1. Prove that multiplication is well-defined on $\mathbb{Q}$. Begin by stating exactly what you need to prove.

2. Prove that $(\mathbb{Q}, +, \cdot)$ is a field by checking all of the required axioms one-by-one (even the ones proven in class).

3. Denote the elements (i.e. equivalence classes) in $\mathbb{Q}$ by $\left[ \frac{a}{b} \right]$, omitting writing $E$ in the superscript. An element $\left[ \frac{a}{b} \right] \in \mathbb{Q}$ is said to be positive if $a$ is positive. Define a binary relation $<$ on $\mathbb{Q}$ by setting, for each $\left[ \frac{a}{b} \right], \left[ \frac{c}{d} \right] \in \mathbb{Q}$,

$$\left[ \frac{a}{b} \right] < \left[ \frac{c}{d} \right] :\Leftrightarrow \left[ \frac{c}{d} \right] - \left[ \frac{a}{b} \right] \text{ is positive.}$$

(a) Prove that the notion of positivity is well-defined on $\mathbb{Q}$. Conclude that $<$ is well-defined.

(b) Prove that $<$ is a total strict order on $\mathbb{Q}$.

(c) Prove that $<$ satisfies Axioms (O3) and (O4) (written on page 12 of Sally’s book). This makes $\mathbb{Q}$ an ordered field.

4. For $n \geq 2$, show that for any $a, b \in \mathbb{Z}$, $a \equiv b \pmod{n}$ if and only if $a - b$ is divisible by $n$.

5. Let $n \geq 2$ and consider the ring $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$. Prove that there is no binary relation $<$ that makes this ring an ordered ring, i.e. there is no total strict order $<$ on $\mathbb{Z}/n\mathbb{Z}$ satisfying Axioms (O3) and (O4).

6. Let $(R, +, \cdot)$ be a ring and let $0_R, 1_R$ denote its additive and multiplicative identities. Prove:
   
   (a) The additive and the multiplicative identities are unique.
   
   (b) Each $x \in R$ has a unique additive inverse.
   
   (c) Each $x \in R$ has at most one multiplicative inverse.
   
   (d) For each $x \in R$, $0_R \cdot x = 0_R = x \cdot 0_R$.
   
   (e) If every nonzero $^1 x \in R$ has a multiplicative-inverse, then $R$ satisfies the cancellation axiom, namely for all $x, y \in R$, if $x \cdot y = 0_R$ then $x = 0_R$ or $y = 0_R$. Thus, a field is a domain.

   If $0_R = 1_R$ then $R$ has only one element, namely, $R = \{0_R\}$. In this case, we call $R$ the zero ring or the trivial ring.

7. Let $(R, +, \cdot)$ be a ring and let $0_R, 1_R$ denote its additive and multiplicative identities. A subset $R_0 \subseteq R$ is called a subring if $(R_0, +, \cdot)$ is a ring. Recall that, unlike the textbook, our definition of ring includes the existence of multiplicative identity; in particular, $R_0$

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$^1$Not equal to $0_R$. 

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should have both additive and multiplicative identities, by definition; denote them by $0_{R_0}$ and $1_{R_0}$, respectively.

(a) Prove that for a subring $R_0 \subseteq R$, $0_{R_0} = 0_R$ and $1_{R_0} = 1_R$.

(b) For a subset $R_0 \subseteq R$, prove that $R_0$ is a subring if and only if $R_0 \ni 1_R$ and for all $a, b \in R_0$, the elements $a - b$ and $a \cdot b$ are also in $R_0$.

(c) Show by example that a subring of a field need not be a field.