Ch. 7, Interest rate forwards and futures

We denote by $X_{t_0} \left( t_1, t_2 \right)$ the value of the quantity $X$, as known at time $t_0$, when some action begins at time $t_1$ and ends at $t_2$, where $t_0 \leq t_1 < t_2$.

We take $t_0$ to be the present time, and often $t_0 = 0$.

If $t_0 = t_1$, we abbreviate $X_{t_0} \left( t_1, t_2 \right)$ to $X(t_1, t_2)$.

Let the price of the zero-coupon bond purchased at $t_1$, maturing at $t_2$, and paying the amount 1 at $t_2$, be $P_{t_0} \left( t_1, t_2 \right)$.

We agree to this deal at time $t_0$, and pay $P_{t_0} \left( t_1, t_2 \right)$ at time $t_1$.

If various forward interest rates are known today (at $t_0$), they can be used to calculate implied future interest rates. (Actual interest rates in the future may be different, but these current forward interest rates can be used to make agreements today on actions to be undertaken in the future).
Let \( r_t (t_1, t_2) \) be the (annualized) interest rate, known at time \( t_1 \), valid from time \( t_1 \) to time \( t_2 \).

To avoid arbitrage (today), certain relations must be satisfied.

Suppose the \( t_j \) are integers and interest is compounded annually (not continuously). Then

\[
(1 + r(0, t_1))^{t_1} P(0, t_1) = 1,
\]

so

\[
P(0, t_1) = \frac{1}{(1 + r(0, t_1))^{t_1}}.
\]

Also,

\[
(1 + r_0 (t_1, t_2))^{t_2 - t_1} (1 + r_0 (0, t_2))^{t_2} P(0, t_1) = (1 + r_0 (0, t_1))^{t_1} P(0, t_2)
\]

and generally, if \( t_0 = 0 \leq t_1 < t_2 < t_3 \),

\[
(1 + r_0 (t_1, t_3))^{t_3 - t_1} (1 + r_0 (t_1, t_2))^{t_2 - t_1} (1 + r_0 (t_2, t_3))^{t_3 - t_2} = P(0, t_1) P(0, t_2) P(0, t_3).
\]

We have

\[
P_0 (t_1, t_2) = \frac{1}{(1 + r_0 (t_1, t_2))^{t_2 - t_1}} = \frac{P(0, t_2)}{P(0, t_1)}.
\]
Coupon bonds

\[ B_f(t_1, t_2, c, n) = \text{price at time } t_1 \text{ of a bond that is issued at time } t_1, \text{ matures at } t_2, \]
\[ \text{pays the amount } 1 \text{ at maturity, and makes } n \text{ payments, each in the amount } c, \text{ at times } \]
\[ t_j = t_1 + \frac{j(t_2 - t_1)}{n}, 1 \leq j \leq n. \]

Principle for all situations like this:

Calculate the present values of all payments involved, and write down the appropriate equality.

In this case, considering present values (i.e., values at time \( t \)), we get

\[ B_f (t, T, c, n) = P_t(t, T) + \sum_{i=1}^{n} c P_t(t, t_i). \]

If all variables but \( c \) are known, we get

\[ c = \frac{B_f(t, T, c, n) - P_t(t, T)}{\sum_{i=1}^{n} P_t(t, t_i)}. \]
Def. A par coupon is a bond whose price is equal to the payment at maturity.

Hence \( B_+ (t, T, \xi, n) = 1 \) for a par coupon, thus

\[
c = \frac{1 - P_+ (t, T)}{\sum_{i=1}^{n} P_+ (t, t_i)}.
\]

If the bond makes \( m \) payments a year for \( n \) periods (held for \( T = n/m \) years) and has maturity yield \( y_m \) per period, then

\[
B_+ (t, T, \xi, n) = \frac{1}{(1+y_m)^n} + \sum_{i=1}^{n} \frac{c}{(1+y_m)^i}.
\]

The maturity yield \( y = my_m \).

If we know prices of sufficiently many coupon bonds, we get a system of equations from which we can deduce prices of zero coupon bonds.

We see that a zero coupon bond is "a coupon bond with no coupons," hence the name.
Forward rate agreements (FRA)

An OTC (over-the-counter) contract that guarantees a borrowing/lending rate on a given notional principal amount.

Can be settled at the initiation ($t_1$) or maturity ($t_2$) of the transaction (agreed to at time $t$) settled in arrears.

FRA is a forward contract based on an interest rate. No actual lending occurs.

Let $r_{FRA}$ be the rate in the contract. For simplicity, suppose $t_2 - t_1 = 1$. At time $t_1$, we know $r_{t_1}(t_1, t_2)$. At $t_2$, one party pays to the other the notional principal times

$$r_{t_1}(t_1, t_2) - r_{FRA}.$$

If settled at $t_1$, the payment is notional times

$$\frac{r_{t_1}(t_1, t_2) - r_{FRA}}{1 + r_{t_1}(t_1, t_2)}.$$

FRA can be used to hedge future interest rate risk.
Price value of a basis point and DV01

Basis point = 0.01

Suppose bond makes m coupon payments per year, each C/m, for T years (number of payments n = mT), and pays M at maturity. Let y = y/m be the per-period yield to maturity. Denote price of bond by B(y). Then

\[ B(y) = \frac{M}{(1+y)^n} + \sum_{i=1}^{n} \frac{C/m}{(1+y/m)^i} \]

Approximation by differentials:

Change in bond price \[ \frac{\Delta B(y)}{\Delta y} = -\frac{\sum_{i=1}^{n} \frac{C/m}{(1+y/m)^i} + \frac{n}{m} \frac{M}{(1+y/m)^n}}{1+y/m} \]

If we consider the change of 1 in basis points, we get price value of a basis point (PVBP) or the dollar value of an OI (DV01), and it is the above divided by 10,000.
Duration

Price sensitivity of a bond as per dollar of bond price is the above change times \(-\frac{1}{B(y)}\).

Modified duration \(D = \frac{1}{B(y)} \left[ \frac{\sum_{i=1}^{n} \frac{i}{m} \frac{C_i}{m} + \frac{M}{(1 + \frac{y}{m})^n}}{1 + \frac{y}{m} \frac{C_i}{m} + \frac{M}{(1 + \frac{y}{m})^n}} \right] \)

Maccanly duration \(D_{mac} = D \cdot \left(1 + \frac{y}{m}\right)\)

\[= \frac{1}{B(y)} \left[ \ldots \right] \]

\(D_{mac} = \text{weighted average of the time (number of periods) until bond payments occur. To see this, note that} \]

\[1 = \frac{1}{B(y)} \left[ \sum_{i=1}^{n} \frac{i}{m} \frac{C_i}{m} + \frac{M}{(1 + \frac{y}{m})^n} \right] \]

The numbers \(\frac{C_i/m}{(1+y/m)^i \cdot B(y)}\) and \(\frac{M}{B(y)(1+y/m)^n}\)

are the weights in this weighted average of the numbers \(i/m\) and \(n/m\), which represent the times from the present until the payments are made. Hence the term duration.
Convexity

Duration corresponds to $d B(y)/dy$.

Convexity = $u = d^2 B(y)/dy^2$.

By definition,

Convexity (at a bond) =

$$\frac{1}{B(y)} \left[ \sum_{i=1}^{n} \frac{i(i+1)}{m^2} \frac{C_i}{(m^2/i^2 + \frac{2}{m})^{n+2}} + \frac{n(n+1)}{m^2} \frac{M}{(1 + \frac{n}{m})^{n+2}} \right].$$

We have (Taylor expansion)

$$B(y + \varepsilon) \approx B(y) - DB(y) \varepsilon + \frac{1}{2} (Convexity) B(y) \varepsilon^2.$$  

This is a more accurate estimate than

$$B(y + \varepsilon) \approx B(y) - DB(y) \varepsilon.$$
Ch. 8 Swaps

General principles

Suppose that at times $t_i$, for $1 \leq i \leq n$, where $0 < t_1 < t_2 < \ldots < t_n$, we are supposed to make interest (or analogous) payments at variable rates $r_i(t_{i-1}, t_i)$ based on possibly variable principal amounts $Q_i$, but we wish to "swap" this to using a fixed interest rates $R$. We swap payments with someone so that we pay according to $R$ and the other party pays $r_i(t_{i-1}, t_i)$.

For this to be fair, the present values of the payments at time 0 must be equal. The present value is the future value times $P(0, t_i) (= P_0(0, t_i))$. Thus we require

$$\sum_{i=1}^{n} Q_i \cdot r_i(t_{i-1}, t_i) \cdot P(0, t_i) = \sum_{i=1}^{n} Q_i \cdot R \cdot P(0, t_0),$$

hence

$$R = \frac{\sum_{i=1}^{n} Q_i \cdot P(0, t_i) \cdot r_i(t_{i-1}, t_i)}{\sum_{i=1}^{n} Q_i \cdot P(0, t_i) \cdot P(0, t_{i-1}) / P(0, t_i) - 1},$$

If all $Q_i$ are equal, then

$$R = \frac{\sum_{i=1}^{n} P(0, t_i) \cdot r_i(t_{i-1}, t_i) \cdot (1 - P(0, t_i))}{\sum_{i=1}^{n} P(0, t_i)}.$$
\[ a_i \uparrow \quad \text{accepting swap} \]

\[ a_j \downarrow \quad \text{as in a mortgage} \quad \text{amortizing swap} \]

\[ \text{Deferred swaps: if payments begin at time } t_k \text{ (and all } a_j \text{ are equal, say), then} \]

\[ R = \frac{\sum_{i=k}^{n} P(0, t_i) F_0, t_i}{\sum_{i=k}^{n} P(0, t_i)} \]

\[ = \frac{P(0, t_{k-1}) - P(0, t_n)}{\sum_{i=k}^{n} P(0, t_i)} \]

We can make swaps for other payments (e.g., payments for commodities). Suppose we want to buy (or sell) a unit of a commodity at future times \( t_i \) at forward prices \( F_0, t_i \) that are not the same for all \( i \).

If we wish to swap these to constant payments, all equal to \( R \), we require that

\[ \sum_{i=1}^{n} R P(0, t_i) = \sum_{i=1}^{n} F_0, t_i P(0, t_i) \]

so

\[ R = \frac{\sum_{i=1}^{n} P(0, t_i) F_0, t_i}{\sum_{i=1}^{n} P(0, t_i)} = \sum_{i=1}^{n} \left( \frac{P(0, t_i)}{\sum_{j=1}^{n} P(0, t_j)} \right)^{F_0, t_i} \]
which is a weighted average of the
forward prices.

Typically the constant payments are
"too much" early on and "too little"
later. Thus the party with fixed payments
is making a loan (on which
interest is received) to the other party.

In a financial settlement, only the cash
difference (between the fixed and variable
payments) needs to be paid by one
party to the other.

In a physical settlement for a commodity,
the commodity is delivered, and the
financial settlement may take place
with the assistance of a swap counterparty.