Extensive parity analysis of broken 5-diamond partitions

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Abstract. We investigate congruences for $\Delta_5(n)$ modulo 2 and characterize the parity of $\Delta_5(4n+1)$ and $\Delta_5(4n+2)$ according to the arithmetic property of n. As a consequence, we obtain various Ramanujan type congruences for $\Delta_5(n)$. We also extend these results to several infinite families of congruences.

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1 Introduction

In 2007, Andrews and Paule [2] introduced a new class of combinatorial objects called broken k-diamond partitions. Define $\Delta_k(n)$ to be the number of broken k-diamond partitions of n. Applying MacMahon's partition analysis, they found that the generating function of $\Delta_k(n)$ satisfies

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3 (q^{4k+2};q^{4k+2})_\infty}, \tag{1.1}$$

where throughout this paper, we adopt the notation

$$(a;q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}), \quad |q| < 1.$$

Andrews and Paule [2] showed that, for all $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}. \tag{1.2}$$

They also posed three conjectures related to $\Delta_2(n)$ and made the comment "The following observations about congruences suggest strongly that there are undoubtedly a myriad of partition congruences for $\Delta_k(n)$". Since then, numerous mathematicians have investigated congruences satisfied by $\Delta_k(n)$ for small values of k. For example, Hirschhorn and Sellers [9] found several parity results for $\Delta_1(n)$ and $\Delta_2(n)$. Radu and Sellers investigated congruences for $\Delta_2(n)$ modulo 3 in [16]. Lin and Wang [12] presented an alternative proof of Radu and Sellers' congruence results for $\Delta_2(n)$ by applying one of Ramanujan's modular equations of degree 5. A number of congruences for $\Delta_2(n)$ modulo powers of 5 have been established in [5, 9, 13].

In 2010, Paule and Radu [13] proposed conjectures related to broken 3-diamond partitions modulo 7 and broken 5-diamond partitions modulo 11. These conjectures have been confirmed by Jameson [10] and Xiong [18].

Recently, Radu and Sellers provided an extensive analysis of the parity of $\Delta_3(n)$ in [15]. Later, Lin [11] gave an elementary proof of Radu and Sellers' parity results for $\Delta_3(n)$ by using a theta function identity due to Ramanujan. For some recent results on broken 3-diamond partitions, we refer the reader to [8, 17].

More recently, Yao [19] considered parity results for $\Delta_{11}(n)$ by employing an identity due to Chan and Toh [4], and the p-dissection formula of $(q;q)_\infty$ given by Cui and Gu [7]. Additionally, Ahmed and Baruah [1] found several new congruences modulo 2 for broken 5-, 7- and 11-diamond partitions by employing Ramanujan's theta functions.

Our goal of this work is to focus on parity results satisfied by $\Delta_5(n)$.

2 Preliminaries

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}. \tag{2.1}$$

In Ramanujan's notation, the Jacobi triple product identity takes the shape

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

The three most important special cases of $f(a, b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{2.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{2.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2.5}$$

where the above three product representations follow from (2.2).

We now list the necessary preliminary results in the following lemmas, which will be used in our later proofs. We start with the p-dissection of $\psi(q)$ and $f(-q)$.

Lemma 2.1. *[7, Theorem 2.1] For an odd prime p,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}), \tag{2.6}$$

and furthermore, for $0 \le k \le \frac{p-3}{2}$, we have

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Lemma 2.2. *[7, Theorem 2.2] For any prime $p \ge 5$,*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+p(6k+1)}{2}}, -q^{\frac{3p^2-p(6k+1)}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}). \tag{2.7}$$

Furthermore, for $-(p-1)/2 \le k \le (p-1)/2$ and $k \neq (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

We also need the following theta function identity.

Lemma 2.3. *[3, p. 69, Equation 36.8] For integers μ and ν with μ even and $\mu > \nu$,*

$$\begin{aligned} \psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) &= \phi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) \\ &+ \sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m}, q^{2\mu-2\nu m}) \\ &+ q^{\mu^3/4-\mu\nu/2}\psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu\nu}, q^{2\mu-\mu\nu}). \end{aligned}$$

For notational convenience, in the rest of the paper we assume that all congruences are modulo 2, unless stated otherwise.

Lemma 2.4. *Let $a(n)$ be defined by*

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty}.$$

Then,

$$\sum_{n=0}^{\infty} a(2n)q^n \equiv \frac{(q^4;q^4)_\infty}{(q^3;q^3)_\infty}.$$

Proof. We have

$$\begin{aligned}\frac{(q^3;q^3)_\infty}{(q;q)_\infty} &\equiv \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty}\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\\ &\equiv \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty}(q;q^6)_\infty(q^5;q^6)_\infty(q^2;q^6)_\infty(q^4;q^6)_\infty\\ &\equiv \frac{1}{(q^6;q^6)_\infty}(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty\\ &\equiv \frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}, \end{aligned} \tag{2.8}$$

where the last congruence follows from (2.1) and (2.2) with $a=-q^5, b=-q$.

Selecting the terms whose powers of q are congruent to 0 modulo 2 in (2.8) and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}a(2n)q^n \equiv \frac{1}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}q^{6n^2-2n}. \tag{2.9}$$

With a replaced by q^8, and b by q^4 in (2.2), we obtain from (2.9),

$$\begin{aligned}\sum_{n=0}^{\infty}a(2n)q^n &\equiv \frac{1}{(q^3;q^3)_\infty}(-q^4;q^{12})_\infty(-q^8;q^{12})_\infty(q^{12};q^{12})_\infty\\ &\equiv \frac{(q^4;q^4)_\infty}{(q^3;q^3)_\infty}.\end{aligned}$$

This completes the proof. ∎

3 Congruences satisfied by $\Delta_5(4n+1)$ and $\Delta_5(4n+2)$

In this section, we establish congruence results related to the generating functions of $\Delta_5(4n+1)$ and $\Delta_5(4n+2)$. Note that, with $k=5$ in (1.1), it follows from the fact $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty$ that

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_5(n)q^n &= \frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty}{(q;q)_\infty^3(q^{22};q^{22})_\infty}\\ &\equiv \frac{1}{(q;q)_\infty(q^{11};q^{11})_\infty}.\end{aligned} \tag{3.1}$$

Theorem 3.1. *If $\psi(q)$ is defined by (2.4), then*

$$\sum_{n=0}^{\infty}\Delta_5(4n+1)q^n \equiv \psi(q), \tag{3.2}$$

$$\sum_{n=0}^{\infty}\Delta_5(4n+2)q^n \equiv q\psi(q^{11}). \tag{3.3}$$

Proof. Applying Lemma 2.3 with $\mu = 6, v = 5$, we have

$$\begin{aligned}\psi(q)\psi(q^{11}) &= \varphi(q^{66})\psi(q^{12}) + qf(q^{88}, q^{44})f(q^{10}, q^2) \\ &\quad + q^{14}f(q^{110}, q^{22})f(q^{20}, q^{-8}) + q^{39}\psi(q^{132})f(q^{30}, q^{-18}). \qquad (3.4)\end{aligned}$$

From (2.2),

$$\begin{aligned}f(q^{88}, q^{44}) &= (-q^{44}; q^{132})_\infty(-q^{88}; q^{132})_\infty(q^{132}; q^{132})_\infty \\ &\equiv (q^{44}; q^{132})_\infty(q^{88}; q^{132})_\infty(q^{132}; q^{132})_\infty \\ &\equiv (q^{44}; q^{44})_\infty.\end{aligned}$$

Similarly, we have

$$\begin{aligned}f(q^{10}, q^2) &\equiv \frac{(q^6; q^6)_\infty(q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty}, \\ f(q^{20}, q^{-8}) &\equiv q^{-8}(q^4; q^4)_\infty, \\ f(q^{30}, q^{-18}) &\equiv q^{-24}.\end{aligned}$$

Substituting the above four congruences into (3.4), and using the fact

$$\varphi(q) = 1 + 2\sum_{n=0}^{\infty} q^{n^2} \equiv 1,$$

we find that

$$\begin{aligned}\frac{(q^2; q^2)_\infty^2(q^{22}; q^{22})_\infty^2}{(q; q)_\infty(q^{11}; q^{11})_\infty} &\equiv \psi(q^{12}) + q(q^{44}; q^{44})_\infty\frac{(q^6; q^6)_\infty(q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty} \\ &\quad + q^6\frac{(q^{66}; q^{66})_\infty(q^{132}; q^{132})_\infty}{(q^{22}; q^{22})_\infty}(q^4; q^4)_\infty + q^{15}\psi(q^{132}). \qquad (3.5)\end{aligned}$$

Applying (3.1), and using (3.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_5(n)q^n &\equiv \frac{1}{(q^4; q^4)_\infty(q^{44}; q^{44})_\infty}\left(\psi(q^{12}) + q^{15}\psi(q^{132})\right) \\ &\quad + q\frac{(q^6; q^6)_\infty(q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty(q^4; q^4)_\infty} + q^6\frac{(q^{66}; q^{66})_\infty(q^{132}; q^{132})_\infty}{(q^{22}; q^{22})_\infty(q^{44}; q^{44})_\infty}. \qquad (3.6)\end{aligned}$$

If we extract those terms whose powers of q are congruent to 1 modulo 4 from (3.6), divide both sides by q, replace q^4 by q, and use (2.4) along with Lemma 2.4, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_5(4n+1)q^n &\equiv \frac{(q^3; q^3)_\infty}{(q; q)_\infty}\sum_{n=0}^{\infty}a(2n)q^n \\ &\equiv \frac{(q^3; q^3)_\infty}{(q; q)_\infty}\cdot\frac{(q^4; q^4)_\infty}{(q^3; q^3)_\infty} \\ &\equiv \psi(q).\end{aligned}$$

Collecting the terms whose powers of q are congruent to 2 modulo 4 from (3.6), dividing both sides by q and replacing q^4 by q, and using Lemma 2.4, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_5(4n+2)q^n &\equiv q\frac{(q^{33};q^{33})_\infty}{(q^{11};q^{11})_\infty}\sum_{n=0}^{\infty}a(2n)q^{11n}\\ &\equiv q\frac{(q^{33};q^{33})_\infty}{(q^{11};q^{11})_\infty}\cdot\frac{(q^{44};q^{44})_\infty}{(q^{33};q^{33})_\infty}\\ &\equiv q\psi(q^{11}).\end{aligned}$$

The proof of Theorem 3.1 is completed. ∎

As a consequence of this theorem, we have the following corollary.

Corollary 3.1. *For $n \geq 0$, we have*

$$\Delta_5(4n+1) \equiv \Delta_5(44n+6), \tag{3.7}$$

and

$$\Delta_5(44n+r) \equiv 0, \tag{3.8}$$

whenever $r = 2, 10, 14, 18, 22, 26, 30, 34, 38, 42$.

Proof. Each term on the right-hand side of (3.3) is of the form q^{11n+1}. Equating the coefficients of $q^{11n+(r-2)/4}$ in (3.3), we obtain (3.8). Meanwhile, extracting those terms whose powers of q are congruent to 1 modulo 11, we arrive at

$$\sum_{n=0}^{\infty}\Delta_5(44n+6)q^{11n+1} \equiv q\psi(q^{11}). \tag{3.9}$$

Dividing both sides of (3.9) by q and replacing q^{11} by q, we deduce that

$$\sum_{n=0}^{\infty}\Delta_5(44n+6)q^n \equiv \psi(q).$$

This combined with (3.2) yields (3.7). ∎

We are now ready to characterize the parity of $\Delta_5(4n+1)$ and $\Delta_5(4n+2)$.

Theorem 3.2. *For $n \geq 0$,*

(1) $\Delta_5(4n+1)$ *is odd if and only if* $n = m(m+1)/2$ *for some integer* $m \geq 0$*;*

(2) $\Delta_5(4n+2)$ *is odd if and only if* $n = 11m(m+1)/2+1$ *for some integer* $m \geq 0$.

Proof. From (2.4) and (3.2), we have

$$\sum_{n=0}^{\infty}\Delta_5(4n+1)q^n \equiv \sum_{m=0}^{\infty}q^{m(m+1)/2}.$$

Equating the coefficients of q^n, we see that $\Delta_5(4n+1)$ is odd if and only if $n = m(m+1)/2$ for some integer $m \geq 0$. Similarly, from (3.3), we can conclude that $\Delta_5(4n+2)$ is odd if and only if $n = 11m(m+1)/2 + 1$ for some integer $m \geq 0$. This completes the proof. ∎

Based on Theorem 3.2, we can derive several Ramanujan type congruences, some of which are listed in the following corollary.

Corollary 3.2. *If $n \geq 0$, we have*

$$\begin{aligned} \Delta_5(12n + r_1) &\equiv 0, \\ \Delta_5(20n + r_2) &\equiv 0, \\ \Delta_5(28n + r_3) &\equiv 0, \end{aligned}$$

where $r_1 = 9, 10$, $r_2 = 2, 9, 14, 17$ and $r_3 = 2, 9, 10, 14, 17, 21$.

Proof. Since $3n + 2$ can not be written as $m(m+1)/2$ or $11m(m+1)/2 + 1$, we have

$$\Delta_5(12n+9) \equiv 0 \quad \text{and} \quad \Delta_5(12n+10) \equiv 0.$$

Since no triangular number is congruent to 2 or 4 modulo 5, $\Delta_5(20n + 9)$ and $\Delta_5(20n + 17)$ are both even. A similar argument can be applied to the remaining cases, and we omit the details here. ∎

Corollary 3.3. *We have*

$$\sum_{n=0}^{\infty} \Delta_5(12n+1)q^n \equiv f(-q). \tag{3.10}$$

Proof. Using the 3-dissection of $\psi(q)$ in (2.6) for $p = 3$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \Delta_5(4n+1)q^n \equiv \psi(q) &\equiv f(q^3, q^6) + q\psi(q^9) \\ &\equiv f(-q^3, -q^6) + q\psi(q^9) \\ &\equiv f(-q^3) + q\psi(q^9). \end{aligned}$$

Extracting the terms with powers of q which are multiples of 3, and then replacing q^3 by q, we see that

$$\sum_{n=0}^{\infty} \Delta_5(12n+1)q^n \equiv f(-q).$$

This completes the proof of the corollary. ∎

4 Elementary proof of a parity result of Radu and Sellers

Relying heavily on modular forms, Radu and Sellers [14] established several congruences satisfied by certain families of broken k-diamond partitions. In fact, what they obtained are parity results on t-cores. The second author with her collaborators [6] extended these results for larger values of k and also obtained lower bounds for the number of n's up to a fixed large N for which $\Delta_k(n)$ is odd. In this section, we present an elementary proof of the following parity result on broken 5-diamond partitions established by Radu and Sellers.

Theorem 4.1. *For $n \geq 0$,*

$$\Delta_5(22n+s) \equiv 0, \tag{4.1}$$

provided that $s = 2, 8, 12, 14, 16$.

Proof. For any $s \in \{2, 8, 12, 14, 16\}$, each $22n+s$ is congruent to s or $s+22$ modulo 44. Thus the desired result follows once we prove that

$$\Delta_5(44n+t) \equiv 0,$$

where $t = 2, 8, 12, 14, 16, 24, 30, 34, 36, 38$. Since we have seen that

$$\Delta_5(44n+t) \equiv 0,$$

for $t = 2, 14, 30, 34, 38$ in Corollary 3.1, we only need to show that

$$\Delta_5(44n+m) \equiv 0,$$

for $m = 8, 12, 16, 24, 36$. Extracting those terms whose powers of q are even in (3.6) and replacing q^2 by q yields

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_5(2n)q^n &\equiv \frac{\psi(q^6)}{(q^2;q^2)_\infty (q^{22};q^{22})_\infty} + q^3 \frac{(q^{33};q^{33})_\infty (q^{66};q^{66})_\infty}{(q^{11};q^{11})_\infty (q^{22};q^{22})_\infty} \\ &\equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty (q^{22};q^{22})_\infty} + q^3 \frac{(q^{33};q^{33})_\infty (q^{66};q^{66})_\infty}{(q^{11};q^{11})_\infty (q^{22};q^{22})_\infty}.\end{aligned} \tag{4.2}$$

From (2.8), we see that

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} \equiv \sum_{n=-\infty}^{\infty} q^{3n^2-2n} \equiv \sum_{n=-\infty}^{\infty} q^{3n^2+2n}.$$

Thus we can rewrite (4.2) as

$$\sum_{n=0}^{\infty} \Delta_5(2n)q^n \equiv \frac{1}{(q^{22};q^{22})_\infty} \sum_{n=-\infty}^{\infty} q^{6n^2+4n} + q^3 \frac{(q^{33};q^{33})_\infty (q^{66};q^{66})_\infty}{(q^{11};q^{11})_\infty (q^{22};q^{22})_\infty}. \tag{4.3}$$

Employing the fact that

$$6n^2 + 4n \equiv 0, 2, 10, 14, 16, 20 \pmod{22},$$

and observing that each term in the second infinite product of (4.3) has the form q^{22n+3} or q^{22n+14}, we conclude that there are no terms of the form q^{22n+r} with $r = 4, 6, 8, 12, 18$ on the right-hand side of (4.3). Equating the coefficients of q^{22n+r} in (4.3), we find that

$$\Delta_5(44n+2r) \equiv 0,$$

whenever $r = 4, 6, 8, 12, 18$. This completes the proof. ∎

5 Generalization to infinite families

In this section, we generalize the results in Section 3 to infinite families of congruences. We start with recalling the congruence (3.2)

$$\sum_{n=0}^{\infty} \Delta_5(4n+1)q^n \equiv \psi(q),$$

and investigate a generalization of this congruence.

Theorem 5.1. *For an odd prime p and integers $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha} n + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv \psi(q). \tag{5.1}$$

Proof. We proceed by induction on α. The case $\alpha = 0$ is the congruence (3.2). Assume that

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha} n + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv \psi(q)$$

is true for some fixed integer $\alpha \geq 0$. We show that the same congruence is true when α is replaced by $\alpha+1$. Using the p-dissection of $\psi(q)$ in (2.6), and then extracting the terms corresponding to $q^{p^2 n + \frac{p^2-1}{8}}$, we obtain

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha+2} n + \frac{p^{2\alpha+2}+1}{2}\right) q^{p^2 n + \frac{p^2-1}{8}}$$
$$= \sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha}\left(p^2 n + \frac{p^2-1}{8}\right) + \frac{p^{2\alpha}+1}{2}\right) q^{p^2 n + \frac{p^2-1}{8}} \equiv q^{\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Now we divide both sides by $q^{\frac{p^2-1}{8}}$ and replace q^{p^2} by q to complete the proof. ∎

Corollary 5.1. *For an odd prime p and integers $\alpha, n \geq 0$, we have*

$$\Delta_5\left(4 \cdot p^{2\alpha+2} n + \frac{p^{2\alpha+2} + 8j \cdot p^{2\alpha+1} + 1}{2}\right) \equiv 0$$

for $1 \leq j \leq p-1$.

Proof. Using (2.6) in (5.1) and extracting the terms of the form $q^{pn + \frac{p^2-1}{8}}$, we obtain

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha+1} n + \frac{p^{2\alpha+2}+1}{2}\right) q^{pn + \frac{p^2-1}{8}}$$
$$= \sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha}\left(pn + \frac{p^2-1}{8}\right) + \frac{p^{2\alpha}+1}{2}\right) q^{pn + \frac{p^2-1}{8}} \equiv q^{\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Canceling $q^{\frac{p^2-1}{8}}$ from both sides and replacing q^p by q, we have

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}+1}{2}\right) q^n \equiv \psi(q^p).$$

The terms appearing on the right side are powers of q^p, and thus for $j = 1, 2, \ldots, p-1$,

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha+1}(pn+j) + \frac{p^{2\alpha+2}+1}{2}\right) q^n \equiv 0,$$

which completes the proof. ∎

Corollary 5.2. *Assume that p is an odd prime and α, n are non-negative integers. If $1 \leq j \leq p-1$ and $\left(\frac{8j+1}{p}\right) = -1$, then*

$$\Delta_5\left(4 \cdot p^{2\alpha+1}n + \frac{(8j+1)p^{2\alpha}+1}{2}\right) \equiv 0.$$

Proof. For $\left(\frac{8j+1}{p}\right) = -1$, we have $j \not\equiv \frac{p^2-1}{8} \pmod p$ and $j \not\equiv \frac{k^2+k}{2} \pmod p$ where $0 \leq k \leq (p-3)/2$. Now, using (2.6) in (5.1) and extracting the terms of the form q^{pn+j} for which $\left(\frac{8j+1}{p}\right) = -1$, we obtain

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot p^{2\alpha}(pn+j) + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv 0.$$

This finishes the proof. ∎

One can generalize the above congruences to a product of finitely many odd primes as we show below.

Theorem 5.2. *Let $p_1, p_2, \ldots, p_r$ be distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be non-negative integers. Then*

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} \cdot n + \frac{\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) q^n \equiv \psi(q). \tag{5.2}$$

Proof. The proof uses induction on r, the number of primes. For $r = 1$, the congruence holds from (5.1). Assuming the result holds for r, we prove the same is true for $r+1$. Using the p-dissection of $\psi(q)$ for $p = p_{r+1}$ in (2.6), we extract the terms of the form $q^{p_{r+1}^2 n + \frac{p_{r+1}^2-1}{8}}$. After canceling $q^{\frac{p_{r+1}^2-1}{8}}$ from both sides and replacing $q^{p_{r+1}^2}$ by q, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} \cdot p_{r+1}^2 n + \frac{\prod_{l=1}^{r} p_l^{2\alpha_l} \cdot p_{r+1}^2+1}{2}\right) q^n \\
&\quad = \sum_{n=0}^{\infty} \Delta_5\left(4 \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} \cdot \left(p_{r+1}^2 n + \frac{p_{r+1}^2-1}{8}\right) + \frac{\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) q^n \equiv \psi(q).
\end{aligned}$$

Now as in the proof of Theorem 5.1, for $p = p_{r+1}$, we apply induction on the exponent of p_{r+1} to obtain the desired result for the case $r+1$, and this completes the induction on r. ∎

Next, we note an analogue of Theorem 3.2 and Corollary 3.2 in this general setting.

Corollary 5.3. *For a positive integer r, let $p_1, p_2, \ldots, p_r$ denote distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be non-negative integers. Then, for any non-negative integer n,*

$$\Delta_5\left(4\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right)$$

is odd if and only if n is a triangular number. Also, we have the following congruences:

$$\begin{aligned}
\Delta_5\left(12\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{17\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) &\equiv 0,\\
\Delta_5\left(20\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{17\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) &\equiv 0,\\
\Delta_5\left(20\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{33\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) &\equiv 0,\\
\Delta_5\left(28\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{17\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) &\equiv 0,\\
\Delta_5\left(28\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{33\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) &\equiv 0,\\
\Delta_5\left(28\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{41\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right) &\equiv 0.
\end{aligned}$$

Proof. The proof follows by applying Theorem 5.2 and (2.4) along with the fact that no triangular number is of the form $3m+2, 5m+2, 5m+4, 7m+2, 7m+4$ or $7m+5$. We omit the details here. ∎

Corollary 5.4. *Let $p_1, p_2, \ldots, p_r$ be distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be non-negative integers. Then for any integer $n \geq 0$ and $1 \leq j \leq r$,*

$$\Delta_5\left(4p_j^2\cdot\prod_{l=1}^{r} p_l^{2\alpha_l}\cdot n+\frac{p_j(8n_j+p_j)\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right)\equiv 0,$$

where $1 \leq n_j \leq p_j - 1$.

Proof. For a fixed j, $1 \leq j \leq r$, using Theorem 5.2 and the p_j-dissection of $\psi(q)$ in (2.6), we note that

$$\sum_{n=0}^{\infty}\Delta_5\left(4\prod_{l=1}^{r} p_l^{2\alpha_l}\left(p_jn+\frac{p_j^2-1}{8}\right)+\frac{\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right)q^n\equiv\psi(q^{p_j}).$$

Therefore,

$$\Delta_5\left(4\prod_{l=1}^{r} p_l^{2\alpha_l}\left(p_j(p_jn+n_j)+\frac{p_j^2-1}{8}\right)+\frac{\prod_{l=1}^{r} p_l^{2\alpha_l}+1}{2}\right)\equiv 0,$$

which is what we wanted to prove. ∎

Corollary 5.5. *Let $p_1, p_2, \ldots, p_r$ be distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be non-negative integers. For any integer $n \geq 0$ and $1 \leq j \leq r$, we have*

$$\Delta_5\left(4p_j \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} \cdot n + \frac{(8n_j+1)\prod_{l=1}^{r} p_l^{2\alpha_l} + 1}{2}\right) \equiv 0,$$

provided that $1 \leq n_j \leq p_j - 1$ and $\left(\frac{8n_j+1}{p_j}\right) = -1$.

Proof. Once again, using (2.6) in Theorem 5.2 and extracting the terms of the form $q^{p_j n + n_j}$, we obtain

$$\sum_{n=0}^{\infty} \Delta_5\left(4 \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} \cdot (p_j n + n_j) + \frac{\prod_{l=1}^{r} p_l^{2\alpha_l} + 1}{2}\right) q^n \equiv 0.$$

This finishes the proof. ∎

Next, using the p-dissection of $f(-q)$, we generalize the congruence in Corollary 3.3.

Theorem 5.3. *For any prime $p \geq 5$ and non-negative integer α,*

$$\sum_{n=0}^{\infty} \Delta_5\left(12p^{2\alpha} n + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv f(-q). \tag{5.3}$$

Proof. We apply induction on α. Note that the case $\alpha = 0$ is (3.10). Now assume the result holds for any α. Using the p-dissection for $f(-q)$ in (2.7) and extracting the terms of the form $q^{p^2 n + \frac{p^2-1}{24}}$, we have

$$\sum_{n=0}^{\infty} \Delta_5\left(12p^{2\alpha}\left(p^2 n + \frac{p^2-1}{24}\right) + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv f(-q),$$

or

$$\sum_{n=0}^{\infty} \Delta_5\left(12p^{2\alpha+2} n + \frac{p^{2\alpha+2}+1}{2}\right) q^n \equiv f(-q).$$

Therefore, the result is true for $\alpha + 1$ as well, which completes the proof. ∎

Corollary 5.6. *Given any prime $p \geq 5$ and integers $\alpha \geq 1, n \geq 0$, we have*

$$\Delta_5\left(12p^{2\alpha} n + \frac{(24j+p)p^{2\alpha-1}+1}{2}\right) \equiv 0,$$

for $j = 1, 2, \ldots, p-1$.

Proof. Using (2.7) in (5.3), we see that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \Delta_5\left(12p^{2\alpha}\left(pn + \frac{p^2-1}{24}\right) + \frac{p^{2\alpha}+1}{2}\right) q^n \equiv f(-q^p),$$

or

$$\sum_{n=0}^{\infty} \Delta_5\left(12p^{2\alpha+1} n + \frac{p^{2\alpha+2}+1}{2}\right) q^n \equiv f(-q^p).$$

Thus, we conclude that for $j = 1, 2, \ldots, p-1$,

$$\sum_{n=0}^{\infty} \Delta_5 \left(12p^{2\alpha+1}(pn+j) + \frac{p^{2\alpha+2}+1}{2} \right) q^n \equiv 0,$$

which proves the claim in the corollary. ∎

Corollary 5.7. *For any prime $p \geq 5$ and integers $\alpha, n \geq 0$,*

$$\Delta_5 \left(12p^{2\alpha+1}n + \frac{(24j+1)p^{2\alpha}+1}{2} \right) \equiv 0,$$

provided that $1 \leq j \leq p-1$ and $\left(\frac{24j+1}{p}\right) = -1$.

Proof. For $\left(\frac{24j+1}{p}\right) = -1$, we have $j \not\equiv \frac{p^2-1}{24} \pmod{p}$ and $j \not\equiv \frac{3k^2+k}{2} \pmod{p}$. Now with the use of (2.7) in Theorem 5.3, it is easy to see that

$$\sum_{n=0}^{\infty} \Delta_5 \left(12p^{2\alpha}(pn+j) + \frac{p^{2\alpha}+1}{2} \right) q^n \equiv 0.$$

This completes the proof. ∎

Furthermore, we have the following generalizations.

Theorem 5.4. *For a positive integer r, let $p_1, p_2, \ldots, p_r \geq 5$ be distinct prime numbers and $\alpha_1, \ldots, \alpha_r$ be non-negative integers. Then,*

$$\sum_{n=0}^{\infty} \Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} \cdot n + \frac{\prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) q^n \equiv f(-q). \tag{5.4}$$

Proof. The proof follows by induction on r. The case $r = 1$ is given in Theorem 5.3. Assuming the result holds for any positive integer r, we show that the same is true for $r+1$. For any prime number $p_{r+1} \geq 5$, using the p_{r+1}-dissection of $f(-q)$ in (2.7), we obtain

$$\sum_{n=0}^{\infty} \Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} \left(p_{r+1}^2 n + \frac{p_{r+1}^2 - 1}{24} \right) + \frac{\prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) q^n \equiv f(-q).$$

This gives

$$\sum_{n=0}^{\infty} \Delta_5 \left(12p_{r+1}^2 \prod_{j=1}^{r} p_j^{2\alpha_j} \cdot n + \frac{p_{r+1}^2 \prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) q^n \equiv f(-q).$$

In order to complete the proof, we apply induction on α_{r+1} as in the proof of Theorem 5.3 and omit the details here. ∎

Corollary 5.8. *For a positive integer $r \geq 1$, let $p_1, p_2, \ldots, p_r \geq 5$ denote distinct prime numbers and let $\alpha_1, \ldots, \alpha_r$ be non-negative integers. For $l = 1, 2, \ldots, r$ such that $\alpha_l \geq 1$, we have, for $n \geq 0$ and $n_l = 1, 2, \ldots, p_l - 1$,*

$$\Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} \cdot n + \frac{(24n_l + p_l)p_l^{2\alpha_l - 1} \prod_{\substack{j=1 \\ j \neq l}}^{r} p_j^{2\alpha_j} + 1}{2} \right) \equiv 0.$$

Proof. By Theorem 5.4, we have for $\alpha_l \geq 0$,

$$\sum_{n=0}^{\infty} \Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} \cdot n + \frac{\prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) q^n \equiv f(-q).$$

Therefore, employing the p_l-dissection of $f(-q)$ in (2.7) and extracting the terms corresponding to $q^{p_l n + \frac{p_l^2-1}{24}}$, we obtain

$$\sum_{n=0}^{\infty} \Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} \left(p_l n + \frac{p_l^2 - 1}{24} \right) + \frac{\prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) q^n \equiv f(-q^{p_l}).$$

This implies

$$\Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} \left(p_l(p_l n + n_l) + \frac{p_l^2 - 1}{24} \right) + \frac{\prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) \equiv 0,$$

or

$$\Delta_5 \left(12 p_l^2 \prod_{j=1}^{r} p_j^{2\alpha_j} \cdot n + \frac{(24 n_l + p_l) p_l^{2\alpha_l + 1} \prod_{\substack{j=1 \\ j \neq l}}^{r} p_j^{2\alpha_j} + 1}{2} \right) \equiv 0.$$

This finishes the proof of the corollary. ∎

Corollary 5.9. *For a positive integer $r \geq 1$, let $p_1, p_2, \ldots, p_r \geq 5$ denote distinct prime numbers and let $\alpha_1, \ldots, \alpha_r$ be non-negative integers. For $l = 1, 2, \ldots, r$, if $1 \leq n_l \leq p_l - 1$ and $\left(\frac{24n_l+1}{p_l}\right) = -1$, we have, for $n \geq 0$,*

$$\Delta_5 \left(12 p_l \prod_{j=1}^{r} p_j^{2\alpha_j} \cdot n + \frac{(24 n_l + 1) \prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) \equiv 0.$$

Proof. As in the proof of Corollary 5.7, by Theorem 5.4 and the p_l-dissection of $f(-q)$ in (2.7), we obtain

$$\sum_{n=0}^{\infty} \Delta_5 \left(12 \prod_{j=1}^{r} p_j^{2\alpha_j} (p_l n + n_l) + \frac{\prod_{j=1}^{r} p_j^{2\alpha_j} + 1}{2} \right) q^n \equiv 0,$$

which upon simplification gives the desired congruence. ∎

Remark. With the help of the fact that $\Delta_5(4n+1) \equiv \Delta_5(44n+6)$, one can easily obtain the parity results of $\Delta_5(4n+2)$ corresponding to all the results of this section. It is interesting to point out that we can readily and extremely generalize the parity results for broken 5-diamond partitions obtained by Ahmed and Baruah [1], which will be shown below.

According to Theorem 5.1, we have, if $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \Delta_5 \left(44 \cdot p^{2\alpha} \cdot n + \frac{44 \cdot p^{2\alpha} + 4}{8} \right) q^n \equiv \psi(q). \tag{5.5}$$

Theorem 2.1 of Ahmed and Baruah [1] is the special case when $p = 3$ in (5.5).

Similarly, corresponding to Theorem 5.2, we have the following result: Let $p_1, p_2, \ldots, p_r$ be distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ be non-negative integers, then

$$\sum_{n=0}^{\infty} \Delta_5 \left(44 \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} \cdot n + \frac{11 \cdot \prod_{l=1}^{r} p_l^{2\alpha_l} + 1}{2} \right) q^n \equiv \psi(q). \tag{5.6}$$

In particular, if we let $r = 2$, $p_1 = 3, p_2 = p$ and $\alpha_1 = 1, \alpha_2 = \alpha$ in (5.6), then Theorem 2.4 of Ahmed and Baruah [1] follows immediately.

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