Math 574: Set Theory

Homework 8

Due: Apr 12 and 13

Terminology. For a filter $\mathcal{F}$ on a set $X$, say that a set $A \subseteq X$ is $\mathcal{F}$-large (resp. $\mathcal{F}$-small) if $A \in \mathcal{F}$ (resp. $A \in \mathcal{F}'$, i.e. $A^c \in \mathcal{F}$). For a property $P$ of elements of $X$ (in other words, $P \subseteq X$), say that $\mathcal{F}$-a.e. $x \in X$ has property $P$, written

$$\forall^\mathcal{F} x \in X \; (x \text{ has property } P),$$

if the set $\{x \in X : x \text{ has property } P\}$ is $\mathcal{F}$-large.

1. For a set $A$, let $[A]^2$ denote the collection of all subsets of $A$ of size exactly 2; think of these as edges between the elements of $A$. Prove the following using an ultrafilter on $\mathbb{N}$.

Infinite Ramsey Theorem. For any finite coloring of $[\mathbb{N}]^2$, i.e. a function $c : [\mathbb{N}]^2 \to k := \{0, 1, \ldots, k-1\}$, there is an infinite monochromatic set $A \subseteq \mathbb{N}$, i.e. $c|_{[A]^2}$ is constant.

Remark: The usage of an ultrafilter here is an overkill, of course, but a nice one.

Hint: Letting $\mu$ be an ultrafilter on $\mathbb{N}$, derive a coloring $c'$ of $\mathbb{N}$ as follows: for each vertex $v \in \mathbb{N}$, look at the edges incident to $v$ and take the $\mu$-large color. Having colored $\mathbb{N}$, one of these colors is $\mu$-large. Build a monochromatic subsequence inside this color by induction.

2. Call a set $D \subseteq \mathbb{Z}$ a difference set or a $\Delta$-set if there is a sequence $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}$ of pairwise distinct elements such that $D := \{z_n - z_m : n > m\}$. Denote the collection of all $\Delta$-sets by $\Delta$.

(a) Show that $\Delta$-sets have the Ramsey property; namely, for any $D \in \Delta$, whenever $D$ is partitioned into two sets, at least one of them contains a $\Delta$-set.

(b) Conclude that $\Delta^* := \{A \subseteq \mathbb{Z} : \forall D \in \Delta \ (A \cap D \neq \emptyset)\}$ is a filter.

3. Let $X$ be a topological space and $\mathcal{F}$ a filter on $\mathbb{N}$. Call $x \in X$ an $\mathcal{F}$-limit of a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$, written

$$\lim_{n \to \mathcal{F}} x_n = x,$$

if for every open neighborhood $U$ of $x$, $x_n \in U$ for $\mathcal{F}$-a.e. $n \in \mathbb{N}$.

(a) For which filter $\mathcal{F}$, the notion of $\mathcal{F}$-limit coincides with the usual limit?

(b) Now let $\mu$ be an ultrafilter on $\mathbb{N}$ and suppose that $X$ is compact (open covers definition). Prove that every sequence in $X$ has a $\mu$-limit.

Definition. Let $(\mathbb{P}, \leq)$ be a partially ordered set, which we call a poset, for short. Say that $p, q \in \mathbb{P}$ are compatible if they admit a common strengthening, i.e. there is $r \in \mathbb{P}$ with $r \leq p$ and $r \leq q$. Otherwise, say that $p$ and $q$ are incompatible, written $p \perp q$.

Definition. For a poset $(\mathbb{P}, \leq)$, a set $F \subseteq \mathbb{P}$ is called a $\mathbb{P}$-filter if

(i) (Upward closed) $p \in F$ and $q \geq p \implies q \in F$;

(ii) (Contains common strengthenings) $p, q \in F \implies$ there is $r \in F$ with $r \leq p$ and $r \leq q$.

Call a $\mathbb{P}$-filter $F$ an ultrafilter (or a strongly maximal filter) if for each $p \in \mathbb{P}$ either $p \in F$ or there is $q \in F$ with $p \perp q$. 

9:49 pm, Apr 8.
4. Let $A, B$ be sets and let $\mathcal{P}$ be the set of all partial functions $A \to B$ with finite domain. We turn $\mathcal{P}$ into a poset under extension (i.e. reverse inclusion), i.e. for any $p, q \in \mathcal{P}$, $p \leq q :\iff p \supseteq q$. Show that the map $F \mapsto \cup F$ is a bijection between the set of ultrafilters and the set of functions $A \to B$.

5. Let $(\mathcal{P}, \leq)$ be a countable poset. Identifying $X := \mathcal{P}(\mathcal{P})$ with $2^\mathcal{P}$ equips $X$ with the product topology, making it a homeomorphic copy of the Cantor space. Let $X_\mathcal{P} \subseteq X$ denote the set of all $\mathcal{P}$-ultrafilters.

   (a) Show that $X_\mathcal{P}$ is a $G_\delta$ subset of $X$, i.e. a countable intersection of open sets.

   **Remark:** $X$ is a Polish space$^1$ (being compact metrizable) and a subset of a Polish space is itself a Polish space (in the relative topology) if and only if it is $G_\delta$ (see, for example, Proposition 1.7 in my DST notes). Thus, $X_\mathcal{P}$ is also a Polish space.

   (b) Prove that the sets

   $$U_p := \{ G \in X_\mathcal{P} : p \in G \}, \ p \in \mathcal{P},$$

   form a basis for the topology on $X_\mathcal{P}$.

   (c) Express in terms of the $U_p$ what it means for a set $A \subseteq X_\mathcal{P}$ to be

   (i) open,

   (ii) dense.

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$^1$A topological space is called Polish if it is separable and admits a compatible complete metric (the two most important properties of $\mathbb{R}$). See Section 1 of my DST notes for a gentle and loving introduction to Polish spaces.