Math 595: Topics on CBERs  

Homework 1  

Due: to be determined

1. Prove that every Polish space $X$ admits a Borel (as a subset of $X^2$) linear order. In fact, show that there is a linear order that is both $F_\sigma$ and $G_\delta$.

**HINT:** Use second-countability.

2. Let $E$ be a finite Borel equivalence relation\(^1\) on a Polish space $X$.

(a) Show that for each $p \in \mathbb{N}^+$, the set $X_p := \{x \in X : |[x]_E| = p\}$ is Borel. Pinpoint each use of Luzin–Novikov.

(b) Prove that $E$ admits a Borel transversal, i.e., a Borel set $S \subseteq X$ that meets every $E$-class in exactly one point.

**HINT:** Use Problem 1 and Luzin–Novikov.

(c) Deduce that $E$-admits a Borel selector, i.e., an $E$-invariant\(^2\) Borel function $s : X \to X$ with $xEs(x)$ for each $x \in X$.

(d) Take a break and listen to Chopin’s Mazurka in A Minor Op.17 No.4 (performed by Vladimir Ashkenazy).

(e) Now show that $E$ is induced by a Borel automorphism $T : X \to X$ (as an action $\mathbb{Z} \curvearrowright X$), i.e., for each $x \in X$, $[x]_E = \{T^n x : n < |[x]_E|\}$.

**HINT:** Do this for each $X_p$ separately.

3. Let $\mu$ be the $\left(\frac{1}{2}, \frac{1}{2}\right)$ coin flip measure on $2^\mathbb{N}$. Using the 99% lemma for this measure, prove that the equivalence relation $E_0$ (eventual equality of the sequences) is $\mu$-ergodic.

4. (Feldman–Moore) For a set $X$, we refer to an pair $(x, y) \in X^2$ as a directed edge with source $x$ and target $y$. We say that edges $(x, y)$ and $-(x, y) := (y, x)$ are parallel. Directed edges $(x, y), (x', y')$ are said to be source-incident (resp., target-incident, mixed-incident) if $x = x'$ (resp., $y = y'$, $x = y'$ or $y = x'$). We say that they are whatsoever-incident if they are incident in one of the three aforementioned ways.

Let $E$ be a CBER on a Polish space $X$.

(a) By the Luzin–Novikov theorem, $G := E \setminus \text{Id}_X = \bigcup_n f_n$, where $f_n : X \to X$ is a Borel partial function. This defines $c_0 : G \to \mathbb{N}$ by $(x, y) \mapsto$ the least $n \in \mathbb{N}$ such that $f_n(x) = y$. Show that $c_0$ is Borel and that for any distinct source-incident edges $e, e' \in G$, $c_0(e) \neq c_0(e')$. Hence, for any target-incident edges $e, e' \in G$, $c_0(-e) \neq c_0(-e')$.

(b) Because $X$ is second countable, we can write $X^2 \setminus \text{Id}_X = \bigcup_m (U_m \times V_m)$, where $U_m, V_m \subseteq X$ are open and $U_m \cap V_m = \emptyset$. This defines $c_1 : G \to \mathbb{N}$ by $(x, y) \mapsto$ the least $m \in \mathbb{N}$ such that $(x, y) \in U_m \times V_m$. Show that $c_1$ is Borel and that for any mixed-incident edges $e, e' \in G$, $c_1(e) \neq c_1(e')$.

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\(^1\)This just means each $E$-class is finite.

\(^2\)For an equivalence relation $E$ on a set $X$, a function $f : X \to Y$ is $E$-invariant if $x_0 E x_1 \Rightarrow f(x_0) = f(x_1)$ for all $x_0, x_1 \in X$. 
(c) Conclude that \( c : G \to \mathbb{N}^3 \) defined by \( e \mapsto (c_0(e), c_0(-e), c_1(e)) \) is a directed edge-coloring of \( G \), in the strong sense that any two whatsoever-incident edges get different colors. Thus, \( G \) admits a Borel directed edge-coloring with countable-many colors.

(d) Show that for any Borel directed edge-coloring \( c : G \to \mathbb{N} \), the map \( c' : G \to \mathbb{N} \) defined by \( e \mapsto \min\{c(e), c(-e)\} \) is a Borel (undirected) edge-coloring of \( G \), in the sense that the color of an undirected edge is well-defined (i.e., parallel directed edges get the same color) and incident undirected edges get different colors (i.e., nonparallel whatsoever-incident directed edges get different colors).

5. Take a break and listen to some Bartok, say, Violin Concerto No 2 (has to be Patricia Kopatchinskaja on violin) or Romanian Folk Dances if you prefer something lighter.

6. Let \( E \) be a CBER on a standard Borel space \( X \). Prove that \( E \) admits a Borel transversal if and only if there is a Borel map \( \pi : X \to \mathbb{N} \) such that for each \( E \)-class \( C \), \( \pi|_C \) is a bijection from \( C \) to an initial segment of \( \mathbb{N} \).

7. For a set \( X \), call a collection \( \mathcal{P} \) of subsets of \( X \) a prepartition of \( X \) if the sets in \( \mathcal{P} \) are pairwise disjoint. Denote \( \text{dom}(\mathcal{P}) := \bigcup \mathcal{P} \) and call it the domain of \( \mathcal{P} \). Now let \( E \) be a CBER on a standard Borel space \( E \) and let \( \mathcal{U} \) be any Borel property of finite nonempty subsets of \( E \)-classes, i.e., \( \mathcal{U} \) is a Borel subset of \( [X]^{<\mathbb{N}}_E \), call the sets in \( \mathcal{U} \) good. Show that there is a Borel (inclusion) maximal prepartition of \( X \) into good sets.

8. Give a full proof that every aperiodic\(^3\) CBER \( E \) admits a Borel subequivalence relation \( F \subseteq E \) whose each equivalence class has 7 elements.

\(^3\)An equivalence relation is aperiodic if every equivalence class is infinite.