Math 432: Set Theory and Topology  HOMEWORK 3  Due: Feb 21 (Thu)

Exercises from Kaplansky’s book.

TO BE ADDED.

Other (mandatory) exercises.

1. Let \( < \) be a strict partial order on a set \( X \) and let \( Y \subseteq X \). Call \( y_0 \in Y \) \( < \)-minimal in \( Y \) if for any \( y \in Y \), \( y \not< y_0 \). Call \( y_1 \in Y \) \( < \)-least (or \( < \)-minimum) in \( Y \) if for any \( y \in Y \), \( y_1 \leq y \).
   
   (a) Prove that if \( y_0 \in Y \) is \( < \)-least then it is \( < \)-minimal.
   
   (b) Give an example of \( (X,<) \) and \( Y \subseteq X \) such that \( Y \) admits a \( < \)-minimal element, but does not admit a \( < \)-least element.
   
   (c) Give an example of \( (X,<) \) and \( Y \subseteq X \) such that \( Y \) does not admit any \( < \)-minimal elements.
   
   (d) Prove that if \( < \) is a total order, then the converse of part (a) holds: if \( y_0 \in Y \) is \( < \)-minimal, then it is \( < \)-least.
   
   (e) Conclude that if \( < \) is total, then every \( Y \subseteq X \) admits at most one \( < \)-minimal element.

2. Determine which pairs of sets are isomorphic as ordered sets with their usual ordering \( < \). Prove your answers.
   
   (a) \( \mathbb{N} \) and \( \{ -\frac{1}{n} : n \in \mathbb{N} - \{0\} \} \)
   
   (b) \( \mathbb{Z} \) and \( \{ \frac{1}{n} : n \in \mathbb{Z} - \{0\} \} \cup \{0\} \)
   
   (c) \( \mathbb{R} \) and \( (0,1) \)
   
   (d) \( \mathbb{Q} \) and \( [0,1) \cap \mathbb{Q} \)
   
   (e) \( (0,2) \) and \( (0,1) \cup (1,2) \)
   
   (f) \( (0,2) \) and \( (0,1) \cup [2,3) \).

3. (a) Let \( (A,<) \) be a well-ordering and let \( f : A \to A \) be an order-homomorphism, i.e.
   
   \[
   a_0 < a_1 \implies f(a_0) < f(a_1)
   \]
   
   for all \( a_0, a_1 \in A \). Prove that \( f \) progressive, i.e., \( a \leq f(a) \) for all \( a \in A \).

   (b) Deduce directly from part (a) that \( (A,<) \not< (A,<) \) for any well-ordering \( (A,<) \).

   Remark. We proved this statement in class as a corollary of the uniqueness lemma for isomorphisms witnessing \( \leq \). The purpose of this exercise is to give a more direct proof.

   (c) Ordering \( \mathbb{N}^2 \) lexicographically, give an example of an order-homomorphism \( f : \mathbb{N}^2 \to \mathbb{N}^2 \) (other than the identity map) such that \( f(n,m) = (n,m) \) for all \( (n,m) \geq_{\text{lex}} (2,0) \).

4. Let \( (A,<) \) and \( (B,<) \) be well orderings.
(a) Prove that there is a set $F$ such that

$$F = \{ f : f \text{ is an order isomorphism between initial segments of } (A, <) \text{ and } (B, <) \}.$$ 

(b) Prove that for any $f, g \in F$, $f \subseteq g$ or $g \subseteq f$.

(c) Conclude that $f := \bigcup F$ is an order isomorphism (in particular, a function) of an initial segment $A'$ of $(A, <)$ with an initial segment of $B'$ of $(B, <)$.

(d) Prove that $A' = A$ or $B' = B$.

MORE TO BE ADDED.