**Math 432: Set Theory and Topology** HOMEWORK 4 **Due: Feb 28 (Thu)**

**Other** (mandatory) **exercises.**

**1.** Below, for the questions asking for examples, make sure your examples are different from those in the notes ("A quick introduction to basic set theory").

(a) Write down the ordinal 5 (the $6^{\text{th}}$ least ordinal) explicitly.

(b) Give examples $A, B$ of transitive sets such that the relation $\in$ on $A$ is not an ordering, but on $B$ it is.

(c) Give an example of a non-transitive set.

**2.** Show that the powerset of a transitive set is transitive.

**3.** Prove that there does **not** exist a set that contains all of the ordinal, i.e., there is no set **Ord** that is equal to $\{x : x \text{ is an ordinal}\}$. You may **not** use Foundation, Powerset, or Axiom of Choice, just use the properties of ordinals proved in class.

**4.** Prove that for an ordinal $\alpha$, $\sup \alpha \in \alpha$ if and only if $\alpha$ is a successor ordinal.

**5.** Let $C$ be a nonempty set.

(a) Prove that there is a set, denoted by $\bigcap C$, that is equal to $\{x : \forall A \in C\ (x \in A)\}$.

(b) Prove that if $C$ is a set of ordinals, then $\bigcap C$ is an ordinal and moreover, it is the least ordinal in $C$, i.e., $\min C = \bigcap C$.

**6.** The following basic statements about ordinals were proven in class. Rewrite their proofs in your own words and be ready to present in the problem session.

Let $\alpha, \beta$ be ordinals.

(a) Prove directly from the definition of an ordinal (without using any ZFC axioms, especially Foundation) that $\alpha \notin \alpha$.

(b) Prove that for any $y \in \alpha$, $y = \alpha_{<y}$.

(c) Conclude that if $\alpha \neq \emptyset$, then the least element of $\alpha$ is $\emptyset$. We denote $0 := \emptyset$.

(d) Prove that every $y \in \alpha$ is itself an ordinal.

(e) Prove that $\in$ is a total order on ordinals, i.e. exactly one of the following holds: either $\alpha = \beta$, or $\alpha \in \beta$, or $\beta \in \alpha$.

(f) Prove that if $\alpha \subseteq \beta$, then either $\alpha = \beta$ or $\alpha \in \beta$.

(g) Prove that for every formula $\varphi(x)$, if there is an ordinal $\alpha$ for which $\varphi(\alpha)$ holds, then there is a least such ordinal.

(h) Prove that a transitive set of ordinals is itself an ordinal.

**7.** Let $\alpha$ denote a natural number and recall that $\mathbb{N}$ denotes the set of all natural numbers.

(a) Prove that any $x \in \alpha$ is itself a natural number.

(b) Prove that $\alpha+1$ is a natural number. Deduce that $\mathbb{N}$ is an inductive set.

(c) Prove that $\mathbb{N}$ is the $\subseteq$-least inductive set, i.e., for any inductive set $I$, $\mathbb{N} \subseteq I$.

(d) Deduce the usual induction theorem: Suppose that $P \subseteq \mathbb{N}$ is such that

   (i) $0 \in P$ and

   (ii) for any $n \in \mathbb{N}$, $n \in P \Rightarrow n+1 \in P$.

   Then $P = \mathbb{N}$.