Math 595: Topics on CBERs   Homework 2   Due: Mar 28, 5–6:15pm, in 443AH

1. **The \( \{1,2,3,\infty\} \) Theorem.** Let \( X \) be a standard Borel space and let \( f : X \to X \) be a Borel function. Let \( G_f := \text{Graph}(f) \), so \( G_f \) is a Borel directed graph, possibly with loops (indeed, it could be that \( f(x) = x \) for some \( x \in X \)). A **vertex coloring** for such a graph is defined by demanding that any two distinct vertices with a directed edge between them get distinct colors. Prove that \( \chi_{B}(G_f) \in \{1,2,3,\aleph_0\} \), following the steps below.

Let \( E_f \) denote the connectedness relation of \( G_f \). A set \( A \subseteq X \) is called **\( f \)-forward recurrent** for \( x \in X \) if for infinitely-many \( n \in \mathbb{N} \), \( f^n(x) \in A \). Say that \( f \) is **periodic** at \( x \in X \) if there are distinct \( m,n \in \mathbb{N} \), \( f^m(x) = f^n(x) \). Call \( f \) **aperiodic** if it is not periodic at any \( x \in X \).

(i) Firstly, re-realize that \( \chi_{B}(G_f) \leq \aleph_0 \), so assume that \( G_f \) has a finite Borel (vertex) coloring.

(ii) Prove that the points at which \( f \) is periodic form an \( E_f \)-invariant Borel set on which \( E_f \) admits a transversal. Thus, this part can be easily colored with 3 colors in a Borel fashion, so we may assume that \( f \) is aperiodic.

(iii) Fix a finite Borel coloring of \( G_f \) and observe that for each \( x \in X \), one of the colors forms an \( f \)-forward recurrent set. Construct a \( G_f \)-independent Borel set \( A \subseteq X \) that is \( f \)-forward recurrent for every \( x \in X \).

(iv) Given such a set \( A \), color \( G_f \) with 3 colors in a Borel fashion.

HINT: \( A \) is one of the colors.

2. Maybe something fun, like Stravinsky, say “Infernal Dance of Kashchei and His Subjects” from Firebird conducted by Pierre Boulez (or Esa-Pekka Salonen’s version).

3. Let \( E \) be a CBER on a standard measure space \((X,\mu)\). Recall that \([E] \) denotes the set of all Borel partial injections \( \pi : X \to X \) such that \( \pi(x)E x \) for all \( x \in \text{dom}(\pi) \). We denote by \([E] \) the subset of \([E] \) of all entire functions, i.e., those whose domain is \( X \).

Prove that the following are equivalent:

1. \( E \) is measure preserving, i.e., for every \( \pi \in [E] \), \( \mu(\text{dom}(\pi)) = \mu(\text{im}(\pi)) \).

2. Every \( \gamma \in [E] \) is measure preserving.

3. Every Borel action \( \Gamma \acts X \) of a countable group \( \Gamma \) with \( E_a \subseteq E \) is measure preserving.

4. There is a Borel measure preserving action \( \Gamma \acts X \) of a countable group \( \Gamma \) with \( E_a = E \).

4. Let \( X \) be Polish and let \( E \) be an analytic equivalence relation on \( X \).

    (a) Show that for an analytic set \( A \), its saturation \([A]_E := \{ x \in X : \exists y \in A(xE y) \} \) is also analytic.
(b) (Burgess) Let $A, B \subseteq X$ be disjoint $E$-invariant analytic sets (i.e., $[A]_E = A$, $[B]_E = B$). Prove that there is an $E$-invariant Borel set $D$ separating $A$ and $B$, i.e., $D \supseteq A$ and $D \cap B = \emptyset$.

5. Prisoners and hats

(a) **Nonsmoothness of $E_0$.** This question illustrates the nonsmoothness of $E_0$, more particularly, how having a selector for $E_0$ (provided by AC) causes unintuitive things. **Problem.** $\omega$-many prisoners are sentenced to death, but they can get out under the following condition. On the day of the execution they will be lined up, i.e., enumerated $(p_n)_{n \in \mathbb{N}}$, so that everybody can see everybody else but themselves. Each of the prisoners will have a red or blue hat put on them, but he/she won’t be told which color it is (although they can see the other prisoners’ hats). On command, all the prisoners at once make a guess as to what color they think their hat is. If all but finitely many prisoners guess correctly, they all go home free; otherwise all of them are executed. The good news is that the prisoners think of a plan the day before the execution, and indeed, all but finitely many prisoners guess correctly the next day, so everyone is saved. How do they do it?

(b) **Non-2-colorability of the Hamming graph.** This question illustrates that the Hamming graph $H$ on $2^\mathbb{N}$ does not admit a reasonable 2-coloring. The Hamming graph $H$ is defined by putting an edge between two binary sequences if they differ by exactly one bit. Thus, $H$ is a cousin of $G_0$ and $E_H = E_0$.

**Problem.** $\omega$-many prisoners are sentenced to death, but they can get out under the following condition. On the day of the execution they will be lined up, i.e., enumerated $(p_n)_{n \in \mathbb{N}}$, so that everybody can see everybody else but themselves. Each of the prisoners will have a red or blue hat put on them, but he/she won’t be told which color it is (although they can see the other prisoners’ hats). On command, each prisoner, one-by-one (starting from $p_0$, then $p_1$, then $p_2$, etc.), makes a guess as to what color they think their hat is. Whoever guesses right, goes home free. The good news is that the prisoners think of a plan the day before the execution, so that at most one prisoner is executed. How do they do it?

6. Odometer. Let $X_0 = \{ x \in 2^\mathbb{N} : \forall \infty n \ x(n) = 0 \}$, $X_1 = \{ x \in 2^\mathbb{N} : \forall \infty n \ x(n) = 1 \}$, and put $X = 2^\mathbb{N} \setminus (X_0 \cup X_1)$. Note that $X_0$ and $X_1$ are $E_0$-classes, so all we did is throwing away from $2^\mathbb{N}$ two $E_0$-classes. Define a continuous action of $\mathbb{Z}$ on $X$ so that the induced orbit equivalence relation $E_Z$ is exactly $E_0|_X$.

7. Something new I learnt last week: Terry Riley’s “The Wheel and Mythic Birds Waltz\(^1\)”.

(There has to be a Kronos Quartet recording, but I can’t find it, please let me know if you can.)

8. **Universality of the shift action.** Let $\Gamma \curvearrowright X$ be a Borel action of a countable group $\Gamma$ on a Polish space $X$. Show that there is a Borel equivariant\(^2\) embedding $f : X \hookrightarrow (2^\mathbb{N})^\Gamma$,\(^3\)

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\(^1\)Greg, I like this waltz!

\(^2\)A map is called equivariant if it commutes with the action, i.e., $\gamma \cdot f(x) = f(\gamma \cdot x)$, for $x \in X$.\(^3\)
where $\Gamma \curvearrowright (2^\mathbb{N})^\Gamma$ by shift as follows: $\gamma \cdot y(\delta) = y(\delta \gamma)$, for $\gamma, \delta \in \Gamma$, $y \in (2^\mathbb{N})^\Gamma$. In particular, $f$ is a Borel reduction of the induced orbit equivalence relations.

9. For any Polish space $X$, let $E_0(X)$ denote the equivalence relation of eventual equality on $X^\mathbb{N}$, i.e., for $x, y \in X^\mathbb{N}$, $x E_0(X) y$ if and only if for all large enough $n \in \mathbb{N}$, $x(n) = y(n)$.

(a) For $\ell : \mathbb{N} \to \mathbb{N}$, let $E_0(\ell)$ be the restriction of $E_0(\mathbb{N})$ to $\mathcal{N}_{\leq \ell} := \{x \in \mathcal{N} : x(n) \leq \ell(n)\}$.

Show that $E_0(\ell) \sqsubseteq_c E_0$.

(b) More generally, prove that $E_0(\mathbb{N}) \sqsubseteq_c E_0$. 