Math 595: Topics on CBERs  

Homework 1  

Due: to be determined

1. Prove that every Polish space $X$ admits a Borel (as a subset of $X^2$) linear order. In fact, show that there is a linear order that is both $F_\sigma$ and $G_\delta$.

HINT: Use second-countability.

2. Let $E$ be a finite Borel equivalence relation$^1$ on a Polish space $X$.
   
   (a) Show that for each $p \in \mathbb{N}^+$, the set $X_p := \{x \in X : |[x]_E| = p\}$ is Borel. Pinpoint each use of Luzin–Novikov.
   
   (b) Prove that $E$ admits a Borel transversal, i.e., a Borel set $S \subseteq X$ that meets every $E$-class in exactly one point.

   HINT: Use Problem 1 and Luzin–Novikov.

   (c) Deduce that $E$-admits a Borel selector, i.e., an $E$-invariant$^2$ Borel function $s : X \to X$ with $x E s(x)$ for each $x \in X$.

   (d) Take a break and listen to Chopin’s Mazurka in A Minor Op.17 No.4 (performed by Vladimir Ashkenazy).

   (e) Along these lines, show that $E$ is induced by a Borel automorphism $T : X \to X$ (as an action $\mathbb{Z} \curvearrowright X$), i.e., for each $x \in X$, $[x]_E = \{T^n x : n < |[x]_E|\}$.

   HINT: Do this for each $X_p$ separately.

MORE TO BE ADDED.

---

$^1$This just means each $E$-class is finite.

$^2$For an equivalence relation $E$ on a set $X$, a function $f : X \to Y$ is $E$-invariant if $x_0 E x_1 \Rightarrow f(x_0) = f(x_1)$ for all $x_0, x_1 \in X$. 