1. Given a partition $C$ of a set $X$, it was proven in class that the binary relation $E_C$ on $X$ is an equivalence relation. Prove this again on your own very carefully (without skipping any steps). Just in case, we recall the definition of $E_C$: for any $x, y \in X$,

$$xE_Cy :\Leftrightarrow \exists A \in C \text{ such that } x, y \in A.$$ 

2. Conversely, given an equivalence relation $E$ on a set $X$, it was proven in class that the set of $E$-classes form a partition $C$ of $X$ and that $E_C$ is exactly $E$. Prove this again on your own following the steps below.

(a) Prove that for each $x, y \in X$, $xEy$ if and only if $[x]_E = [y]_E$.

(b) Prove that $\bigcup_{x \in X}[x]_E = X$.

(c) Prove that for each $x, y \in X$, if $[x]_E \cap [y]_E \neq \emptyset$ then $[x]_E = [y]_E$.

(d) Conclude that $E_C = E$.

3. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(z) := z^2$. Define the relation $E_f$ on $\mathbb{Z}$ by putting

$$xE_fy :\Leftrightarrow f(x) = f(y).$$

It was proven in class (for any function on any set) that this is an equivalence relation. Explicitly describe and list all $E_f$-equivalence classes.

4. Let $G := (V, E)$ be an undirected graph with no loops, i.e. $V$ is the set of vertices and $E \subseteq V^2$ is the set of edges, which is irreflexive and symmetric. For vertices $x, y \in X$, we say that $y$ is adjacent to $x$ (or $y$ is a neighbor of $x$) if there is an edge $(x, y) \in E$. Assuming that $V$ is finite, the degree of each vertex $v \in V$, denoted by $\deg_G(v)$, is the number of neighbors of $x$. Prove that the sum of all degrees, i.e. $\sum_{v \in V} \deg_G(v)$, has to be an even number.

5. There was a party of 170 people in which every person shook some other people’s hands (at most one hand per person). It is possible that a person didn’t shake anyone’s hand. Prove that there are two people that shook equal number of hands.

Hint: Prove by contradiction. Use the statement of Question 4.