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Math 595: Topics on CBERs HOMEWORK 2 **Due: Mar 28, 5–6:15pm, in 443AH**

1. The $\{1,2,3,\infty\}$ Theorem. Let X be a standard Borel space and let $f : X \to X$ be a Borel function. Let $G_f := \mathrm{Graph}(f)$, so G_f is a Borel directed graph, possibly with loops (indeed, it could be that $f(x) = x$ for some $x \in X$). A *vertex coloring* for such a graph is defined by demanding that any two distinct vertices with a directed edge between them get distinct colors. Prove that $\chi_B(G_f) \in \{1,2,3,\aleph_0\}$, following the steps below.

Let E_f denote the connectedness relation of G_f. A set $A \subseteq X$ is called *f-forward recurrent for $x \in X$* if for infinitely-many $n \in \mathbb{N}$, $f^n(x) \in A$. Say that f is *periodic at $x \in X$* if there are distinct $m,n \in \mathbb{N}$, $f^m(x) = f^n(x)$. Call f *aperiodic* if it is not periodic at any $x \in X$.

(i) Firstly, re-realize that $\chi_B(G_f) \leqslant \aleph_0$, so assume that G_f has a finite Borel (vertex) coloring.

(ii) Prove that the points at which f is periodic form an E_f-invariant Borel set on which E_f admits a transversal. Thus, this part can be easily colored with 3 colors in a Borel fashion, so we may assume that f is aperiodic.

(iii) Fix a finite Borel coloring of G_f and observe that for each $x \in X$, one of the colors forms an f-forward recurrent set. Construct a G_f-independent Borel set $A \subseteq X$ that is f-forward recurrent for every $x \in X$.

(iv) Given such a set A, color G_f with 3 colors in a Borel fashion.

HINT: A is one of the colors.

2. Maybe something fun, like Stravinsky, say "Infernal Dance of Kashchei and His Subjects" from Firebird conducted by Pierre Boulez (or Esa-Pekka Salonen's version).

3. Let E be a CBER on a standard measure space (X,μ). Recall that $[[E]]$ denotes the set of all Borel partial injections $\pi : X \rightharpoonup X$ such that $\pi(x)\, E\, x$ for all $x \in \mathrm{dom}(\pi)$. We denote by $[E]$ the subset of $[[E]]$ of all entire functions, i.e., those whose domain is X.

Prove that the following are equivalent:

(1) E is measure preserving, i.e., for every $\pi \in [[E]]$, $\mu(\mathrm{dom}(\pi)) = \mu(\mathrm{im}(\pi))$.

(2) Every $\gamma \in [E]$ is measure preserving.

(3) Every Borel action $\Gamma \curvearrowright^a X$ of a countable group Γ with $E_a \subseteq E$ is measure preserving.

(4) There is a Borel measure preserving action $\Gamma \curvearrowright^a X$ of a countable group Γ with $E_a = E$.

4. Let X be Polish and let E be an analytic equivalence relation on X.

(a) Show that for an analytic set A, its saturation $[A]_E := \{x \in X : \exists y \in A (x\, E\, y)\}$ is also analytic.

(b) (Burgess) Let $A, B \subseteq X$ be disjoint E-invariant analytic sets (i.e., $[A]_E = A$, $[B]_E = B$). Prove that there is an E-invariant Borel set D separating A and B, i.e., $D \supseteq A$ and $D \cap B = \emptyset$.

5. Prisoners and hats

(a) **Nonsmoothness of $\mathbb{E}_0$.** This question illustrates the nonsmoothness of $\mathbb{E}_0$, more particularly, how having a selector for $\mathbb{E}_0$ (provided by AC) causes unintuitive things.

Problem. ω-many prisoners are sentenced to death, but they can get out under the following condition. On the day of the execution they will be lined up, i.e., enumerated $(p_n)_{n\in\mathbb{N}}$, so that everybody can see everybody else but themselves. Each of the prisoners will have a red or blue hat put on them, but he/she won't be told which color it is (although they can see the other prisoners' hats). On command, all the prisoners at once make a guess as to what color they think their hat is. If all but finitely many prisoners guess correctly, they all go home free; otherwise all of them are executed. The good news is that the prisoners think of a plan the day before the execution, and indeed, all but finitely many prisoners guess correctly the next day, so everyone is saved. How do they do it?

(b) **Non-2-colorability of the Hamming graph.** This question illustrates that the *Hamming graph H* on $2^{\mathbb{N}}$ does not admit a reasonable 2-coloring. The Hamming graph H is defined by putting an edge between two binary sequences if they differ by exactly one bit. Thus, H is a cousin of G_0 and $E_H = \mathbb{E}_0$.

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6. Odometer. Let $X_0 = \left\{x \in 2^{\mathbb{N}} : \forall^\infty n \ x(n) = 0\right\}$, $X_1 = \left\{x \in 2^{\mathbb{N}} : \forall^\infty n \ x(n) = 1\right\}$, and put $X = 2^{\mathbb{N}} \setminus (X_0 \cup X_1)$. Note that X_0 and X_1 are $\mathbb{E}_0$-classes, so all we did is throwing away from $2^{\mathbb{N}}$ two $\mathbb{E}_0$-classes. Define a continuous action of $\mathbb{Z}$ on X so that the induced orbit equivalence relation $E_{\mathbb{Z}}$ is exactly $\mathbb{E}_0|_X$.

7. Something new I learnt last week: Terry Riley's "The Wheel and Mythic Birds Waltz[1]". (There has to be a Kronos Quartet recording, but I can't find it, please let me know if you can.)

8. Universality of the shift action. Let $\Gamma \curvearrowright X$ be a Borel action of a countable group Γ on a Polish space X. Show that there is a Borel equivariant[2] embedding $f : X \hookrightarrow (2^{\mathbb{N}})^\Gamma$,

[1] Greg, I like this waltz!

[2] A map is called *equivariant* if it commutes with the action, i.e., $\gamma \cdot f(x) = f(\gamma \cdot x)$, for $x \in X$.

where $\Gamma \curvearrowright (2^{\mathbb{N}})^\Gamma$ by shift as follows: $\gamma \cdot y(\delta) = y(\delta\gamma)$, for $\gamma, \delta \in \Gamma$, $y \in (2^{\mathbb{N}})^\Gamma$. In particular, f is a Borel reduction of the induced orbit equivalence relations.

9. For any Polish space X, let $\mathbb{E}_0(X)$ denote the equivalence relation of eventual equality on $X^{\mathbb{N}}$, i.e., for $x, y \in X^{\mathbb{N}}$, $x \, \mathbb{E}_0(X) \, y$ if and only if for all large enough $n \in \mathbb{N}$, $x(n) = y(n)$.

(a) For $\ell : \mathbb{N} \to \mathbb{N}$, let $\mathbb{E}_0(\ell)$ be the restriction of $\mathbb{E}_0(\mathbb{N})$ to $\mathcal{N}_{\leqslant \ell} := \{x \in \mathcal{N} : x(n) \leqslant \ell(n)\}$. Show that $\mathbb{E}_0(\ell) \sqsubseteq_c \mathbb{E}_0$

(b) More generally, prove that $\mathbb{E}_0(\mathbb{N}) \sqsubseteq_c \mathbb{E}_0$.