Math 574: Set Theory  

Homework 4  

Due: Mar 8 and 9  

1. Prove the following metatheorem: Every finite class of \( U \) is a set (i.e. is realized by an element of \( U \)). Explain why this is a metatheorem (i.e. not a first-order statement in \( U \)).  

Remark: This question should have been assigned much earlier, but it didn’t occur to me until now.  

2. Prove directly that Pairing and Powerset axioms hold in \( V \).  

3. Prove that the following class-relations and class-functions, defined in \( T := \text{ZF} - \text{Pow} - \text{Infty} \), are all \( \Delta_0 \) and hence absolute for any transitive model of \( T \). You may choose to prove only half of these, but please choose the ones you doubt most.  

Notation and terminology. Below, for class-functions, we just write \( F(\vec{x}) \) instead of \( \vec{x} \mapsto F(\vec{x}) \). Recall that a class-function is \( \Delta_0 \) if, by definition, its graph \( y = F(\vec{x}) \) is \( \Delta_0 \). This includes the 0-ary functions, i.e. constants such as \( \emptyset, \omega \).  

(a) \( \{x\} \).  

(b) \( (x, y) \).  

(c) \( z \) is an ordered pair.  

(d) \( \emptyset \).  

(e) \( x \cup y, x \cap y, x \setminus y \).  

(f) \( \bigcup x, \bigcap x \).  

(g) \( S(x) := x \cup \{x\} \).  

(h) \( x \) is a successor (of some set).  

(i) \( x \) is transitive.  

(j) \( \in \) is a linear order on \( x \).  

(k) \( x \times y \).  

(l) \( R \) is a relation, i.e. is a set of ordered pairs.  

(m) \( \text{dom}(R) := \{x : \exists y (x, y) \in R\} \) and \( \text{ran}(R) := \{y : \exists x (x, y) \in R\} \), i.e. the class-functions \( F(R, X) \) (resp. \( F(R, Y) \)) defined by setting it to hold if \( R \) is a relation (i.e. a set of ordered pairs) and \( X = \text{dom}(R) \) (resp. \( Y = \text{ran}(R) \)).  

Remark: The definition of \( \text{dom}(R) \) is written as a class on purpose: you have to rewrite it so that it is a set and the class-function \( R \mapsto \text{dom}(R) \) is \( \Delta_0 \).  

(n) \( f \) is a function.  

(o) \( f(x) \), i.e. the class-function \( F(f, x, y) \), which is set to hold exactly when \( f \) is a function, \( x \) is in \( \text{dom}(f) \), and \( y = f(x) \).  

(p) \( f \) is a one-to-one function.  

TO BE CONTINUED...