Exercises from Kaplansky's textbook.

Sec 1.4: 4, 9, 16

1. Redo HW2 Exercise 1, especially part (a).

2. Carefully reprove the following statements indicating each use of the Axiom of Choice.

(a) A function $f : X \to Y$ is injective if and only if it has a left inverse.

(b) A function $f : X \to Y$ is surjective if and only if it has a right inverse.

3. Let $G := (V, R)$ be an *undirected graph*, i.e., V is a set (of vertices) and R (the set of edges) is a symmetric relation on V. A *G-path* (or a *path in G*) is a sequence $v_0, v_1, \ldots, v_n$ of vertices such that there is an edge between every pair of consecutive vertices, i.e., $v_i \, R \, v_{i+1}$ for each $i \in \{0, 1, \ldots, n-1\}$. For vertices $u, v \in V$, a *G-path from u to v* is a G-path that starts with u and ends with v.

Define a binary relation E_G on V by setting

$$u \, E_G \, v :\Longleftrightarrow \text{there is a } G\text{-path from } u \text{ to } v$$

for $u, v \in V$.

(a) Prove that E_G is an equivalence relation.

(b) Call a set $U \subseteq V$ *G-connected* if for any $u, v \in U$, there is a G-path from u to v all of whose vertices lie in U. Prove that the E_G-classes are exactly the $\subseteq$-maximal G-connected sets, called the *connected components* of G.

(c) Define a graph $G = (V, R)$ by taking $V := \mathbb{Z}^2$, putting

$$(x_0, y_0) \, R \, (x_1, y_1) :\Longleftrightarrow (x_1, y_1) - (x_0, y_0) = \pm(1, 1)$$

for $(x_0, y_0), (x_1, y_1) \in \mathbb{Z}^2$. Write each class as an image of a function on $\mathbb{Z}$, i.e., for each $(x, y) \in \mathbb{Z}^2$, define a function $f_{(x,y)} : \mathbb{Z} \to \mathbb{Z}^2$ such that $[(x, y)]_{E_G} = f_{(x,y)}(\mathbb{Z})$. How many E_G-classes (finitely-many or not) are there?

4. For any set X, prove that $\{\mathscr{P}(x) : x \in X\}$ is a set in the following axiom systems:

(a) ZF without Replacement.

(b) ZF without Union.

5. (a) Prove that there is no set R such that $R = \{x : x \notin x\}$.

(b) Deduce that there is no set of all sets, i.e., there is no set X such that $X = \{x : x = x\}$.

6. Do all of the problems on HW3 (especially 2 and 3) to get some practice with partial orderings and well orderings.