Recall the definition of a probability space; a triple $(\Omega, B, P)$ where $\Omega$ is a set, $B$ is a set of subsets of $\Omega$ (a $\sigma$-algebra), such that:

- $\emptyset \in B$
- if $B \in B$ then $\Omega \setminus B \in B$ (so $\Omega = \Omega \setminus \emptyset \in B$)
- if $B_1, B_2, \ldots \in B$ then $\bigcup_{i=1}^{\infty} B_i \in B$

and $P : B \to [0, 1]$ (the probability) is a function such that:

- $P(\Omega) = 1$
- if $B_1, B_2, \ldots$ are disjoint elements of $B$, then

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$

(countable additivity).
Random variables

A (real-valued) random variable $X$ is a function $X: \Omega \to \mathbb{R}$ such that for all $b \in \mathbb{R}$, the set

$$\{ \omega \in \Omega : X(\omega) < b \} \in \mathcal{B}.$$ 

The cumulative distribution function $F_X$ of $X$ is the function $F_X: \mathbb{R} \to \mathbb{R}$ given by

$$F_X(x) = P(\{ \omega \in \Omega : X(\omega) \leq x \}).$$

Often in practice, $F_X$ is differentiable, and then

$$f_X = F'_X$$

is called the probability density function

and then

$$F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt.$$ 

The expected value of $X$ is

$$E[X] = \int_{-\infty}^{\infty} t \cdot f_X(t) \, dt = \int_{-\infty}^{\infty} t \, dF_X(t)$$

and the variance of $X$ is

$$\sigma^2 = \sigma^2[X] = \text{Var}[X] = \int_{-\infty}^{\infty} (t - E[X])^2 \, dF_X(t).$$
The normal distribution with parameters 
\( \mu \) (mean or expectation) and 
\( \sigma \) (standard deviation) has the cumulative distribution function
\[
N_{\mu,\sigma}(x) = \int_{-\infty}^{x} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{t-\mu}{\sigma} \right)^2} dt
\]
and the probability density function
\[
\phi(x;\mu,\sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}.
\]
Thus our \( N(x) \) is the same as \( N_{\mu,\sigma}(x) \).

If a random variable has the distribution given by \( N_{\mu,\sigma}(x) \) (i.e., if \( F_X(x) = N_{\mu,\sigma}(x) \) for all \( x \in \mathbb{R} \)), we write
\[
X \sim N(\mu, \sigma^2).
\]
In what follows we often consider a random variable \( Z \) such that
\[
Z \sim N(0, 1).
\]
If \( X \sim N(\mu, \sigma^2) \), and if we define
\[
Z = \frac{X - \mu}{\sigma},
\]
then
\[
Z \sim N(0, 1).
\]

Thus for \( b \in \mathbb{R} \),
\[
P(\{X \leq b\}) = P(\{ \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma} \})
\]
\[
= P(\{Z \leq \frac{b - \mu}{\sigma} \}) = N(\frac{b - \mu}{\sigma}).
\]

Also
\[
P(\{X > b\}) = 1 - P(\{X \leq b\})
\]
\[
= 1 - N(\frac{b - \mu}{\sigma}) = N(-\frac{b - \mu}{\sigma}) = N(\frac{\mu - b}{\sigma})
\]
\[
\text{since} \quad N(x) + N(-x) = 1 \quad \text{for all} \ x \in \mathbb{R}.
\]
If \( X_1, X_2 \) are random variables with 
\[
E(X_i) = \mu_i, \quad \text{Var}(X_i) = \sigma_i^2,
\]
then the 
\[
\rho_{12} = \frac{1}{\sigma_1 \sigma_2} \mathbb{E} \left[ (X_1 - \mu_1)(X_2 - \mu_2) \right]
\]
The covariance of \( X_1, X_2 \) is \( \text{Cov}[X_1, X_2] = \sigma_{12} = \rho_{12} \sigma_1 \sigma_2 \).

If \( X_1, \ldots, X_n \) are random variables,
\[
E \left[ \sum_{i=1}^{n} a_i X_i \right] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu_i,
\]
\[
\text{Var} \left[ \sum_{i=1}^{n} a_i X_i \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}.
\]
The normal distribution has the special property that if \( X_i \sim \text{N}(\mu_i, \sigma_i^2) \), then
\[
\sum_{i=1}^{n} a_i X_i \sim \text{N} \left( \sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij} \right).
\]
The Central Limit Theorem

Let $X_1, X_2, \ldots$ be random variables with the same expected value $\mu$ and the same variance $\sigma^2$. Suppose that the $X_i$ are independent (if $i \neq j$ and $a, b \in \mathbb{R}$ then
\[ P(\min \{X_i \leq a \text{ and } X_i \leq b\}) = P(X_i \leq a) P(X_i \leq b) \]
and identically distributed ($X_i = X_j$).

Then, as $n \to \infty$, we have
\[
\frac{\sum_{i=1}^{n} X_i}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]

\[
\lim_{n \to \infty} P \left( \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma} \leq x \right) = N(0, 1)
\]
for all $x \in \mathbb{R}$.

In this sense,
\[
\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma}
\]
tends to the standard normal distribution $N(0, 1)$.

There are also stronger forms of the Central Limit Theorem where the $\mu$ and $\sigma$ need not be the same for all $X_i$ (for example).
Exponentials of certain random variables.

Let $X, Y$ be random variables such that

$$Y = e^X,$$

so $Y > 0$ and

$$X = \ln Y.$$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$E[Y] = E[e^X] = e^{\mu + \frac{1}{2} \sigma^2}$$

and

$$\text{Var}[Y] = \text{Var}[e^X] = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1).$$

Note that this is based on the assumption that $X \sim \mathcal{N}(\mu, \sigma^2)$. 