For the normalized (mean = 0, standard deviation = 1) normal distribution, the probability density function is

$$ \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} $$

and the cumulative distribution function is

$$ N(x) = \int_{-\infty}^{x} \varphi(t) \, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt. $$

For all real $x$, we have $0 < N(x) < 1$, and

$$ \lim_{x \to -\infty} N(x) = 0, \quad \lim_{x \to \infty} N(x) = 1. $$

We have

$$ 1 - N(x) = \int_{x}^{\infty} \varphi(t) \, dt = \int_{-\infty}^{\infty} \varphi(u) \, du = N(-x) $$

So

$$ N(x) + N(-x) = 1 $$

for all $x \in \mathbb{R}$. We have

$$ N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. $$
We consider prices of calls and puts on a stock depending on the following 6 parameters:

\[ S = \text{current stock price} \quad K = \text{strike price} \]
\[ T = \text{time to expiration} \quad \sigma = \text{volatility} \]
\[ r = \text{risk-free interest rate} \quad \delta = \text{dividend rate} \]

We write, for brevity, \( C \) for the price of a call omitting these parameters from the notation.

Recall the put-call parity for European (not American) options:

\[ C - P = S e^{-\delta T} - K e^{-r T} \]

Here we also have the interpretation (which this comes from, in fact) that:

\[ K e^{-r T} = \text{present value of } (K \text{ at time } T) \]
\[ S e^{-\delta T} = \text{present value of forward price of stock} \]
\[ = \text{present value of forward price of stock.} \]
Black–Scholes formula for the price of a European call option:

\[ C = S \, e^{-rT} \, N(d_1) - Ke^{-rT} \, N(d_2) \]

where

\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

One can also write

\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \]

Put–call parity then implies the price of a European put option:

\[ P = C + Ke^{-rT} - S \, e^{-rT} = Ke^{-rT} \, N(-d_2) - S \, e^{-rT} \, N(-d_1) \]

Since \( 1 - N(x) = N(-x) \).
One interpretation of the formulas:

A portfolio replicating the call option should have

\[ \Delta = e^{-\delta T} N(d_1) \]

shares and the amount

\[-Ke^{-\delta T} N(d_2)\]

in the bond.

Since \( S \) varies with time (and so does \( T \), which is time to expiration), a continuous

"rebalancing" of the portfolio would be required.

In practice, one could rebalance only finitely many times, and each time there would be
transaction costs (not considered in the formula).
Modifications to the formula when applied to other situations (we discuss European call options only)

1) Options on stocks with discrete dividends:
   replace \( S e^{-\delta T} = F_0^P(S) \) by \( S - PV(DV) \)

   where \( PV(DV) \) is the present value of all dividends paid during the time interval from \( t = 0 \) to \( t = T \).

2) Options on currencies:
   - replace \( \delta \) by the risk-free interest rate in the other currency
   - the counterpart of \( S \) is the amount \( X \) of \( \$ \) needed to buy one unit of the other currency

3) Options on futures:
   If the forward price is \( F \) then the prepaid forward price is \( F e^{-rT} \). Replace \( S e^{-\delta T} \) by \( F e^{-rT} \).

   So (e.g.):
   \[
   d_1 = \left( \ln \frac{F}{K} + \frac{1}{2} \sigma^2 T \right) / (\sigma \sqrt{T}) \quad (\text{not: } \delta = \Phi) \\
   \text{(Now } P = C + (K-F)e^{-rT} \text{.)} 
   \]
Option Greeks (for European options)
(one could replace $T$ by $T-t$ in formulas below)

1) Delta

$$\Delta = \frac{\partial C}{\partial S} = e^{-\delta T} N(d_1) > 0$$

measures the option price change when the stock price increases by $\$1$

Put $\Delta = \partial P/\partial S = -e^{-\delta T} N(-d_1) < 0$

2) Gamma (same for calls and puts)

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta T} N'(d_1) e^{\delta T} e^{-d_1/2}}{S_0 \sqrt{T}} > 0$$

measures the change in $\Delta$ when stock price $\uparrow \$1$

3) Vega measures the change in option price when volatility increases by 1 percentage point, so

Call vega = Put vega = $\frac{\partial C}{\partial \sigma} = \frac{S e^{-\delta T} \sqrt{T} N'(d_1)}{100} > 0$

4) Theta ($\Theta$) measures the change in option price when $T$ decreases by 1 day

$$\begin{align*}
\Theta & = \frac{\partial C}{\partial T} = \frac{-S e^{-\delta T} N(d_1) - r K e^{-r T} N(d_2)}{2 \sqrt{T}} - \frac{K e^{-r T} N'(d_2)}{2 \sqrt{T}} \\
\text{Put} \Theta & = \frac{\partial P}{\partial T} = \text{Call} \Theta + r K e^{-r T} - S_0 e^{-\delta T}
\end{align*}$$

Divide above by 365 to get the day theta.
5) Rho ($\rho$) measures the change in option price when $r$ increases by 1 percentage point ($= 100$ basis points).

\[
\text{Call } \rho = \frac{1}{100} \frac{\partial C}{\partial r} = \frac{1}{100} TK e^{-rT} N(d_2) > 0
\]

\[
\text{Put } \rho = \frac{1}{100} \frac{\partial P}{\partial r} = -\frac{1}{100} TK e^{-rT} N(-d_2) < 0
\]

6) Psi ($\Psi$) measures the change in option price when the continuous dividend yield $\delta$ increases by 1 percentage point ($= 100$ basis points), so

\[
\text{Call } \Psi = \frac{1}{100} \frac{\partial C}{\partial \delta} = -\frac{1}{100} TS e^{-\delta T} N(d_1) < 0
\]

\[
\text{Put } \Psi = \frac{1}{100} \frac{\partial P}{\partial \delta} = \frac{1}{100} TS e^{-\delta T} N(-d_1) > 0
\]

Charts in the book show examples of the behavior of these quantities.
Ex. Suppose we are able to choose \( K = S \). Then

\[
d_1 = \frac{r - \delta}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T}, \quad d_2 = \frac{r - \delta}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T}.
\]

Also suppose further that \( \delta = 0 \). Now in cells,

\[
\Delta = N(d_1) = N\left( \frac{r}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right) > \frac{1}{2},
\]

\[
\Gamma = \frac{e^{-d_1^2/2}}{\text{So}\sqrt{T}2\pi} = \frac{1}{\text{So}\sqrt{T}2\pi} \exp\left( -\frac{1}{2} \left( \frac{r}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right) \right) = \frac{1}{\text{So}\sqrt{T}2\pi} \exp\left( -\frac{r^2}{2 \sigma^2 T} \right).
\]

\[
\text{Vega} = \frac{1}{100} \frac{\text{So}\sqrt{T} e^{-d_1^2/2}}{\sqrt{2\pi}} = \frac{\text{So}\sqrt{T}}{100 \sqrt{2\pi}} \exp\left( -\frac{r^2}{2 \sigma^2 T} \right).
\]

\[
\theta = rK e^{-rT} N\left( \frac{r - 1}{\sigma \sqrt{T}} \right) \frac{K \sigma e^{-\delta T}}{2 \sqrt{T} \sqrt{2\pi}} \exp\left( \frac{r^2 - 1 - \delta^2 T}{2 \sigma^2 T} \right).
\]

\[
g = \frac{TK e^{-\delta T}}{100} N\left( \frac{r - 1}{\sigma \sqrt{T}} \right)
\]

\[
k = \frac{-TS}{100} N\left( \frac{r}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right)
\]

Above, we could replace \( K \) by \( S \), or \( S \) by \( K \).

E.g., \( \lim \theta = rK = rS \), on an annualized basis while

\[TV0(\text{day } t) \rightarrow 365.\]
Implied Volatility

Suppose that we know, for a certain stock, the quantities $S, K, r, d, T$.
Suppose that the market price of a given option is $C_0$. numerically.
We can solve the equation

$$C = C_0$$

where $C$ is the Black-Scholes price, for the unknown quantity $\sigma$.
The number $\sigma$ so obtained is called the implied volatility (implied by the actual option price).

If we do this (e.g.) keeping $S, r, d, T$ fixed and varying $K$, we usually get different answers for $\sigma$ (see the chart on page 371 of the book).
This shows that in real life, there is no constant $\sigma$, hence the Black-Scholes model is not an accurate description of reality.
Calendar Spreads

Buying and selling options for two different expiration dates.

An example. Suppose \( S=K=40 \).

Sell a call for \( T=0.25 \) (3 months) and
buy a call for \( T=1 \) (1 year).

There is an initial net payment, and max. loss is
equal to this payment (assuming the entire
position is closed after 3 months).

If at time \( T \), \( S \leq K \) (\( S \) close to \( K \) but not
exactly), then the sold call expires worthless
and the bought call can be sold for more
than the initial payment, resulting in
a profit.

If at time \( T \), \( S > K \), then the sold call loses
\( S-K \) and the bought call can be sold
for a bit more than \( S-K \). The money
received may or may not cover all of the
initial payment. If \( S-K > 0 \) is large, then
the two calls almost cancel out each other,
and the loss is the initial payment.
Greeks for portfolios

Let a portfolio have $N$ options (1 ≤ $i$ ≤ $n$) on the same stock, with $n_i$ options of type $i$, each with Delta $A_i$. Then the Delta of the portfolio is

$$\Delta \text{portfolio} = \sum_{i=1}^{N} n_i A_i$$

A similar formula applies to the other Greeks.

Optim elasticity

change in stock price = $\Delta S$

change in optim price = $\Delta C$

optim elasticity $\Omega = \frac{\text{relative change in optim price}}{\text{relative change in stock price}}$

$$\frac{\Delta C}{C} = \frac{\Delta S}{S}$$

For a call, $\Omega > 1$.

For a put, $\Omega < 0$.

Volatility of an option

$$\sigma_{\text{option}} = |\Omega| \sigma_{\text{stock}}$$
(lectra)

Risk premium and $\beta$ of an option

$r = \text{risk-free interest rate}$

$d = \text{return on the stock}$

$y = \text{return on the option}$

$y = \frac{\Delta S}{C} \alpha + (1 - \frac{\Delta S}{C}) r$  since the replicating portfolio has $\Delta S$ stocks. Since $\Omega = \frac{\Delta S}{C}$, we have

$y = \Omega \alpha + (1 - \Omega) r$

or $y - r = \Omega (\alpha - r) .

By definition, $\beta_{\text{stock}}$ is given by

$\alpha - r = \beta_{\text{stock}} \left( \frac{r_{p}}{r} - r \right)$

where $r_{p}$ is the rate of return of a comparison asset (e.g., a stock index). Hence

$\frac{y - r}{r_{p} - r} = \beta_{\text{option}} = \Omega \beta_{\text{stock}}$.
The Sharpe ratio of the stock (by def.) is

\[ \alpha - r = \text{risk premium} \]
\[ \sigma \quad \text{volatility} \]

Hence the Sharpe ratio of a call option is (2.31)

\[ \frac{\alpha - r - \Omega(\alpha - r)}{\sigma} = \frac{\alpha - r}{\sigma} \quad \text{same as for the underlying stock.} \]

Elasticity of a portfolio

Let a portfolio have \( n_i \) call options (value \( C_i \)) for \( 1 \leq i \leq N \). The total value is \( \sum_{i=1}^{N} n_i C_i \).

If the stock price increases by $1, the change in portfolio value is \( \sum_{i=1}^{N} n_i \Delta C_i \). Hence

\[ \Omega_{\text{portfolio}} = \frac{\sum_{i=1}^{N} n_i \Delta C_i}{\sum_{i=1}^{N} n_i C_i} = \frac{\sum_{i=1}^{N} n_i C_i \Delta \omega_i}{\sum_{i=1}^{N} n_i C_i} = \sum_{i=1}^{N} \omega_i \Omega_i \]

where

\[ \omega_i = \frac{n_i C_i}{\sum_{i=1}^{N} n_i C_i} \]

\( \omega_i \) is the fraction of the portfolio invested in option \( i \). So by definition,