Ch. 10 Binomial option pricing: Basic concepts

The law of one price

Two portfolios with the same payoff (at the same time in the future) must have the same price today.

If not, there is an arbitrage opportunity.

Binomial trees

We consider discrete time. We assume that after being in a certain state at time $t$, there are only two possibilities for our state at the next time $t+1$.

We describe this as follows, for one step:

If we take two steps (starting at the known state today), we may get in general, the following non-recombined tree:

and so on for further steps
However, it may happen that we may arrive at some future states in more than one way:

and so on.

If this keeps happening, we say that we have a **recombinant tree** (or a lattice).

We now apply the idea of using a binary tree to get a rough idea of option values. We assume that once at stock price is given at time $t_i$, there are only two possibilities for the stock price $S_{i+1}$ at time $t_{i+1}$:

$$S_{i+1} = uS_i \quad \text{or} \quad S_{i+1} = dS_i$$

where $0 < d < 1 < u$ ("up" and "down").

Since $uud =udd = ddu$, and so on, this gives rise to a recombinant tree.

We work such a tree, from top to bottom backwards in time, to get option prices at each stage, for expiration at $T = t_n$. 
We know all stock prices. E.g. with $S=S_0$:

At each step

we construct a portfolio whose payoff at each of $uS'$ and $dS'$ is the same as the payoff of the option (call or put) that we are considering.

The portfolio is of the form

$$AS + B$$

where we have $A$ shares of the stock and the amount $B$ invested in a risk-free bond that has annualized interest rate $r$. 
Here both $\Delta$ and $B$ are any real numbers $(>0, =0, <0)$. For convenience we assume we can buy (if $\Delta > 0$) or sell short (if $\Delta < 0$) any fractional number of shares.

We start at the end, e.g., here and use stock prices at 1,2,3 to construct the portfolios at node 1 to get the option price at node 1.

Then we use 3,4 to get into node 5. Then we use 1,5 to get 6.

If this is the entire tree, we are done, since we wanted to know the option price at node 6 (the beginning time).

If the tree is larger, we begin at the right end and move to the left (backwards in time) until we reach the beginning (the present time) of the tree.

Now we explain how to find $\Delta, B$, and the option price. These steps are then applied throughout the tree.
European call option

\[ S_0 \rightarrow \text{stock prices } S, uS, dS \]

0 < d < u

Strike price \( K \)

Payoff at nodes on the right if this is \( \max \{ uS - K, 0 \} (uS - K)^+ \) Explanation:

\[ \{ C_u = (uS - K)^+ \}

Call these \( C_u, C_d \) for brevity, so \( C_d = (dS - K)^+ \)

at node 1

If we start with \( \Delta \) shares of stock and the amount \( B \) in the bond, and if the time difference is \( h \), then at the time \( h \), the payoff of this portfolio is \( (f = \text{dividend rate}) \)

at node 2: \( \Delta \cdot uS \cdot e^{sh} + Be^{rh} \)

at node 3: \( \Delta \cdot dS + e^{sh} + Be^{rh} \)
We want the portfolio to have the same payoff as the option. So we require that

\[ A u \Delta e^{\delta h} + B e^{\nu h} = C_u \]

\[ A d \Delta e^{\delta h} + B e^{\nu h} = C_d \]

This is a pair of linear equations for the variables \( A \) and \( B \). Solving, we get

\[ A = e^{\delta h} \frac{C_u - C_d}{S(u-d)} \quad B = e^{\nu h} \frac{u C_d - d C_u}{u-d} \]

The law of one price says that the price of the option at node 1 must be the value of this portfolio at node 1, i.e.,

\[ \Delta S + B = e^{\nu h} \left( \frac{e^{(r-s)h}}{u-d} C_u + \frac{u-e^{(r-s)h}}{u-d} C_d \right) \]

In addition to \( 0 < d < e \), we must have

\[ d < e^{(r-s)h} < u \]

to avoid arbitrage. Note: above one can calculate the option price directly from the formula without finding \( A, B \) separately (now that we have the formula for \( \Delta S + B \)).
At the rightmost level of the tree, corresponding to expiration, the payoffs $C_u, C_d$ are the payoffs if the option is exercised (zero if option is worthless).

At each earlier stage, $C_u, C_d$ are the option prices that we have calculated during the previous step.

Exception: if the option is an American option and its payoff, if exercised immediately, is greater than the calculated option price, then we replace the payoff ($C_u$ or $C_d$) by this "intrinsic" value of the option.

In this way, we can use a binomial tree of any size for any option (call or put, European or American) to find an option price.

Note: The above is based on simplifying assumptions (only 2 possible stock prices, etc.) so it cannot correspond to real life exactly, but it gives an idea of the principles involved.
Define \( p^* = \frac{e^{(\mu - \delta)h}}{u-d} \), so \( 0 \leq p^* \leq 1 \) and

\[
C = C_0 + B = e^{-rh} \left[ p^* C_u + (1-p^*) C_d \right],
\]

the present value of a cash flow that
will, at time \( h \), be \( C_u \) with probability \( p^* \)
and \( C_d \) with probability \( 1-p^* \).

This is risk-neutral pricing and

\[ p^* = \text{risk-neutral probability} \]
Use of Volatility

Assume that the returns of a stock over disjoint time periods (of the same length) are independent and identically distributed.

\[ \sigma = \text{annualized standard deviation of returns} \quad (n=1) \]

\[ \sigma_h = \sigma \sqrt{h} = \text{st. dev. over time period of length } h \]

Define \( u = e^{(r-s)h + \sigma \sqrt{h}} \)

\[ d = e^{(r-s)h - \sigma \sqrt{h}} \]

Then \( d < e^{(r-s)h} < u \).

In practice, often \( \sigma = 0.2 \) or 0.3.
(Often written as 20\% or 30\%)

In options on currencies, \( d \) is replaced by \( r_y \), the risk-free interest rate in the other currency.
For options on futures contracts,

\[ u = e^{\sigma \sqrt{h}}, \quad d = e^{-\sigma \sqrt{h}}, \quad p^* = \frac{1-d}{u-d} \]

\[ \Delta = \frac{Cu-Cd}{F(u-d)}, \quad \psi = e^{-r_h} \left( \frac{1-d}{u-d} \right) \left( \frac{Cu}{u-d} + \frac{Cd}{d} \right) \]

where \( F \) is the current price of the futures contract (and \( uF, dF \) are the only possible prices after one step).

Options on commodities can be more complicated or non-existent, depending on the commodity.

Options on bonds will be discussed in another chapter.
Early exercise of options

Points to consider (in favor or against)

For calls:
(+): getting the stock, hence getting dividends earlier
(-): paying the strike price earlier (interest cost)
(-): losing "insurance" (if the stock price decreases a lot, early exercise may turn out to have been a poor choice)

For puts:
(-): losing dividends earlier
(+): receiving the strike price earlier
(-): losing "insurance"

If volatility $\sigma = 0$, it is optimal to exercise a call early if $S > \frac{rK}{\sigma}$

($S =$ current stock price, $K =$ strike price, $r =$ risk-free interest rate, $\sigma =$ continuous dividend rate)

If $\sigma > 0$, the lower bound for $S$ is greater, and remaining time to expiration matters.
Pricing an option using risk probabilities

Use a binomial tree for one step with the same notation as before. Suppose \( S = 0 \) (no dividends).

Then,
\[
\Delta = \frac{C_u - C_d}{S(u-d)}, \quad B = e^{-rT} \frac{uC_u - dC_d}{u-d},
\]

option price = \( \Delta S + B = e^{-rT} \left( \frac{e^{\sigma \sqrt{T}} - d}{u-d} + \frac{u - e^{-rT}}{u-d} \right) \).

Suppose the true (perhaps unknown) probability of the stock going up is \( p \), going down is \( 1-p \) (\( 0 < p < 1 \)). Suppose the continuously compounded (perhaps unknown) expected return on the stock is \( \mu \). Then we must have

\[
p u S + (1-p) d S = e^{\mu T} S,
\]

so \( p = \frac{e^{\mu T} - d}{u-d} \) and we must have \( d < e^{\mu T} < u \).

The actual expected payoff of the option is

\[
(*) \quad p \cdot C_u + (1-p) \cdot C_d = \frac{e^{\mu T} - d}{u-d} C_u + \frac{u - e^{\mu T}}{u-d} C_d.
\]

How should this payoff at time \( t = T \), be discounted to its present value at \( t = 0 \)?

Suppose the appropriate rate is \( r \), so we'd multiply (*) by \( e^{-rT} \).

Idea: the discount factor should be the same for the equivalent stock-bond portfolio, hence
\[ e^{xh} = \frac{AS - e^{-xh} + B}{\Delta S + B} e^{xh} \]

Thus, \( e^{xh} \) times \( \Delta S + B \) is

\[
\frac{\Delta S + B}{AS e^{xh} + B e^{xh}} \left[ \frac{e^{xh} - d}{u - d} C_u + \frac{u - e^{xh}}{u - d} C_d + \frac{e^{xh} - e^{-xh}}{u - d} (C_u - C_d) \right] \]

\[
e^{xh} (\Delta S + B) \quad \text{(see previous page)} \]

\[
= e^{xh} (\Delta S + B) \frac{\Delta S + B}{AS e^{xh} + B e^{xh}} \left[ e^{xh} (\Delta S + B) + (e^{xh} - e^{-xh}) \Delta S \right]
\]

\[
e^{xh} \Delta S + B e^{xh}
\]

\[
= \Delta S + B,
\]

The same option price as we obtained before. Thus, the actual (perhaps unknown) values of \( \alpha \) (and \( p \)) did not matter.
Lognormal Distribution

One assumes that the continuously compounded returns on the stock are normally distributed.

Approximation to this by the binomial model:

\[ S_{t+h} = S_t e^{(\mu - \sigma^2)h + \sigma \sqrt{h}} \]

so

\[ \ln \frac{S_{t+h}}{S_t} \approx (\mu - \sigma^2)h + \sigma \sqrt{h} \]

drift volatility