1. Let \( (X,d_X) \) and \( (Y,d_Y) \) be metric spaces. Define the following functions \( X \times Y \to \mathbb{R}^+ \):
\[
    \begin{align*}
    d_\infty \left( (x_1,y_1),(x_2,y_2) \right) &:= \max \{ d_X(x_1,x_2), d_Y(y_1,y_2) \} \\
    d_1 \left( (x_1,y_1),(x_2,y_2) \right) &:= d_X(x_1,x_2) + d_Y(y_1,y_2) \\
    d_2 \left( (x_1,y_1),(x_2,y_2) \right) &:= \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}. 
    \end{align*}
\]

(a) Prove \( d_\infty \) and \( d_1 \) are metrics on \( X \times Y \).

**Remark:** \( d_2 \) is also a metric on \( X \times Y \), but the proof of the triangle inequality is a bit tedious, so it is left as an optional exercise.

(b) Show that these metrics are equivalent in the following sense: any pair \( (d_i,d_j) \in \{d_1,d_2,d_\infty\} \) of them admits a constant \( C_{ij} > 0 \) such that for all \( (x,y),(x',y') \in X \times Y \),
\[
    \frac{1}{C_{ij}} d_i \left( (x,y),(x',y') \right) \leq d_j \left( (x,y),(x',y') \right) \leq C_{ij} d_i \left( (x,y),(x',y') \right).
\]

**Hint:** It’s enough to show for the pairs \( d_1,d_\infty \) and \( d_2,d_\infty \). (Why?)

(c) Show that the topology on \( X \times Y \) defined by \( d_\infty \) (equivalently, any of these metrics, but you don’t have to show this) is just the product topology. (First, unfold the definitions to see exactly what you have to show.)

2. Define the function \( d : (\mathbb{N}^\mathbb{N})^2 \to \mathbb{R}^+ \) by setting it for distinct \( x,y \in \mathbb{N}^\mathbb{N} \) to be \( d(x,y) := 2^{-\Delta(x,y)} \), where \( \Delta(x,y) \) is the largest \( n \in \mathbb{N} \) such that \( x|_n = y|_n \), and 0 for \( x = y \).

(a) Show that \( d \) is an ultrametric on \( \mathbb{N}^\mathbb{N} \), i.e., \( d(x,z) \leq \max \{ d(x,y), d(y,z) \} \).

**Hint:** Draw pictures.

(b) Show that every open ball is of the form \( U_s := \{ x \in \mathbb{N}^\mathbb{N} : x \supseteq s \} \) for some \( s \in \mathbb{N}^{<\mathbb{N}} \).

(c) Show that the sets \( U_s \) as above are clopen, i.e., both open and closed.

3. For each of the following, determine the boundary and closure of the set \( A \) in the metric space \( (X,d) \) where \( X \subseteq \mathbb{R} \) given below and \( d \) is the standard metric on \( \mathbb{R} \). Prove your answers.

(a) \( X := [1,\infty), \ A := \left\{ q \in \mathbb{Q} : q^2 > 2 \right\} \).

(b) \( X := (\mathbb{R}, d) \), \( A := \left\{ \frac{1}{n} : n \in \mathbb{N} \setminus \{0\} \right\} \).

(c) \( X := [-1,1] \cup \{2\}, \ A := (-1,1) \).

4. Call a set \( Q \) in topological space \( X \) dense if it intersects every nonempty open set. Prove:

(a) A set \( Q \) in \( X \) is dense if and only if \( \overline{Q} = X \).

(b) \( \mathbb{Q} \) is dense in \( \mathbb{R} \) (with the standard topology).

(c) The set \( Q \) of eventually 0 sequences in \( \mathbb{N}^\mathbb{N} \), i.e., \( Q := \left\{ x \in \mathbb{N}^\mathbb{N} : \forall \infty n \in \mathbb{N} \ x(n) = 0 \right\} \), is dense in \( \mathbb{N}^\mathbb{N} \). Here \( \forall \infty n \) stands for \( \exists m \forall n \geq m \).
5. Consider $\mathbb{R}$ with its standard metric.
   (a) Show that every open set is a union of open intervals with rational endpoints.
   (b) What is the cardinality of the set $U$ of all open intervals with rational endpoints?
   (c) How many open sets are there in $\mathbb{R}$? More precisely, letting $T$ denote the topology of $\mathbb{R}$, i.e., the set of all open subsets of $\mathbb{R}$, show that $T \equiv \mathbb{R}$.

   HINT: Define a surjection $\mathcal{P}(U) \twoheadrightarrow T$. 