Ch. 13 Market-making and Delta-hedging

Using Delta

Suppose that a market-maker is holding a position in a certain stock as well as in a number of options for that stock. A market-maker (as an intermediary) does not want to be exposed to market risk (due to changes in the stock price and corresponding changes in the option prices). Recall that the total delta of the portfolio (based on this one stock) is

\[ \Delta_{portfolio} = \sum_{i} n_i \Delta_i \]

if there are \( n_i \) options of type \( i \), each with delta \( \Delta_i \) (\( n_i \) itself may be >0 or <0 depending on if they are long or short on the option). We may include the stock here for some \( i \), recalling that the delta of a (long) stock is \( = 1 \).

If \( \Delta_{portfolio} \) is not (approximately) zero, the market-maker may want to enter into some additional position to get the total delta to be close to zero.
Delta changes with the stock price. So if we use a certain delta now, it will no longer be accurate after \( S \) has changed by more than just a little.

In particular, if there is a sudden large change in \( S \), the previous delta may be quite wrong now. Thus, a delta-hedged position (where Delta is close to zero) may fail to be so after a large change in \( S \).

To deal with this, frequent re-balancing of the market-maker's portfolio may be required. The market-maker may mark-to-market (calculate the current profit or loss in case the portfolio, hypothetically, were to be liquidated immediately) the portfolio often (say, several times a day, or daily) to discern whether rebalancing is required.

Using the gamma (second derivative), e.g., effectively using a Taylor polynomial of degree 2, gives a more accurate understanding of the situation of the portfolio. See the book for many numerical examples of this (pages 390-392).
Note that the gamma of a stock is zero. The gamma of long calls and puts is > 0, and short is < 0.

If we desire to make the total gamma of a portfolio (close to) zero, to further reduce dependence on changes in S, we cannot do this using the stock. If, now, \( \Gamma > 0 \), we need to short some options, and if now \( \Gamma < 0 \) we need to buy some options. This makes it harder to obtain gamma-neutrality \( (\Gamma = 0) \).
Using theta (Θ)

The main Greeks used by market-makers to avoid market risk are δ, ν, Γ, Θ. If we now take the view that time to expiration $T$ is fixed and current time $t$ varies, we can write (assuming $r,σ,δ$ fixed) the change in the price of a call option (similar remarks apply to puts) as

$$C(S_{t+h}, T-t-h) = C(S_t, T-t) +$$

$$+ \epsilon \Delta(S_t, T-t) + \frac{1}{2} \epsilon^2 \Gamma(S_t, T-t) +$$

$$+ h \Theta(S_t, T-t)$$

where $\epsilon = S_{t+h} - S_t$. Since option prices change also due to the passage of time (taken into account by the $\Theta$-term), a need for rebalancing may arise also from considering the combined $\Theta$ of the portfolio.
Chapter 14  Exotic options

Asian options
- based on averages
- E.g. let the last stock price in time period $i$ $(1 \leq i \leq N)$ be $S_i$.
- Use the arithmetic mean $\frac{1}{N} \sum_{i=1}^{N} S_i$ or the geometric mean $(\prod_{i=1}^{N} S_i)^{1/N}$, denote either by $A$

- Use $A$ to replace the strike price or the stock price $S_T$ at expiration time $T$
- Hence get the following payoffs:
  - average price call $(A-K)^+$
  - put $(K-A)^+$
  - average strike call $(S_T - A)^+$
  - put $(A - S_T)^+$

And the average can be arithmetic or geometric.

Averages are more stable than the stock price, so these options are cheaper than the standard ones.

If $N=1$, we get the standard options.

Asian options can be used by companies as insurance in situations where an activity takes place bit by bit over a period of time such as receiving payments in a foreign currency (e.g. Asian currency options).
Barrier options

We specify not only the strike price \( K \) but also a barrier \( B \) (usually \( B+K \)).

The following applies to calls and puts:

- **Knock-out options**: The option becomes ineffective if the asset price reaches the barrier. Going below a barrier = down-and-out; above = up-and-out.

- **Knock-in options**: become effective only if the barrier is touched ("down-and-in", "up-and-in").

- **Rebate options**: make a fixed payment if the barrier is reached, either at that time or at expiration time (deferred rebate).

  "Up rebates" and "down rebates" depending on if the barrier is above or below the current asset price.

**Parity relation**:

\[
\text{Knock-in option} + \text{Knock-out option} = \text{Ordinary option}
\]

Hence barrier options are cheaper than ordinary options.
Compound options

These are options to buy an option.

E.g., if $0 < t_s < T$, we may buy an call that allows us to buy, at $t = t_s$, for the price $x$ (the strike price of the compound option), a call option on the stock with strike price $K$ and expiration date $T$.

The payoff at time $t_s$ is

$$( C(S_{t_s}, K, T-t_s) - x )^+$$

actual price of the call option at time $t_s$.

Parity relation.

These compound options have the same expiration.

Call on Call + Put on Call = and the same strike price $x$.

$-e^{-rT}$

$e^{-rT}$

present value of strike price of the compound option.

There is a similar relation for

Call on Put - Put on Put.
Gape options

In addition to specifying a strike price $K$, we specify a level $K_1$ such that the payoff of the option is non-zero only if at expiration time $T$, we have

$S_T > K$, for calls

$S_T < K_1$, for puts

and then the payoff is

$S_T - K$ for calls

$K - S_T$ for puts.

For standard options, $K_1 = K$.

We usually have $K_1 \geq K$ for calls, $K_1 \leq K$ for puts.

If $K_1 < K$ for calls or $K_1 > K$ for puts, then the payoff ($S_T - K$ for calls, $K - S_T$ for puts) can be negative, which means that the buyer of the option must pay more money to the seller at expiration (exercise is compulsory for such options). The term "option" is not best possible for such financial products.
Exchange options on outperformance options

The payoff at \( t = T \) is

\[
(S_T - K_T)^+
\]

where \( K_T \) is the price at \( t = T \) of a risky asset. Thus the payoff is > 0 only if the stock outperforms a companion asset that has a variable price, e.g., a suitably normalized price (to make it comparable to \( S_T \) in a reasonable way) of a stock index, or a different stock.