Exercises from Kaplansky’s book.

Sec 1.4: 8\(^1\), 17\(^2\)

Sec 1.5: 5\(^3\)

Other (mandatory) exercises.

1. For each of the following functions, explicitly define a left and a right inverse of it, if such exist. If any of these inverses doesn’t exist, prove it.
   (a) \(f : \mathbb{N} \to \mathbb{N}\) defined by \(f(n) := n^2\).
   (b) \(f : \mathbb{Z} \to \mathbb{Z}\) defined by \(f(n) := n^2\).
   (c) \(f : \mathbb{Z} \to \{n \in \mathbb{N} : \exists m \in \mathbb{N} n = m^2\}\) defined by \(f(n) := n^2\).

2. Prove that if a function \(f : X \to Y\) has an inverse (two-sided), then this inverse must be unique.

3. (a) For a function \(f : X \to Y\) define a binary relation \(E_f\) on \(X\) by
   \[x_0 E_f x_1 \iff f(x_0) = f(x_1),\]
   for \(x_0, x_1 \in X\). Prove that \(E_f\) is an equivalence relation. Explicitly describe the \(E_f\)-classes.
   (b) Conversely, show that all equivalence relations arise in this fashion. More precisely, for any equivalence relation \(E\) on a set \(X\), find a set \(Y\) and a surjection \(f : X \to Y\) such that \(E_f = E\).
   HINT: Think of the quotient \(X/E\).

4. Read Notation 1.2 and Caution 1.3 in my Intro to Set Theory notes. Use the axioms of ZFC to prove the following.
   (a) There is a set \(S\) satisfying \(S = \emptyset\), i.e., for all sets \(x\), \(x \not\in S\). We denote this set \(S\) by \(\emptyset\) and call it the emptyset.
   (b) For each set \(x\), there is a set \(S\) satisfying \(S = \{x\}\).
   (c) For any sets \(X, Y\), there is a set \(S\) satisfying
   \[S = \{z : z \in X \lor z \in Y\}\]
   We denote this set \(S\) by \(X \cup Y\) and call it the union of \(X\) and \(Y\).
   CAUTION: Union axiom alone doesn’t imply this.

\(^1\)HINT: It is enough (why?) to prove that \(f\) is has an inverse (two-sided).

\(^2\)In part (c), by a set \(X \subseteq A\) being the largest subset of \(A\) with a given property (in this case, the property is that \(f(X) = X\)), they mean that any other \(Y \subseteq A\) with the same property is a subset of \(X\).

\(^3\)Nonvoid means nonempty.
(d) For any sets $x,y$, there is a set $S$ satisfying $S = \{\{x\}, \{x, y\}\}$. We denote this set $S$ by $(x, y)$ and call it the *ordered pairing* of $x, y$ or just an *ordered pair*.

(e) For any sets $X, Y$, there is a set $S$ satisfying

$$S = \{z : x \in X \land y \in Y \land z = (x, y)\}.$$  

We denote this set by $X \times Y$ and call it the *Cartesian product* of $X$ and $Y$.

**CAUTION:** Comprehension only gives the existence of sets of the form

$$\{z \in Z : x \in X \land y \in Y \land z = (x, y)\}$$

for a set $Z$, so to apply it, one has to first prove the existence of an appropriate $Z$.

(f) For any sets $X, Y$, write down a formula $\varphi(f)$ such that for any set $f$, $\varphi(f)$ says that $f$ is a function from $X$ to $Y$. Prove that there is a set $S$ satisfying $S = \{f : \varphi(f)\}$. We denote this set $S$ by $Y^X$ and call it the *set of all functions from $X$ to $Y$*.