Math 432: Set Theory and Topology HOMEWORK 2 **Due: Feb 7 (Thu)**

Exercises from Kaplansky's book.

Sec 1.4: 8[1], 17[2]

Sec 1.5: 5[3]

Other (mandatory) **exercises.**

1. For each of the following functions, explicitly define a left and a right inverse of it, if such exist. If any of these inverses doesn't exist, prove it.

(a) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) := n^2$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) := n^2$.

(c) $f : \mathbb{Z} \to \left\{n \in \mathbb{N} : \exists m \in \mathbb{N}\ n = m^2\right\}$ defined by $f(n) := n^2$.

2. Prove that if a function $f : X \to Y$ has an inverse (two-sided), then this inverse must be unique.

3. (a) For a function $f : X \to Y$ define a binary relation E_f on X by

$$x_0 \, E_f \, x_1 :\Longleftrightarrow f(x_0) = f(x_1),$$

for $x_0, x_1 \in X$. Prove that E_f is an equivalence relation. Explicitly describe the E_f-classes.

(b) Conversely, show that all equivalence relations arise in this fashion. More precisely, for any equivalence relation E on a set X, find a set Y and a surjection $f : X \to Y$ such that $E_f = E$.

HINT: Think of the quotient X/E.

4. Read Notation 1.2 and Caution 1.3 in my Intro to Set Theory notes. Use the axioms of ZFC to prove the following.

(a) There is a set S satisfying $S = \{\}$, i.e., for all sets x, $x \notin S$. We denote this set S by $\emptyset$ and call it the *emptyset*.

(b) For each set x, there is a set S satisfying $S = \{x\}$.

(c) For any sets X, Y, there is a set S satisfying

$$S = \{z : z \in X \vee z \in Y\}.$$

We denote this set S by $X \cup Y$ and call it the *union* of X and Y.

CAUTION: Union axiom alone doesn't imply this.

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[1] HINT: It is enough (why?) to prove that f is has an inverse (two-sided).

[2] In part (c), by a set $X \subseteq A$ being the *largest* subset of A with a given property (in this case, the property is that $f(X) = X$), they mean that any other $Y \subseteq A$ with the same property is a subset of X.

[3] Nonvoid means nonempty.

(d) For any sets x,y, there is a set S satisfying $S = \{\{x\},\{x,y\}\}$. We denote this set S by (x,y) and call it the *ordered pairing* of x,y or just an *ordered pair*.

(e) For any sets X,Y, there is a set S satisfying

$$S = \{z : x \in X \wedge y \in Y \wedge z = (x,y)\}.$$

We denote this set by $X \times Y$ and call it the *Cartesian product* of X and Y.

CAUTION: Comprehension only gives the existence of sets of the form

$$\{z \in Z : x \in X \wedge y \in Y \wedge z = (x,y)\}$$

for a set Z, so to apply it, one has to first prove the existence of an appropriate Z.

(f) For any sets X,Y, write down a formula $\varphi(f)$ such that for any set f, $\varphi(f)$ says that f is a function from X to Y. Prove that there is a set S satisfying $S = \{f : \varphi(f)\}$. We denote this set S by Y^X and call it the *set of all functions from X to Y*.