Suppose that we buy a call and sell a put on the same stock (price $S_T$ at time $T$) for the same strike price $K$ and the same expiration date $T$.

Let these option prices (now) be $C(K,T)$ and $P(K,T)$.

Suppose we hold these options to expiration. No matter what $S_T$ is, we buy the stock at price $K$ (spending $K$ at time $T$).

The present value of this transaction is

$$C(K,T) - P(K,T) + \text{PV}(K)$$

and as a result, we get the stock at time $T$, we get the same result by a long forward contract whose cost has present value

$$\text{PV}(F_{0,T})$$

To avoid arbitrage, these present values must be equal, hence

$$C(K,T) - P(K,T) = \text{PV}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K).$$

This relation is called the put-call parity.
If \( F_{0,T} \) is known and we choose \( K = F_{0,T} \), then we are entering into a synthetic forward contract whose cost now (just like for a forward) should be zero. Hence we should have \( C(K,T) - P(K,T) = 0 \) when \( K = F_{0,T} \).

This parity normally does not hold for American option whose early exercise is permitted.

The above remarks are valid for European options on items other than stocks also.

Now consider options on a dividend-paying stock, and denote by \( PV_{0,T}(\text{Div}) \) the present value (at time \( t = 0 \)) of the dividends paid during the time period from 0 to \( T \). Also note that

\[
 PV(F_{0,T}) = e^{-rT} F_{0,T} = S_0 \quad \text{(from Ch. 3)}
\]

The counterpart of (9.1) is

\[
(9.2) \quad C(K,T) - P(K,T) = S_0 - e^{-rT} K - PV_{0,T}(\text{Div}).
\]

For continuous dividends, we have

\[
 S_0 - PV_{0,T}(\text{Div}) = S_0 e^{-\delta T}
\]

in which case

\[
(9.3) \quad C(K,T) = P(K,T) + S_0 e^{-\delta T} - Ke^{-rT}.
\]

The same equation can be written in many ways.
If \( K = S_0 \) (at-the-money option) and \( d = 0 \), then
\[
S_0 e^{-rT} - Ke^{-rT} = S_0 (1 - e^{-rT}) \geq 0,
\]
so
\[
C(K,T) \geq P(K,T).
\]

Usually
\[
C(K,T) > P(K,T) \quad \text{if} \quad K = S_0,
\]

i.e.,
at-the-money calls are more expensive than at-the-money puts. Why?

If we buy the call and sell the put, we buy the stock at price \( K \), but not now, but at the later time \( T \). We get to keep the amount \( K \) for time \( T \), so we should be able to get interest on it (the one who sells to us at \( T \) does not get that interest but has committed to selling to us at \( T \)). Thus
\[
C(K,T) - P(K,T) > 0
\]
corresponds to this interest.
Writing (9.2) as
\[ S_0 = C(K,T) - P(K,T) + PV_{0,T}(\text{DIV}) + e^{-rT}K \]

describes how the option transaction creates a synthetic stock.

Writing (9.2) as
\[ S_0 + P(K,T) - C(K,T) = PV_{0,T}(\text{DIV}) + PV_{0,T}(K) \]

where the left hand side corresponds to shorting the stock, buying a put, and selling a call (so as \( t = T \) we have the cash \( K \), no matter what \( S_T \) is), while the right hand side corresponds to buying, at \( t = 0 \), a bond that pays \( S_0 + K + PV_{0,T}(\text{DIV}) \) at \( t = T \).

describes a synthetic bond (book: synthetic Treasury bill or T-bill). This is called a conversion, and doing the opposite is called a reverse conversion.

Synthetic options. A call on a put is equivalent to the transaction described by the right hand side of the equation:

\[ C(K,T) = S_0 - PV_{0,T}(\text{DIV}) - PV_{0,T}(K) + P(K,T) \]

\[ P(K,T) = C(K,T) - S_0 + PV_{0,T}(K) + PV_{0,T}(\text{DIV}) \]
Options on currencies

Same as above, with dividend rate d replaced by the risk-free interest rate for the other currency (e.g., r_y). So with today's exchange rate $\bar{x}_0 = 1$ yen (say), for an option for one yen at time T:

$$C(K,T) - P(K,T) = \bar{x}_0 e^{-r_Y T} - Ke^{-r_T T}$$

Options on bonds Current bond price = B_0

$$C(K,T) - P(K,T) = (B_0 - PV_{0,T}(Coupons)) - PV_{0,T}(K)$$

The above options correspond to the exchange of two assets, e.g., cash and a stock.

Similarly, any two assets could be considered (neither one of which is cash), and analogous formulas would apply. The consideration of such exchanges would show more clearly a certain equivalence of calls and puts: in any exchange of two assets, each party gives something and receives something and could therefore take the view that it is buying (B) or selling (A). Each view may be justified. This may be even clearer if each asset is a currency ( "buying $ by selling yen") or
Comparison of options (same stock, same strike price $K$, same expiration date $T$ unless otherwise specified)

1) by type: American options can be exercised early, hence are more valuable than European ones:

\[ C_{\text{Amer}}(K, T) \geq C_{\text{Eur}}(K, T) \]

\[ P_{\text{Amer}}(K, T) \geq P_{\text{Eur}}(K, T) \]

2) by strike price ($K_1 < K_2$):

\[ C(K_1, T) \geq C(K_2, T) \]

\[ P(K_2, T) \geq P(K_1, T) \]

3) by expiration date ($T_1 < T_2$)

A later expiration date gives an American option more time to be exercised any time, hence

\[ C_{\text{Amer}}(K, T_1) \leq C_{\text{Amer}}(K, T_2) \]

\[ P_{\text{Amer}}(K, T_1) \leq P_{\text{Amer}}(K, T_2) \]

For European options, these relations need not hold always. Individual circumstances dictate the relationship.
Maximum and minimum option prices
At any time t, with $S_0$ abbreviated as $S$

1) $S \geq C_{\text{Am.}}(S, K, T) \geq C_{\text{Eur.}}(S, K, T) \geq \max\left\{ 0, \ PV_{\text{Eur.}}(F_{0,T}) - PV_{\text{Eur.}}(K) \right\}$

2) $K \geq P_{\text{Am.}}(S, K, T) \geq P_{\text{Eur.}}(S, K, T) \geq \max\left\{ 0, \ PV_{\text{Eur.}}(K) - PV_{\text{Eur.}}(F_{0,T}) \right\}$

Early exercise for American options

Puts: Usually not worth it (better to sell the put)
Barring exceptional circumstances (e.g., company goes bankrupt, $S = 0$, put has already attained its maximum value so no need to wait to take that money).

Calls: Suppose there are no dividends.

Put-call parity implies that (rewrite (9.3), $S = 0$)

$C_{\text{Eur.}}(K, T) = (S_0 - K) + P(K, T) + K(1 - e^{-rT}) > S_0 - K$

and similarly at any time $t$, $0 < t < T$,

$C_{\text{Am.}}(K, T) \geq C_{\text{Eur.}}(K, T) > S_t - K = \text{value if exercised}$,
Thus, before exercising, it is always better to sell the call than to exercise it.

**Terminology**

If $S_t - K > 0$, then $S_t - K$ = **intrinsic value** of the call.

If $S_t - K < 0$, then $S_t - K$ = **intrinsic value** = 0 and **time value** = $C(K, T)$.

For puts, if $S_t - K < 0$, then $K - S_t$ = **intrinsic value** and $P(K, T) - (K - S_t)$ = **time value**.

If $S_t - K > 0$, then **intrinsic value** = 0 and **time value** = $P(K, T)$. 
For both American and European options (these subscripts omitted),

1) Convexity of the option price with respect to the strike price ($K_1 < K_2 < K_3$, same $T$, so the notation for $T$ is omitted):

\[
\frac{C(K_1) - C(K_2)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_3)}{K_3 - K_2}
\]

2nd der.,

\[
\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}
\]

If not, arbitrage using asymmetric butterflies.

2) (1st der.) $K_1 < K_2$

\[
C(K_1) - C(K_2) \leq K_2 - K_1
\]

if not, there is an arbitrage using call or put spreads

\[
P(K_2) - P(K_1) \leq K_2 - K_1
\]

Exercise and moneyness

If it is optimal to exercise an option, it is also optimal to exercise an otherwise identical option that is more in-the-money. True for both calls and puts.