Math 574: Set Theory                Homework 7                Due: Apr 5 and 6

1.  
   (a)  \( V_n = L_n \) for each \( n \in \omega \); in particular, \( V_\omega = L_\omega \).
   
   (b)  \( V_{\omega+1} \neq L_{\omega+1} \), in fact, \( |V_{\omega+1}| > |L_{\omega+1}| \).

   REMARK: This is true even when \( V = L \).

2.  
   Let \( F(x) \) be a \( \Delta_0 \) class-function and \( R(y, \vec{z}) \) be a \( \Delta_0 \) class-relation. In class we proved that the relation \( R(F(x), \vec{z}) \) is \( \Sigma_1 \) in general. However, prove that for the following class-functions \( F(x) \), \( R(F(x), \vec{z}) \) is \( \Delta_0 \).

   (a)  \( F(x) := \bigcup x \) and \( F(x) := \bigcap x \).

   (b)  \( F(x) := \text{dom}(x) \) if \( x \) is a function, and \( \emptyset \), otherwise. Also, same with \( \text{dom}(x) \) replaced by \( \text{im}(x) \).

   (c)  (Optional) \( F(x) \) is an arbitrary \( \Delta_0 \) class-function such that the relation \( z \in F(x) \) is also \( \Delta_0 \) and for some \( n \in \mathbb{N} \) (a genuine finite number, not an element of \( V \)), \( \forall x F(x) \subseteq cl_n(x) \), where

   \[
   cl_n(x) := \bigcup \bigcup \ldots \bigcup x. \quad \text{n times}
   \]

3.  
   Let \( F(x) \) be a \( \Sigma_1 \) class-function and let \( M \) be a transitive model of a large enough finite fragment of ZF. Suppose that for each \( x \) in \( M \) there is \( y \) in \( M \) such that \( (F(x) = y)^M \) holds. Prove:

   (a)  \( F(x) \) is absolute for \( M \).

   (b)  If \( \varphi(y, \vec{z}) \) is an absolute formula for \( M \), then so is \( \exists y (y = F(x)) \land \varphi(y, \vec{z}) \).

   REMARK: If you think this is absolutely trivial, you are right.

4.  
   Prove that the following class-functions satisfy the hypothesis of Question 3:

   (a)  \( F(x, n) := x^n \) if \( n \in \omega \), and \( \emptyset \), otherwise.

   HINT: \( y = x^n \) if and only if there is a certificate \( c : \omega \to x \) such that \( c(0) = \emptyset \) and for each \( k < n \ldots \)

   (b)  \( F(x) := x^{<\omega} \).

   CAUTION: The class-function \( F(x) := x^\omega \) is very nonabsolute.

   Conclude that these class-functions are absolute for transitive models of a large enough finite fragment of ZF.

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\(^1\)Thanks to Christian Schulz for suggesting this question.