = \bigcup_n F_n$, where each $F_n \subseteq X \times X$ is nowhere dense and $F_n \subseteq F_{n+1}$. In order to get a desired embedding $\mathcal{C} \hookrightarrow X$, we will construct a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ on X of vanishing diameter (with respect to a fixed complete metric d for X) with the following properties:

(i) U_s is nonempty open and $\overline{U_{s^\frown i}} \subseteq U_s$, for each $s \in 2^{<\mathbb{N}}$, $i \in \{0,1\}$;

(ii) $(U_s \times U_t) \cap F_n = \emptyset$, for all distinct $s, t \in 2^n$ and $n \in \mathbb{N}$.

Granted this construction, let $f : \mathcal{C} \rightharpoonup X$ be the associated map. By Proposition 5.4, the domain of f is all of $\mathcal{C}$, and f is continuous and injective. Hence f is a topological embedding since $\mathcal{C}$ is compact. To show that f is also a reduction of $\mathrm{Id}(\mathcal{C})$ to E, we need to fix distinct $\sigma, \tau \in \mathcal{C}$ and show that $(f(\sigma), f(\tau)) \notin E$. To this end, let $k \in \mathbb{N}$ be such

[29] We call an equivalence relation *countable* if every equivalence class is countable; not to be confused with the number of equivalence classes being countable.

that $\sigma|_n \neq \tau|_n$, for all $n \geqslant k$. But for each $n \geqslant k$, $f(\sigma) \in U_{\sigma|_n}$ and $f(\tau) \in U_{\tau|_n}$, so by (ii), $(f(\sigma), f(\tau)) \notin F_n$. Since the F_n are increasing, $(f(\sigma), f(\tau)) \notin F_n$ for all $n < k$ as well, so $(f(\sigma), f(\tau)) \notin E$.

To construct such a scheme, first note that X is perfect: indeed, if $x \in X$ is an isolated point, then $\{(x,x)\}$ is open in X^2 and $\{(x,x)\} \subseteq E$, contradicting E being meager. This allows us to construct a Cantor scheme with property (i) as done in the proof of the perfect set theorem for Polish spaces. As for (ii), one has to iteratively use the following fact: for a nowhere dense set $F \subseteq X^2$ and any nonempty open sets $U, V \subseteq X$, there are nonempty open sets $U' \subseteq U$ and $V' \subseteq V$ such that $(U' \times V') \cap F = \emptyset$. We leave the details of this construction as an exercise. □

20. Concrete classifiability (smoothness)

In this section we make it precise what it means to classify mathematical objects (matrices, measure-preserving transformations, unitary operators, Riemann surfaces, etc.) up to some notion of equivalence (isomorphism, conjugacy, conformal equivalence, etc.). We will consider some examples and nonexamples, as well as discuss related (famous) dichotomy theorems.

20.A. **Definitions.**

Definition 20.1. An equivalence relation E on a Polish space X is called *concretely classifiable* (or *smooth*) if $E \leqslant_B \mathrm{Id}(\mathbb{R})$. By the Borel isomorphism theorem, $\mathbb{R}$ can be replaced by any other uncountable Polish space.

Note that *smooth equivalence relations are necessarily Borel*: indeed, if $f : X \to \mathbb{R}$ is a Borel reduction of E to $\mathrm{Id}(\mathbb{R})$, then the function $f_2 : X^2 \to \mathbb{R}^2$ by $(x,y) \mapsto (f(x), f(y))$ is Borel and $E = f_2^{-1}(\Delta_{\mathbb{R}})$, where $\Delta_{\mathbb{R}}$ is the diagonal in $\mathbb{R}^2$. But $\Delta_{\mathbb{R}}$ is closed in $\mathbb{R}^2$, so E is Borel being a preimage of Borel.

A special case of smoothness is when we can select a canonical representative from each equivalence class.

Definition 20.2. Let E be an equivalence relation on a Polish space X. A map $s : X \to X$ is called a *selector for E* if for all $x \in X$, $s(x) \in [x]_E$, and s is a reduction of E to $\mathrm{Id}(X)$, i.e. $xEy \Leftrightarrow s(x) = s(y)$. A set $Y \subseteq X$ is called a *transversal for E* if it meets every E-class at exactly one point, i.e. for each $x \in X$, $[x]_E \cap Y$ is a singleton.

Proposition 20.3. *An analytic equivalence relation E on a Polish space X admits a Borel selector if and only if it admits an analytic transversal.*[30]

Proof. If $s : X \to X$ is a Borel selector for E, then it is clear that $s(X)$ is an analytic transversal. For the converse, let $Y \subseteq X$ be an analytic transversal and define $s : X \to X$ by $x \mapsto$ the unique $y \in Y$ with xEy. To prove that s is Borel, we recall Corollary 12.8 and observe that the graph of s is analytic: for each $(x,y) \in X^2$, $s(x) = y$ if and only if xEy and $y \in Y$. □

Thus, the chain of implications for analytic equivalence relations is as follows:

$$\text{Borel transversal} \Rightarrow \text{analytic transversal} \Leftrightarrow \text{Borel selector} \Rightarrow \text{smooth}.$$

Concerning the reverse direction of the first implication, we have the following:

[30] Thanks to Aristotelis Panagiotopoulos for pointing out that assuming the existence of a merely analytic transversal still implies the existence of a Borel selector.

Proposition 20.4. *For orbit equivalence relations of Borel actions of Polish groups, any analytic transversal is actually Borel.*

Proof. Let $G \curvearrowright X$ be a Borel action of a Polish group G on a Polish space X, and let $Y \subseteq X$ be an analytic transversal for E_G. Then, for $x \in X$,

$$x \notin Y \iff \exists g \in G (gx \in Y \text{ and } gx \neq x),$$

so Y^c is analytic as well, and hence Y is Borel by Souslin's theorem (Corollary 12.7). □

As for the implication "Borel selector ⇒ smooth", it is a theorem of Burgess that the reverse implication is also true for the orbit equivalence relations of continuous actions of Polish groups. Here we will record a special case of this[31].

Definition 20.5. An equivalence relation E on a Polish space X is called *countable* if each E-class is countable.

Proposition 20.6. *A countable equivalence relation E on a Polish space X is smooth if and only if it admits a Borel selector.*

Proof. The nontrivial direction follows immediately from Corollary 13.8. □

20.B. **Examples of concrete classification.** We start by listing some well known examples of equivalence relations from different areas of mathematics that admit concrete classification.

Examples 20.7.

(a) *Isomorphism of finitely generated abelian groups.* Let $\mathcal{L}_g = \{\cdot, 1\}$ be the language of groups. Then the set $Y \subseteq X_{\mathcal{L}_g}$ of all finitely generated abelian groups is Σ^0_3 ($\exists$ finitely many elements such that $\forall$ group elements γ $\exists$ a combination equal to γ), and hence standard Borel. We know from algebra that every $\Gamma \in Y$ is isomorphic to a group of the form $\mathbb{Z}^n \bigoplus \mathbb{Z}_{q_1} \bigoplus \mathbb{Z}_{q_2} \bigoplus \ldots \bigoplus \mathbb{Z}_{q_k}$, where $q_1 \leqslant q_2 \leqslant \ldots \leqslant q_k$ are powers of primes. The map $\Gamma \mapsto \mathbb{Z}^n \bigoplus \mathbb{Z}_{q_1} \bigoplus \mathbb{Z}_{q_2} \bigoplus \ldots \bigoplus \mathbb{Z}_{q_k}$ from Y to Y is a selector for $\mathrm{Iso}(Y)$ and it can be shown to be Borel, witnessing the smoothness of $\mathrm{Iso}(Y)$.

(b) *Similarity of matrices.* Let $M_n(\mathbb{C})$ denote the Polish space of complex $n \times n$ matrices and $\sim$ denote the similarity relation on $M_n(\mathbb{C})$, which is Σ^1_1 by definition. For each $A \in M_n(\mathbb{C})$, let $J(A)$ denote its Jordan canonical form. We know from linear algebra that $A \sim B \Leftrightarrow J(A) = J(B)$, in other words, J is a selector for $\sim$. Moreover, one can show that it is Borel, so $\sim$ is smooth. In particular, $\sim$ is a Borel equivalence relation, which wasn't apparent at all from its definition.

(c) *Isomorphism of Bernoulli shifts.* Let (X, μ) be a probability space (X can be finite) and let $\mu^{\mathbb{Z}}$ denote the product measure on $X^{\mathbb{Z}}$. Let $S : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ denote the shift automorphism, i.e. for $f \in X^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, $T(f)(n) = f(n-1)$. The dynamical system $(X^{\mathbb{Z}}, \mu^{\mathbb{Z}}, S)$ is called a *Bernoulli shift*. By the measure isomorphism theorem, every Bernoulli shift is isomorphic to $([0,1], \lambda, T)$, where λ is the Lebesgue measure and T some measure-preserving automorphism of $([0,1], \lambda)$. In this case, we would call T a *Bernoulli shift* as well, and let $B \subseteq \mathrm{Aut}([0,1], \lambda)$ be the set of all Bernoulli shifts. Ornstein showed that B is a Borel subset of $\mathrm{Aut}([0,1], \lambda)$, and hence is a standard

[31] The fact this is a special case is due to the Feldman–Moore theorem 22.2 and Corollary 11.23

Borel space. Furthermore, to each $T \in \mathrm{Aut}([0,1],\lambda)$, one can attach a real number $e(T) \in \mathbb{R} \cup \{\infty\}$ called the *entropy of the dynamical system* $([0,1],\lambda,T)$, which somehow measures the probabilistic unpredictability of the action of T. This notion of entropy is defined by Kolmogorov and it follows from the definition that it is an isomorphism invariant. For the Bernoulli shifts however (i.e. $T \in B$), it is a celebrated theorem of Ornstein that entropy is a complete invariant! In other words, for $T_1, T_2 \in B$, $([0,1],\lambda,T_1) \simeq ([0,1],\lambda,T_2) \Leftrightarrow e(T_1) = e(T_2)$. It can also be checked that the function $T \mapsto e(T)$ is Borel, hence a Borel reduction of the isomorphism relation of Bernoulli shifts to $\mathrm{Id}(\mathbb{R} \cup \{\infty\})$, witnessing the smoothness of the former.

The following proposition gives a new batch of examples.

Proposition 20.8. *Let E be an equivalence relation on a Polish space X. If each E-class is G_δ and the E-saturations of open sets are Borel, then E is smooth. Moreover, if each E-class is actually closed, then E admits a Borel selector.*

Proof. The map $\rho : X \to \mathcal{F}(X)$ by $x \mapsto \overline{[x]_E}$ is a reduction of E to $\mathrm{Id}(\mathcal{F}(X))$ because if $\overline{[x]_E} = \overline{[y]_E}$, then both $[x]_E, [y]_E$ are dense G_δ subsets of $\overline{[x]_E}$, hence comeager in $\overline{[x]_E}$ (in the relative topology of $\overline{[x]_E}$). By the Baire category theorem, $[x]_E \cap [y]_E \neq \emptyset$, so $[x]_E = [y]_E$. It remains to show that ρ is Borel, which follows from the fact that if $U \subseteq X$ is open, then $\rho^{-1}(\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}) = [U]_E$ is Borel by the hypothesis.

If moreover, each E-class is closed, then composing ρ with a Borel selector for $\mathcal{F}(X)$ (see Theorem 13.16) gives a Borel selector for E. $\square$

Lemma 20.9. *Let G be a group and let it act $G \curvearrowright X$ by homeomorphisms on a Polish space X. Then the saturations of open subsets of X are open.*

Proof. For open $U \subseteq X$, $[U]_G = \bigcup_{g \in G} gU$ is open because each gU is open being a homeomorphic image of U. $\square$

This lemma, together with the above proposition, gives:

Corollary 20.10. *Let a group G act by homeomorphisms on a Polish space X and let E_G denote the induced orbit equivalence relation. If every orbit is G_δ, then E_G is smooth. If every orbit is closed, then E_G admits a Borel selector.*

Examples 20.11.

(a) *Orbit equivalence relation E_K induced by a continuous action of a compact group K on a Polish space X admits a Borel selector*. This is because every orbit $[x]_K$ is equal to $K \cdot x$, and hence is compact being a continuous image of a compact space K. In particular, $[x]_K$ is closed.

(b) *For a closed subgroup $H < G$ of a Polish group G, the H-coset equivalence relation E_H admits a Borel selector*. Indeed, each E_H-class is just an H-coset Hg, for some $g \in G$ and hence is closed.

(c) *For a discrete subgroup $\Gamma < G$ of a Polish group G, the Γ-coset equivalence relation E_Γ admits a Borel selector*. This is a special case of the previous example because discrete subgroups of Polish groups are closed. Indeed, the relative topology of Γ is discrete and hence Polish. But by a homework exercise, Polish subgroups of Polish groups are closed.

As an instance of the last example, the $\mathbb{Z}$-coset equivalence relation on $\mathbb{R}$ in fact admits a Borel transversal, namely, the interval $[0,1)$.

20.C. **Characterizations of smoothness.**

Definition 20.12. Let E be an equivalence relation on a Polish space X and let $\mathcal{F}$ be a family of subsets of X. We say that $\mathcal{F}$ *generates* E if

$$xEy \iff \forall A \in \mathcal{F}(x \in A \Leftrightarrow y \in A).$$

Theorem 20.13 (Combinatorial characterization of smoothness)**.** *An equivalence relation E on a Polish space X is smooth if and only if it is generated by a countable Borel family.*

Proof. For the forward direction, let $f : X \to \mathbb{R}$ be a Borel reduction of E to $\mathrm{Id}(\mathbb{R})$ and let $\{U_n\}_n$ be a countable open basis for $\mathbb{R}$. Then it is easy to check that the family $\left\{f^{-1}(U_n)\right\}_n$ generates E.

Conversely, if $\{B_n\}_n$ is a countable Borel family generating E then it is easy to check that the function $f : X \to \mathcal{C}$, defined by $x \mapsto$ the characteristic function of $\{n \in \mathbb{N} : x \in B_n\}$, is a Borel reduction of E to $\mathrm{Id}(\mathcal{C})$. ☐

Using this, one also gets:

Theorem 20.14 (Topological characterization of smoothness)**.** *An equivalence relation E on a Polish space $(X,\mathcal{T})$ is smooth if and only if there is a Polish topology $\mathcal{T}_E \supseteq \mathcal{T}$ on X (and hence automatically $\mathcal{B}(\mathcal{T}_E) = \mathcal{B}(\mathcal{T})$) such that E is closed in $(X^2, \mathcal{T}_E^2)$.*

Proof. Outlined in a homework problem. ☐

The following proposition gives a class of examples of closed equivalence relations on Polish spaces.

Proposition 20.15. *Orbit equivalence relations induced by continuous actions of compact groups are closed.*

Proof. Let G be a compact group, X a topological space and consider a continuous action $G \curvearrowright X$, i.e. $a : G \times X \to X$ is continuous. Then the graph $\mathrm{Graph}(a)$ of the function a is a closed subset of $G \times X^2$ and $E_G = \mathrm{proj}_{2,3}(\mathrm{Graph}(a))$. Therefore, E_G is closed by the tube lemma[32] since G is compact. ☐

20.D. **Nonsmooth equivalence relations.**

Definition 20.16. An equivalence relation E on a Polish space X (resp. measure space $(X,\mathcal{B},\mu)$) is called *generically ergodic* (resp. *μ-ergodic*) if every invariant Baire measurable (resp. μ-measurable) subset of X is either meager (resp. μ-null) or comeager (resp. μ-conull).

We call a (continuous or measurable) group action $G \curvearrowright X$ *generically ergodic* (resp. *μ-ergodic*) if such is the induced orbit equivalence relation E_G.

[32]The tube lemma states that for topological spaces K, Y, if K is compact then projections of closed subsets of $K \times Y$ onto Y are closed.

Proposition 20.17. *Let E be an equivalence relation on a Polish space X and let $f : X \to \mathcal{C}$ be a Baire measurable homomorphism of E to $\mathrm{Id}(\mathcal{C})$. If E is generically ergodic, then there is $y \in \mathcal{C}$ such that $f^{-1}(y)$ is comeager. Letting μ be a Borel measure on X, the analogous statement holds for μ-ergodic E.*

Proof. We only prove the topological statement since the proof of the measure-theoretic statement is analogous. First note that for any $A \subseteq \mathcal{C}$, $f^{-1}(A)$ is E-invariant by the virtue of f being a homomorphism. By recursion on n, we now define an increasing sequence $(s_n)_n \subseteq 2^{<\mathbb{N}}$ such that $|s_n| = n$ and $f^{-1}(N_{s_n})$ is comeager. Put $s_0 = \emptyset$, and suppose s_n is defined and satisfies the requirements. Since $f^{-1}(N_{s_n}) = f^{-1}(N_{s_n{}^\frown 0}) \cup f^{-1}(N_{s_n{}^\frown 1})$, for at least one $i \in \{0,1\}$, $f^{-1}(N_{s_n{}^\frown i})$ must be nonmeager, and hence comeager because $f^{-1}(N_{s_n{}^\frown i})$ is invariant and Baire measurable. Set $s_{n+1} = s_n{}^\frown i$. Having finished the construction of $(s_n)_n$, put $y = \bigcup_n s_n$. Then $f^{-1}(y) = f^{-1}(\bigcap_n N_{s_n}) = \bigcap_n f^{-1}(N_{s_n})$ is comeager. □

Corollary 20.18. *Let E be an equivalence relation on a Polish space X. If E is generically ergodic (resp. ergodic) and every E-class is meager, then E is not smooth. Letting μ be a nontrivial Borel measure on X, the analogous statement holds for μ-ergodic E.*

Proof. If $f : X \to \mathcal{C}$ is a Baire measurable reduction of E to $\mathrm{Id}(\mathcal{C})$, then the preimage of every point $y \in f(X)$ is an E-class, and hence is meager, contradicting the previous proposition. □

Proposition 20.19. *Let Γ be a group acting on a Polish space X by homeomorphisms, i.e. each $\gamma \in \Gamma$ acts as a homeomorphism of X. The following are equivalent:*

(1) *E_Γ is generically ergodic.*
(2) *Every invariant nonempty open set is dense.*
(3) *For comeager-many $x \in X$, the orbit $[x]_\Gamma$ is dense.*
(4) *There is a dense orbit.*
(5) *For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \emptyset$.*

Proof. The only implications worth proving are the following:
(2)⇒(3): Fixing a countable basis $\{U_n\}_n$, note that $D := \bigcap_n [U_n]_\Gamma$ is comeager and for every $x \in D$, the orbit $[x]_\Gamma$ intersects every U_n, so is dense.
(5)⇒(1): Let $A \subseteq X$ be invariant and Baire measurable. If neither of A, A^c are meager, then, by the Baire alternative, there are nonempty open sets U, V such that $U \Vdash A$ and $V \Vdash A^c$. Let $\gamma \in \Gamma$ be such that $W := (\gamma U) \cap V \neq \emptyset$. Because γ is a homeomorphism, $\gamma U \Vdash \gamma A$ and hence $\gamma U \Vdash A$ because $\gamma A = A$. Thus, $W \Vdash A$ and $W \Vdash A^c$, contradicting W being nonmeager. □

Corollary 20.20. *If a group Γ acts by homeomorphisms on a Polish space X such that every orbit is meager (e.g. when Γ is countable and X is perfect) and there is a dense orbit, then E_Γ is nonsmooth. In particular, if G is a Polish group and $\Gamma < G$ is a countable dense subgroup, then the orbit equivalence relation E_Γ of the left translation action $\Gamma \curvearrowright G$ is nonsmooth.*

Proof. The second statement is immediate from the first, and the first statement follows from Corollary 20.18 and Proposition 20.19. □

Examples 20.21.

(a) *The Vitali equivalence relation* $\mathbb{E}_v$ *is nonsmooth*. Indeed, $\mathbb{E}_v$ is the orbit equivalence relation of the translation action of $\mathbb{Q}$ on $\mathbb{R}$.

(b) *The irrational rotation* $\mathbb{E}_\alpha$ *of* S^1 *is nonsmooth*. Indeed, let Γ be the subgroup of S^1 generated by $e^{2\pi\alpha i}$. It is clear that $\mathbb{E}_\alpha$ is precisely the orbit equivalence relation induced by the translation action $\Gamma \curvearrowright S^1$, and it follows from irrationality of α that Γ is dense.

(c) $\mathbb{E}_0$ *is nonsmooth*. Indeed, each $\mathbb{E}_0$-class is countable dense and $\mathbb{E}_0$ is induced by a continuous action of a countable group as described in Example 17.5(d). Moreover, like in the previous two examples, we can even view $\mathbb{E}_0$ as the orbit equivalence relation induced by the translation action of a countable dense subgroup $\Gamma < (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, namely, $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{\oplus\mathbb{N}} := \left\{q \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} : \forall^\infty n \; q(n) = 0\right\}$.

It follows from Silver's dichotomy (or Mycielski's theorem for $\mathbb{E}_v, \mathbb{E}_\alpha, \mathbb{E}_0$) that all Borel nonsmooth equivalence relations E are strictly above $\mathrm{Id}(\mathcal{C})$ in the Borel reducibility hierarchy; in particular, $\mathrm{Id}(\mathcal{C}) <_B \mathbb{E}_v, \mathbb{E}_\alpha, \mathbb{E}_0$.

20.E. **General Glimm–Effros dichotomies.** For an equivalence relation E on a Polish space X, it is clear that if $\mathbb{E}_0 \leqslant_B E$ then E is nonsmooth. The following striking theorem shows that this is the only impediment to smoothness!

Dichotomy 20.22 (Harrington–Kechris–Louveau '90)**.** *For any Borel equivalence relation E on a Polish space X, either E is smooth or $\mathbb{E}_0 \sqsubseteq_c E$.*

This theorem shows, in particular, that $\mathbb{E}_0$ is the $\leqslant_B$-minimum element (up to $\sim_B$) among all nonsmooth Borel equivalence relations; in other words, it is the minimum unsolvable classification problem among the Borel ones. Moreover, an application of Mycielski's theorem to $\mathbb{E}_0$ gives $\mathrm{Id}(\mathcal{C}) <_B \mathbb{E}_0 \sqsubseteq_c E$, for any Borel nonsmooth equivalence relation E.

Because orbit equivalence relations induced by continuous actions of Polish groups are in general analytic, the Harrington–Kechris–Louveau dichotomy doesn't apply to them. However, the following theorem shows that for a large class of orbit equivalence relations the dichotomy still holds. Before we state it, recall Corollary 20.10, which states whenever the action is such that every orbit is G_δ, then the orbit equivalence relation is smooth.

Dichotomy 20.23 (Becker–Kechris '9?)**.** *Let a Polish group G act continuously on a Polish space X so that every G_δ orbit is also F_σ. Then either every orbit is G_δ and hence E_G is smooth, or else, $\mathbb{E}_0 \sqsubseteq_c E_G$.*

We will prove the latter theorem in the next subsection. The proof of the Harrington–Kechris–Louveau dichotomy is somewhat harder as it involves the *Gandy–Harrington topology*[33], however the construction of the embedding of $\mathbb{E}_0$ is similar in spirit to that in the Becker–Kechris theorem.

[33] This is a topology on $\mathcal{N}$ defined by the means of recursion theory and it is finer than the usual Polish topology.

The reader may be wondering why the above theorems are referred to as *generalized Glimm–Effros dichotomies*. This is because they generalize the following dichotomy of Effros, which in its turn, supersedes the theorem of Glimm below.

Dichotomy 20.24 (Effros)**.** *Let a Polish group G act continuously on a Polish space X so that the orbit equivalence relation E_G is F_σ. Then either E_G is smooth, or else, $\mathbb{E}_0 \sqsubseteq_c E_G$.*

Dichotomy 20.25 (Glimm)**.** *Let a locally compact Polish group G act continuously on a Polish space X. Then either E_G is smooth, or else, $\mathbb{E}_0 \sqsubseteq_c E_G$.*

To see why Effros's dichotomy implies Glimm's, first note that every locally compact Polish space is σ-compact[34], so if G is a locally compact Polish group, we can write $G = \bigcup_n K_n$, where each K_n is compact. Also, if $a : G \times X \to X$ is the map of the action, then

$$E_G = \mathrm{proj}_{2,3}(\mathrm{Graph}(a)) = \mathrm{proj}_{2,3}(\bigcup_n \mathrm{Graph}(a)|_{K_n}) = \bigcup_n \mathrm{proj}_{2,3}(\mathrm{Graph}(a)|_{K_n}),$$

where $\mathrm{Graph}(a)|_{K_n} := \mathrm{Graph}(a) \cap (K_n \times X^2)$. By the continuity of the action, $\mathrm{Graph}(a) \subseteq G \times X^2$ is closed and hence $\mathrm{proj}_{2,3}(\mathrm{Graph}(a)|_{K_n})$ is closed as well, by the tube lemma. Therefore, E_G is F_σ.

20.F. **Proof of the Becker–Kechris dichotomy.** First, we reduce the Becker–Kechris dichotomy to proving the following analogue of Mycielski's theorem for orbit equivalence relations[35]:

Theorem 20.26 (Becker–Kechris)**.** *Let E_G be the orbit equivalence relation induced by a continuous action of a Polish group G on a Polish space X. If E_G is meager and there is a dense orbit*[36]*, then $\mathbb{E}_0 \sqsubseteq_c E_G$.*

Let us explain how the Becker–Kechris dichotomy boils down to this.

Reduction of the Becker–Kechris dichotomy to Theorem 20.26. Let $G \curvearrowright X$ and E_G be as in the statement of the dichotomy. If every orbit is G_δ, we are done; so suppose there is an orbit $[x]_G$ that is not G_δ.

Claim 1. We may assume without loss of generality that $[x]_G$ is dense.

Proof of Claim. Let $Y = \overline{[x]_G}$. Note that Y is invariant because if $x_n \to y$ for $x_n \in [x]_G$, $y \in Y$, and $g \in G$, then $gx_n \to gy$ by the continuity of the action, and hence $gy \in Y$ as well. Thus we may assume $X = Y$ to start with. ⊠

Claim 2. We may assume without loss of generality that every orbit is dense.

Proof of Claim. Let

$$Z = \{y \in X : [y]_G \text{ is dense in } X\}.$$

This set is G_δ because fixing a countable basis $\{U_n\}_n$ of nonempty open sets, we see that for $y \in X$,

$$y \in Z \iff \forall n([y]_G \cap U_n \neq \emptyset) \iff \forall n(y \in [U_n]_G),$$

and $[U_n]_G = \bigcup_{g \in G} gU_n$ is open. Clearly Z is invariant and $[x]_G \subseteq Z$, so, by moving from X to Z we have achieved that every orbit is dense. Note that an orbit in Z is G_δ relative to X

[34] This is because there is a countable basis of precompact open sets, so their closures cover the space.

[35] This is what is often referred to as the *Becker–Kechris theorem*.

[36] Recall that, by Proposition 20.19, this is equivalent to E_G being generically ergodic.

if and only if it is G_δ relative to Z, so we still have that $[x]_G$ is not G_δ in Z, and every G_δ orbit in Z is also F_σ relative to Z. Hence, we may assume that $Z = X$ to start with. ⊠

Claim 3. No orbit is G_δ.

Proof of Claim. If there was a G_δ orbit $[z]_G \subseteq X$, then it would be different from $[x]_G$, so $[x]_G \subseteq X \setminus [z]_G$, and hence $X \setminus [z]_G$ is dense. Moreover, by the hypothesis, $[z]_G$ is also F_σ, so $X \setminus [z]_G$ is G_δ. But then both $[z]_G$ and $X \setminus [z]_G$ are dense G_δ, contradicting the Baire category theorem. ⊠

We now invoke (without proof) the following surprising characterization of when exactly an orbit is G_δ (the proof is not very hard, see [Gao09, Theorem 3.2.4]).

Theorem 20.27 (Effros). *Let $G \curvearrowright X$ be a continuous action of a Polish group G on a Polish space X. For every $x \in X$, $[x]_G$ is G_δ if and only if $[x]_G$ is not meager in itself (i.e. in the relative topology of $[x]_G$).*

Note that the forward direction simply follows from the fact that G_δ subsets are Polish and hence Baire, but what is surprising is that for orbits the converse is also true.

This theorem together with the last claim implies that every orbit is meager in itself, which then implies that it is meager in X since being meager transfers upward (see Part (a) of Proposition 6.7). But E_G is Baire measurable (being analytic), so by the Kuratowski–Ulam theorem, it must be meager, and hence, by Theorem 20.26, $\mathbb{E}_0 \sqsubseteq_c E_G$. □

Proof of Theorem 20.26. The proof is similar to that of Mycielski's theorem with the an extra complication coming from the complexity of $\mathbb{E}_0$ over that of $\mathrm{Id}(\mathcal{C})$. Write $E_G = \bigcup_n F_n$, where each $F_n \subseteq X \times X$ is symmetric and nowhere dense, and $F_n \subseteq F_{n+1}$. In order to get a desired embedding $\mathcal{C} \hookrightarrow X$, we will construct a sequence $(g_{s,t})_{s,t \in 2^n, n \in \mathbb{N}} \subseteq G$ and a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ of vanishing diameter (with respect to a fixed complete metric d for X) with the following properties for all $n \in \mathbb{N}$, $s, t \in 2^n$, and $i \in \{0, 1\}$:

(i) U_s is nonempty open and $\overline{U_{s^\frown i}} \subseteq U_s$;
(ii) $(U_{s^\frown 0} \times U_{t^\frown 1}) \cap F_n = \emptyset$;
(iii) $g_{s^\frown p, t^\frown p} = g_{s,t}$, for all $p \in 2^{<\mathbb{N}}$;
(iv) $g_{s,t} U_s = U_t$.

Because of (iv), we refer to the group elements $g_{s,t}$ as *links*. Granted this construction, let $f : \mathcal{C} \rightharpoonup X$ be the associated map. By Proposition 5.4, the domain of f is all of $\mathcal{C}$, and f is continuous and injective. Hence f is a topological embedding since $\mathcal{C}$ is compact. To show that f is also a reduction of $\mathbb{E}_0$ to E, fix $x, y \in \mathcal{C}$.

Suppose $x \mathbb{E}_0 y$. Then $x = s^\frown z$ and $y = t^\frown z$, for some $s, t \in 2^n$ and $z \in \mathcal{C}$. By (iii) and (iv), we have $g_{s,t} U_{x|_m} = U_{y|_m}$ for all $m \geqslant n$, so $g_{s,t} f(x) = f(y)$ and hence $f(x) E_G f(y)$.

Now suppose $x \not\mathbb{E}_0 y$. Then for infinitely many $n \in \mathbb{N}$, we have $x|_{n+1} = (x|_n)^\frown i$ and $y|_{n+1} = (y|_n)^\frown \bar{i}$, so (ii) and the symmetry of F_n yield $(U_{x|_{n+1}} \times U_{y|_{n+1}}) \cap F_n = \emptyset$, and hence $(f(x), f(y)) \notin F_n$. Since the sequence $(F_n)_n$ is increasing, it follows that $(f(x), f(y)) \notin \bigcup_n F_n = E_G$.

We now turn to the construction of $(g_{s,t})_{s,t \in 2^n, n \in \mathbb{N}}$ and $(U_s)_{s \in 2^{<\mathbb{N}}}$. To make the construction of $(g_{s,t})_{s,t \in 2^n, n \in \mathbb{N}}$ easier, we additionally enforce that for all $n \in \mathbb{N}$ and $s, t, p \in 2^n$, we have

(v) $g_{s,s} = 1_G$, $g_{s,t}^{-1} = g_{t,s}$ and $g_{s,p} = g_{t,p} g_{s,t}$.

Putting $g_{\emptyset,\emptyset} := 1_G$ and $U_\emptyset := X$, assume inductively that for $n \in \mathbb{N}$, the sequences $(g_{s,t})_{s,t\in 2^n}$ and $(U_s)_{s\in 2^n}$ have been defined and satisfy all of the conditions. Right away, condition (iii) forces us to define

- $g_{s^\frown i,t^\frown i} := g_{s,t}$, for all $s,t \in 2^n$ and $i \in \{0,1\}$.

Next, note (as we did in Mycielski's theorem) that the meagerness of E_G implies that X is perfect. This allows us to find first approximations of $U_{(0^n)^\frown 0}$ and $U_{(0^n)^\frown 1}$, namely, disjoint nonempty open sets $U, V \subseteq_c U_{0^n}$ of diameter at most 2^{-n}. To address condition (ii), we will use the following fact, which is immediate from definitions.

Fact 20.28. *For nowhere dense $F \subseteq X^2$, nonempty open $U, V \subseteq X$ and $g,h \in G$, there are nonempty open $U' \subseteq U$ and $V' \subseteq V$ such that $(gU' \times hV') \cap F = \emptyset$.*

For $s,t \in 2^n$, thinking of $g_{(0^n)^\frown 0,s^\frown 0} U$ and $g_{(0^n)^\frown 1,t^\frown 1} V$ as approximations of $U_{s^\frown 0}$ and $U_{t^\frown 1}$, shrink U, V by iteratively applying Fact 20.28 to $g_{(0^n)^\frown 0,s^\frown 0} U$, $g_{(0^n)^\frown 1,t^\frown 1} V$, and F_n, and achieve

$$(g_{(0^n)^\frown 0,s^\frown 0} U \times g_{(0^n)^\frown 1,t^\frown 1} V) \cap F_n = \emptyset, \text{ for all } s,t \in 2^n.$$

Having addressed (ii), we use the existence of a dense orbit to get $g \in G$ with $gU \cap V \neq \emptyset$ (see Proposition 20.19). We are now in a position to define

- $g_{(0^n)^\frown 0,(0^n)^\frown 1} := g$,
- $U_{0^{n+1}} := U \cap g^{-1} V$,

so $U_{0^{n+1}} \subseteq U$ and $g_{(0^n)^\frown 0,(0^n)^\frown 1} U_{0^{n+1}} \subseteq V$. Finally, conditions (v) and (iv) force us to define the rest as follows:

- $g_{(0^n)^\frown 1,(0^n)^\frown 0} := g^{-1}_{(0^n)^\frown 0,(0^n)^\frown 1}$,
- $g_{s^\frown i,t^\frown \bar{i}} := g_{(0^n)^\frown \bar{i},t^\frown \bar{i}}\, g_{(0^n)^\frown i,(0^n)^\frown \bar{i}}\, g_{s^\frown i,(0^n)^\frown i}$,
- $U_p = g_{(0^n)^\frown 0,p} U_{(0^n)^\frown 0}$,

for all $s,t \in 2^n$, $p \in 2^{n+1}$ and $i \in \{0,1\}$. It follows from the definitions that

$$U_{s^\frown 0} \subseteq g_{(0^n)^\frown 0,s^\frown 0} U \text{ and } U_{t^\frown 1} \subseteq g_{(0^n)^\frown 0,t^\frown 1} U_{0^{n+1}} = g_{(0^n)^\frown 1,t^\frown 1} g_{(0^n)^\frown 0,(0^n)^\frown 1} U_{0^{n+1}} \subseteq g_{(0^n)^\frown 1,t^\frown 1} V,$$

so $(U_{s^\frown 0} \times U_{t^\frown 1}) \cap F_n = \emptyset$ by above, hence condition (ii) is fulfilled. □

21. Definable graphs and colorings

The study of definable equivalence relations is tightly connected to the study of definable graphs and chromatic numbers.

21.A. Definitions and examples.

We think of graphs as sets of edges, more precisely:

Definition 21.1. A directed graph $\mathcal{G}$ on a set X is just a relation on X. We call it *undirected* (or just a *graph*) if it is irreflexive and symmetric.

Just like with equivalence relations, we can define the notion of homomorphisms between graphs as follows.

Definition 21.2. Let $\mathcal{G}, \mathcal{H}$ be graphs (directed or undirected) on sets X, Y, respectively. A function $f : X \to Y$ is called a *homomorphism from $\mathcal{G}$ to $\mathcal{H}$* if for all $x_0, x_1 \in X$,

$$x_0 \mathcal{G} x_1 \Rightarrow f(x_0) \mathcal{H} f(x_1).$$

We write $\mathcal{G} \to \mathcal{H}$ to mean that there is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$, and we add a subscript $\to_c$ (resp. $\to_{\mathcal{B}}$) to mean that there is a continuous (resp. Borel) homomorphism.

Here are some examples of Borel or analytic graphs on Polish spaces.

Examples 21.3.

(a) *Generation by a function*. Let X be a Polish space and $f : X \to X$ a Borel function. Then $\mathcal{G}_f = \mathrm{Graph}(f)$ is a Borel graph on X. This is a directed graph with the property that each vertex has exactly one outgoing edge.

(b) *Generation by a semigroup action*. A pointed semigroup is a pair (Γ, S), where Γ is a semigroup and $S \subseteq \Gamma$ is a generating set for Γ. To this we associate a directed graph, called the *Cayley graph* and denoted by $\mathrm{Cay}(\Gamma, S)$, defined as follows: for $\gamma, \delta \in \Gamma$,

$$\gamma \,\mathrm{Cay}(\Gamma, S) \delta :\Leftrightarrow \exists \sigma \in S (\sigma\gamma = \delta).$$

Letting X be a Polish space, consider an action $\Gamma \curvearrowright X$ by Borel functions. The Cayley graph $\mathrm{Cay}(\Gamma, S)$ induces a directed graph $\mathcal{G}_\Gamma$ on X as follows: for $x, y \in X$,

$$x\mathcal{G}_\Gamma y \iff \exists \sigma \in S (\sigma \cdot x = y).$$

Clearly, if S is countable, then $\mathcal{G}_\Gamma$ is Borel. Note that this generalizes the previous example, taking $\Gamma = \mathbb{N}$ and $S = \{1\}$. Note that each connected component need not be a homomorphic image of $\mathrm{Cay}(\Gamma, S)$; take a surjective but noninjective function f in the previous example, then the connected components do not have a "beginning" (unlike $\mathbb{N}$) due to surjectivity and they are not just lines/chains due to noninjectivity.

(c) *Generation by a group action*. An important special case of the previous example is when we have a Borel action of a pointed group (Γ, S) on a Polish space X, where S is symmetric (i.e. $S^{-1} = S$) and $1_\Gamma \notin S$. In this case, $\mathrm{Cay}(\Gamma, S)$, and hence also $\mathcal{G}_\Gamma$, are undirected; moreover, the connected components of $\mathcal{G}_\Gamma$ are precisely the orbits of the action, and each connected component $[x]_\Gamma$ is indeed a homomorphic image of $\mathrm{Cay}(\Gamma, S)$ by the map $\gamma \mapsto \gamma \cdot x$. In fact, if the action is free, then each connected component is isomorphic to $\mathrm{Cay}(\Gamma, S)$.

For example, if $\Gamma = \mathbb{Z}$ and $S = \{\pm 1\}$, then $\mathcal{G}_\mathbb{Z}$ is a collection of lines or cycles (there won't be cycles if the action is free). More generally, if Γ is the free group $\mathbb{F}$ on $n \leqslant \omega$ generators and S is the canonical symmetric generating set, then $\mathrm{Cay}(\mathbb{F}_n, S)$ is a $2n$-regular tree (because $|S| = 2n$). Thus, if the action of $\mathbb{F}_n \curvearrowright X$ is free, then $\mathcal{G}_{\mathbb{F}_n}$ is a forest of $2n$-regular trees.

Finally, note that if Γ is a Polish group and S is Borel, then the action map $a : G \times X \to X$ being Borel implies that $\mathcal{G}_\Gamma$ is analytic. If S is countable, then it is actually Borel.

(d) *Generation by a metric*. A metric d on a Polish space X generates an undirected graph $\mathcal{G}_d$ on X as follows: for $x, y \in X$,

$$x\mathcal{G}_d y \iff d(x, y) = 1.$$

Clearly, $\mathcal{G}_d$ is closed.

The examples above already reveal some connection between graphs and equivalence relations. More generally, any equivalence relation E on a Polish space X can be thought of as undirected graph after subtracting the diagonal; namely, let $\mathcal{G}_E = E \setminus \mathrm{Id}(X)$. Conversely, any graph $\mathcal{G}$ on X induces the equivalence relation $E_\mathcal{G}$ of being in the same connected component, i.e. for $x, y \in X$,

$$xE_\mathcal{G} y \iff \exists n \in \mathbb{N}\, \exists z \in X^n [z(0) = x \wedge z(n-1) = y \wedge \forall i < n-1 (x_i \mathcal{G} x_{i+1} \vee x_{i+1} \mathcal{G} x_i)].$$

It is clear from the definition that if $\mathcal{G}$ is analytic, then so is $E_{\mathcal{G}}$.

21.B. **Chromatic numbers.**

Definition 21.4. For a graph $\mathcal{G}$ on a set X, a function $c : X \to Z$, for some set Z, is called a *coloring of $\mathcal{G}$* if for all $x, y \in X$,

$$x\mathcal{G}y \Rightarrow c(x) \neq c(y).$$

Letting $\mathcal{G}_C(Z)$ denote the complete undirected graph on Z, i.e. $\mathcal{G}_C(Z) = Z^2 \setminus \mathrm{Id}(Z)$, we note that $c : X \to Z$ is a coloring of $\mathcal{G}$ if it is a homomorphism from $\mathcal{G}$ to $\mathcal{G}_C(Z)$.

We refer to Z as the *set of colors* (or the *color set*) and we call this function c a *Z-coloring* if we want to emphasize the color set. For each $z \in Z$, the set $c^{-1}(z)$ is referred to as the *set of vertices having color z*. Note that every set $c^{-1}(z)$ is *independent*, i.e. there are no edges between the vertices in $c^{-1}(z)$, i.e. $\mathcal{G}|_{c^{-1}(z)} = \emptyset$.

Note that the identity function on X is always a coloring for any graph on X. The question is: can we do better? Namely, find a coloring $c : X \to Z$ with $|Z| < |X|$.

In descriptive set theory, we are concerned with colorings from certain classes $\mathbf{\Gamma}$ of functions that have additional regularity properties, e.g. Borel, Baire measurable, μ-measurable (for some Borel measure μ on X), etc. We refer to these as $\mathbf{\Gamma}$-*colorings*.

Definition 21.5. Let $\mathbf{\Gamma}$ be a class of functions between Polish spaces. For a graph $\mathcal{G}$ on a Polish space X, define its $\mathbf{\Gamma}$ *chromatic number* $\chi_{\mathbf{\Gamma}}(\mathcal{G})$ as the smallest cardinality of a Polish space Z for which there is a $\mathbf{\Gamma}$ coloring $c : X \to Z$. In particular, the *Borel chromatic number of $\mathcal{G}$* is denoted by $\chi_{\mathcal{B}}(\mathcal{G})$.

Note that by the perfect set property of Polish spaces, the only possible chromatic numbers are $0, 1, 2, \ldots, \aleph_0, 2^{\aleph_0}$.

The usual notion of chromatic number from combinatorics coincides with $\mathbf{\Gamma}$ being the class of all functions, and we will refer to this as just *the chromatic number*. Depending on $\mathbf{\Gamma}$, the $\mathbf{\Gamma}$ chromatic number may be different for the same graph. For example, the chromatic number for any acyclic graph $\mathcal{G}$ is 2 because one would just select one vertex in every connected component (the resulting set S will be a transversal for $E_{\mathcal{G}}$), and color by red (resp. blue) all vertices whose graph-distance from S is even (resp. odd).[37] However, this algorithm does not yield a Borel, or even Baire or μ-measurable, coloring because it involves choosing a point from every connected component, which, as we already know, cannot always be done definably (e.g. $\mathbb{E}_v$, $\mathbb{E}_0$, $\mathbb{E}_\alpha$). In fact, the following example shows that it cannot be done even for a simple graph such as a forest of $\mathbb{Z}$-lines.

21.6. *Irrational rotation is not Borel 2-colorable.* Consider an irrational rotation $T_\alpha : S^1 \to S^1$ of the unit circle, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and T_α acts as multiplication by $e^{2\pi\alpha i}$. This is a special case of example (a) above and we let $\mathcal{G}_\alpha$ denote the induced graph. Since the action of T_α is free, each connected component of $\mathcal{G}_\alpha$ is just a $\mathbb{Z}$-line, so the usual chromatic number of $\mathcal{G}_\alpha$ is 2.

However, we claim that the Baire measurable chromatic number is bigger than 2! Indeed, suppose for contradiction that there is a Baire measurable $\mathbb{Z}$-coloring $c : S^1 \to \{0, 1\}$, and let $A = c^{-1}(0)$, so $A^c = c^{-1}(1)$. Note that $T_\alpha(A) = A^c$, so A and A^c are homeomorphic. Moreover, they are Baire measurable and are invariant under the action of $T_\alpha^2 = T_{2\alpha}$. But 2α is still irrational, so the action of $T_{2\alpha}$ is still generically ergodic (see Example 20.21(b)), so both A

[37] Note that this algorithm works also for graphs with no odd cycles.

and A^c have to be meager or comeager simultaneously, contradicting the Baire category theorem.

Similarly, one can also show that the Lebesgue measurable chromatic number of $\mathcal{G}_\alpha$ is bigger than 2 using the ergodicity of T_α with respect to the Lebesgue measure on S^1.

Nevertheless, one can construct a Borel 3-coloring of $\mathcal{G}_\alpha$ (as outlined in homework exercises), showing that the Borel (as well as Baire/Lebesgue measurable) chromatic number is precisely 3.

21.C. $\mathcal{G}_0$**—the graph cousin of** $\mathbb{E}_0$**.** In the sequel, we will define a graph counterpart of $\mathbb{E}_0$ on $\mathcal{C}$ in the sense that $E_{\mathcal{G}_0} = \mathbb{E}_0$. We will show that even though $\mathcal{G}_0$ is acyclic, it's Borel (or even Baire or μ-measurable) chromatic number is $2^{\aleph_0}$.

For $S \subseteq 2^{<\mathbb{N}}$, define the graph $\mathcal{G}_S$ on $\mathcal{C}$ by

$$\mathcal{G}_S := \left\{ (s^\frown i^\frown z, s^\frown \bar{i}^\frown z) : s \in S, i \in \{0,1\}, z \in \mathcal{C} \right\},$$

where $\bar{i} := 1 - i$. In other words, we use elements of S as portals to flip the next bit. In particular, each edge in $\mathcal{G}_0$ is associated with a unique $s \in S$.

Note that for $x, y \in \mathcal{C}$,

$$x \mathcal{G}_S y \iff \exists s \in S [x, y \supseteq s \wedge x(|s|) \neq y(|s|) \wedge \forall n > |s| (x(n) = y(n))],$$

so $\mathcal{G}_S$ is F_σ and $E_{\mathcal{G}_S} \subseteq \mathbb{E}_0$. Also note that $\mathcal{G}_S$ *does not have any cycles of odd length*; indeed, if $x_0, x_1, \ldots, x_n$, $x_0 = x_n$ is a cycle in $\mathcal{G}_S$, then in order to start with x_0 and come back to it, each bit needs to be flipped even number of times and hence the number of edges n must be even.

The next three lemmas demonstrate how various properties of S affect $\mathcal{G}_S$.

Lemma 21.7. *If S contains at most one $s \in 2^{<\mathbb{N}}$ of every length, then $\mathcal{G}_S$ is acyclic.*

Proof. Assume for contradiction that there is a cycle (with no repeating vertex) and consider the longest $s \in S$ associated with its edges. We leave the details as an exercise. □

Lemma 21.8. *If S contains at least one $s \in 2^{<\mathbb{N}}$ of every length, then $E_{\mathcal{G}_S} = \mathbb{E}_0$.*

Proof. For each $n \in \mathbb{N}$, show by induction on n that for each $s, t \in 2^n$ and $x \in \mathcal{C}$, there is a path in $\mathcal{G}_0$ from $s^\frown x$ to $t^\frown x$, i.e. $s^\frown x$ can be transformed to $t^\frown x$ by a series of appropriate bit flips. We leave the details as an exercise. □

Call a set $S \subseteq 2^{<\mathbb{N}}$ *dense* if for every $t \in 2^{<\mathbb{N}}$ there is $s \in S$ with $s \supseteq t$.

Lemma 21.9. *If S is dense, then for every nonmeager Baire measurable $A \subseteq \mathcal{C}$, $\mathcal{G}_S|_A \neq \emptyset$, i.e. there are $x, y \in A$ with $x \mathcal{G}_S y$. The analogous statement is true for a μ-measurable $A \subseteq \mathcal{C}$ of positive measure, where μ is the fair coin flip measure (i.e. the Haar measure) on $\mathcal{C}$.*

Proof. We only prove the Baire category statement as the proof of the measure-theoretic statement is analogous (using the Lebesgue density theorem instead of the Baire alternative). By the Baire alternative, there is a nonempty open $U \subseteq \mathcal{C}$ with $U \Vdash A$. Because S is dense, there is $s \in S$ such that $N_s \subseteq U$, so $N_s \Vdash A$. Define a *bit-flip map* $f : N_s \to N_s$ by $s^\frown i^\frown z \mapsto s^\frown \bar{i}^\frown z$. Clearly f is a homeomorphism of N_s, so $N_s \Vdash f(A)$ as well, and hence there is $y \in A \cap f(A)$, so $y = f(x)$ for some $x \in A$. But $x \mathcal{G}_S y$ and both $x, y \in A$. □

Corollary 21.10. *If S is dense, then the Baire measurable (as well as μ-measurable) chromatic number of $\mathcal{G}_S$ is $2^{\aleph_0}$.*

Proof. Assume for contradiction that $c : \mathcal{C} \to \mathbb{N}$ is a Baire measurable coloring. Then for each $n \in \mathbb{N}$, $c^{-1}(n)$ is Baire measurable and $\mathcal{G}_S$-independent (i.e. $\mathcal{G}_S|_{c^{-1}(n)} = \emptyset$). But one of $c^{-1}(n)$ has to be nonmeager, contradicting the previous lemma. $\square$

Thus, for $\mathcal{G}_S$ to have all of the above properties, we need $S \subseteq 2^{<\mathbb{N}}$ to be dense and contain exactly one element of each length. Here is how to define such a set S: enumerate $(t_n)_n \subseteq 2^{<\mathbb{N}}$ so that $|t_n| \leqslant n$, and for each $n \in \mathbb{N}$, choose $s_n \in 2^n$ extending t_n. It is clear then that $S = \{s_n\}_n$ is as desired. For this S, we write $\mathcal{G}_0$ for $\mathcal{G}_S$.

To summarize, $\mathcal{G}_0$ is an acyclic F_σ graph on $\mathcal{C}$, $E_{\mathcal{G}_0} = \mathbb{E}_0$, and the Baire measurable (as well as μ-measurable) chromatic number of $\mathcal{G}_0$ is $2^{\aleph_0}$.

21.D. **The Kechris–Solecki–Todorčević dichotomy.** Letting $\mathbf{\Gamma}$ denote a class of functions between Polish spaces closed under composition, note that for two graphs $\mathcal{G}, \mathcal{H}$ on Polish spaces X, Y, respectively, if there is a $\mathbf{\Gamma}$ graph homomorphism $f : X \to Y$ from $\mathcal{G}$ to $\mathcal{H}$ then $\chi_{\mathbf{\Gamma}}(\mathcal{G}) \leqslant \chi_{\mathbf{\Gamma}}(\mathcal{H})$ because for any $\mathbf{\Gamma}$ coloring $c : Y \to Z$ of $\mathcal{H}$, the composition $c \circ f$ is a $\mathbf{\Gamma}$ coloring of $\mathcal{G}$. In particular, if there is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{H}$, then the Baire measurable chromatic number of $\mathcal{H}$ is $2^{\aleph_0}$. The following dichotomy shows that for analytic graphs this is the only obstruction to being countably Borel colorable.

The $\mathcal{G}_0$-dichotomy 21.11 (Kechris–Solecki–Todorčević)**.** *For any analytic graph $\mathcal{G}$ on a Polish space X, either $\chi_{\mathcal{B}}(\mathcal{G}) \leqslant \aleph_0$, or else, $\mathcal{G}_0 \to_c \mathcal{G}$ (and hence $\chi_{\mathcal{B}}(\mathcal{G}) = 2^{\aleph_0}$).*

The original proof of this dichotomy (see [KST99]) used basic recursion theory via the Gandy–Harrington topology. Later on in 2008, Ben Miller found a classical proof using only Baire category arguments. We won't give this proof in these notes, but it can be found in [Mil09].

22. Some corollaries of the $\mathcal{G}_0$-dichotomy

In the next two subsections, we show how the $\mathcal{G}_0$-dichotomy implies Silver's dichotomy as well as the Luzin–Novikov theorem. These implications are due to Ben Miller [Mil09, Theorem 11 and Exercise 19]. In the last subsection, we will prove the Feldman–Moore theorem using the Luzin–Novikov theorem.

22.A. **Proof of Silver's dichotomy.** Let us first recall the theorem:

Dichotomy (Silver '80)**.** *Any co-analytic equivalence relation E on a Polish space X has either countably many or perfectly many classes. In other words, either $E \leqslant_B \mathrm{Id}(\mathbb{N})$, or $\mathrm{Id}(\mathcal{C}) \leqslant_B E$.*

To prove this dichotomy, note that $\mathcal{G} = E^c$ is an undirected analytic graph on X and apply the Kechris–Solecki–Todorčević dichotomy to $\mathcal{G}$.

Case 1: $\chi_{\mathcal{B}}(\mathcal{G}) \leqslant \aleph_0$. Note that for $x, y \in X$, if $[x]_E \neq [y]_E$ then $x \mathcal{G} y$. Thus, taking a transversal $Y \subseteq X$ for E (using AC) and letting $c : X \to \mathbb{N}$ be a Borel coloring of $\mathcal{G}$, we see that $\mathcal{G}|_Y$ is the complete graph on Y and hence $c|_Y$ is injective. Thus, Y must be countable, and hence, so is X/E.

Case 2: $\exists \varphi : \mathcal{G}_0 \to_c \mathcal{G}$. Let E' be the pullback of E via the map φ, i.e. $E' = (\varphi \times \varphi)^{-1}(E)$. Note that E' is an equivalence relation on $\mathcal{C}$ and, by definition, the map φ is a continuous reduction of E' to E.

Claim. E' is meager.

Proof of Claim. Otherwise, by Kuratowski–Ulam, one of the E'-equivalence classes $C \subseteq \mathcal{C}$ must be nonmeager, so by Lemma 21.9, there are $x, y \in C$ such that $x\mathcal{G}_0 y$ and hence $\varphi(x)\mathcal{G}\varphi(y)$ because φ is a graph homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$. One the other hand, $xE'y$ implies $\varphi(x)E\varphi(y)$ because φ is a reduction of E' to E, contradicting $\mathcal{G} = E^c$. ⊠

This claim allows us to apply Mycielski's theorem to E' and get $\mathrm{Id}(\mathcal{C}) \sqsubseteq_c E' \leqslant_c E$. Thus, $\mathrm{Id}(\mathcal{C}) \leqslant_c E$, which concludes the proof of Silver's theorem.

22.B. **Proof of the Luzin–Novikov theorem.** To prove the Luzin–Novikov theorem, we will need the following uniform version of the Kechris–Solecki–Todorčević dichotomy:

Uniform $\mathcal{G}_0$-dichotomy 22.1. *Let X, Y be Polish spaces and $\mathcal{G} \subseteq X \times Y^2$ be an analytic set, whose every X-fiber is an undirected graph, i.e. $\mathcal{G}_x$ is an undirected graph for every $x \in X$. Then*

either: there is a Borel function $c : X \times Y \to \mathbb{N}$ so that $c_x := c(x, \cdot)$ is a coloring for $\mathcal{G}_x$, for every $x \in X$;

or else: $\mathcal{G}_0 \to_c \mathcal{G}_{x_0}$, for some $x_0 \in X$.

Proof. Let $\mathcal{G}'$ be the graph on $X \times Y$ defined as follows: for $(x, y), (x', y') \in X \times Y$,

$$(x, y)\mathcal{G}'(x', y') :\Leftrightarrow x = x' \wedge y\mathcal{G}_x y'.$$

In particular, $\mathcal{G}'$ is an undirected analytic graph and we apply the $\mathcal{G}_0$-dichotomy to $\mathcal{G}'$. It is clear that if $c : X \times Y \to \mathbb{N}$ is a countable Borel coloring of $\mathcal{G}'$, then c_x is a coloring for $\mathcal{G}_x$, for every $x \in X$, so we are done. Thus, assume that we have the other option, namely, $\varphi : \mathcal{G}_0 \to_c \mathcal{G}'$. Note that the φ-image of each $\mathbb{E}_0$-class has to be contained in one X-fiber of $\mathcal{G}$ because $E_{\mathcal{G}_0} = \mathbb{E}_0$ and connected components have to map to connected components. Hence, the function $\mathrm{proj}_1 \circ \varphi$ is constant on each $\mathbb{E}_0$-class. But each $\mathbb{E}_0$-class is dense in $\mathcal{C}$ and $\mathrm{proj}_1 \circ \varphi$ is continuous, so $\mathrm{proj}_1 \circ \varphi$ must be a constant function. Letting x_0 be its unique value concludes the proof. □

Proof of the Luzin–Novikov Theorem 13.6. Define $\mathcal{G} \subseteq X \times Y^2$ so that for each $x \in X$, $\mathcal{G}_x$ is the complete graph on B_x, i.e.

$$y_0 \mathcal{G}_x y_1 :\Leftrightarrow y_0 \neq y_1 \wedge y_0 \in B_x \wedge y_1 \in B_x.$$

Clearly $\mathcal{G}$ is Borel, so the uniform $\mathcal{G}_0$-dichotomy applies. If it is the first option, i.e. there is a Borel $c : X \times Y \to \mathbb{N}$ such that c_x is a coloring of $\mathcal{G}_x$, for every $x \in X$, then $B_n = c^{-1}(n)$ is as desired. It remains to show that the second option can never happen; indeed, if for some $x_0 \in X$ we had $\mathcal{G}_0 \to_c \mathcal{G}_{x_0}$, then $\mathcal{G}_0$ would be countably Borel colorable since $\mathcal{G}_{x_0}$ is countable, a contradiction. □

22.C. **The Feldman–Moore theorem and $\mathbb{E}_\infty$.** The following is one of the most important applications of the Luzin–Novikov theorem.

Theorem 22.2 (Feldman–Moore)**.** *Every countable Borel equivalence relation E on a Polish space X is a countable union of graphs of Borel involutions on X, i.e. $E = \bigcup_{n \in \mathbb{N}} \mathrm{Graph}(\gamma_n)$, where each $\gamma_n : X \to X$ is a Borel involution. In particular, E is the orbit equivalence relation of a Borel action of a countable group generated by involutions.*

Proof. Here we give a rather classical proof, but we outline a more modern (graph theoretic) proof in the exercises due to Anton Bernshteyn.

First let us fix some notation. For $R \subseteq X^2$, put $R^{-1} := \{(x, y) : (y, x) \in R\}$. We view a Borel function graph $f \subseteq X^2$ as a partial function $f : X \rightharpoonup X$ with domain $\mathrm{dom}(f) := \mathrm{proj}_1(f)$. By the Luzin–Souslin Theorem 13.3, $\mathrm{dom}(f)$ is a Borel set and f is a Borel function on it. Its range $\mathrm{ran}(f) := \mathrm{proj}_2(f)$ is analytic in general, but it is Borel if f is injective.

Now Luzin–Novikov allows us to write E as a disjoint union of Borel partial functions $E = \bigsqcup_n f_n$. We will use these partial functions to build a set of Borel involutions of X, and the group generated by them will be the desired group Γ with its natural action on X.

Note that these f_n may not be injective. We fix this by noting that $E = E^{-1} = \bigsqcup_m f_m^{-1}$, so by replacing $(f_n)_n$ with $(f_n \cap f_m^{-1})_{n,m}$, we may assume without loss of generality that each f_n is injective.

Next, we would like to extend each f_n to a Borel involution of X. We could do so if $\mathrm{dom}(f_n)$ and $\mathrm{ran}(f_n)$ were disjoint; indeed, we would define an extension $\tilde{f}_n : X \to X$ by

$$\tilde{f}_n(x) := \begin{cases} f_n(x) & \text{if } x \in \mathrm{dom}(f_n) \\ f_n^{-1}(x) & \text{if } x \in \mathrm{ran}(f_n) \\ x & \text{otherwise} \end{cases},$$

which would clearly be a Borel involution. Thus, all we have to do is make the domain and the range of each f_n disjoint and we do it as follows: the Hausdorffness and second-countability of X together allow us to write $X^2 \setminus \mathrm{Id}(X) = \bigcup_n U_n \times V_n$, where $U_n, V_n \subseteq X$ are disjoint open. Hence,

$$E = \mathrm{Id}(X) \cup (E \cap \bigcup_m U_m \times V_m) = \mathrm{Id}(X) \cup \bigcup_{n,m} \big(f_n \cap (U_m \times V_m)\big),$$

so every graph in the latter union has its domain and range disjoint, and again, by replacing $(f_n)_n$ with $\big(f_n \cap (U_m \times V_m)\big)_{n,m}$, we may assume that every f_n is already like this.

Extending each f_n to a Borel involution $\tilde{f}_n : X \to X$ as above, we let Γ be the group (under composition) generated by $\left\{\tilde{f}_n\right\}_n$, so the natural action of Γ on X is Borel and $E_\Gamma = E$ because $E = \bigcup_{\gamma \in \Gamma} \mathrm{Graph}(\gamma)$. $\square$

This theorem allows us to define a universal countable Borel equivalence relation in the following sense:

Definition 22.3. For a class $\mathbf{\Gamma}$ of equivalence relations on Polish spaces (e.g. Borel, analytic, smooth, countable Borel), an equivalence relation $E_{\mathbf{\Gamma}}$ on a Polish space X is called a *universal* $\mathbf{\Gamma}$ *equivalence relation* if any equivalence relation $E \in \mathbf{\Gamma}$ is Borel reducible to $E_{\mathbf{\Gamma}}$.

For example, $\mathrm{Id}(\mathcal{C})$ is a universal smooth equivalence relation. Using a $\mathcal{C}$-universal set for $\Sigma_1^1(\mathcal{N}^2)$, one can define a universal analytic equivalence relation as outlined in a homework problem. Furthermore, using the Feldman–Moore theorem and the fact that any countable group is a homomorphic image of $\mathbb{F}_\omega$, the free group on ω generators, one can show that the orbit equivalence relation $E_{\mathbb{F}_\omega}$ of the shift action of $\mathbb{F}_\omega$ on $(\mathcal{C})^{\mathbb{F}_\omega}$ is a universal countable Borel equivalence relation. The proof of this fact is also outlined in a homework problem. Lastly, with a bit of coding, one can show that in fact even the orbit equivalence relation induced by the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is already a universal countable Borel equivalence relation, commonly known as $\mathbb{E}_\infty$.

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