Ch. 5 Financial forwards and futures

Prepaid forward contract on a stock

Stock price $S_t$ at time $t$
Current time $t=0$
Expiration date: $t=T$
Risk-free interest rate: $r$
Continuous dividend rate: $d$

Price of prepaid forward contract if $S=0$

$$P_{0,T} = S_0$$

Why?

E.g., by considering arbitrage opportunities

If $F_{0,T} > S_0$, buy stock now at $S_0$ and short the prepaid forward, receiving $F_{0,T} - S_0 > 0$. At time $T$, use the stock to fulfill the obligation of the forward contract to deliver the stock.
If $F_{0,T}^p < S_0$, buy the prepaid forward and short the stock, receiving $S_0 - F_{0,T}^p > 0$.

At time $T$, receive stock from forward and use the stock to close the short sale.

Price of prepaid forward contract with dividends ($d > 0$)

In general,

$$F_{0,T}^p = S_0 - \sum PV(\text{dividends})$$

With continuous dividends, (stated poisson)

$$F_{0,T}^p = S_0 e^{-dT}$$

Price of a forward contract on a stock

$$F_{0,T} = FV\left( F_{0,T}^p \right)$$

Hence

$$F_{0,T} = e^{rT} (S_0 e^{-dT}) = S_0 e^{(r-d)T}$$

If no dividends ($d = 0$), then

$$F_{0,T} = S_0 e^{rT}$$
A synthetic forward contract

Def: A zero-coupon bond is
- buy bond now with cash
- at a future time, receive the nominal value of the bond
- no intermediate interest payments

E.g. pay $95 now, get $100 later.
The difference is interest, from which one can calculate the implied interest rate.
Here $95 = bond price.
Higher price = lower interest (yields)

Forward = Stock - zero-coupon bond

Why?

For forward, payoff at $T$ is $S_T - F_{0,T}$

Buy a traded position on stock, pay $S_0 e^{-rT}$.
Borrow $S_0 e^{-rT}$.
At time $T$, repay loan at $e^{rT} (S_0 e^{-rT}) = S_0 e^T$ and get stock. Payoff at $T$ is

\[ S_T - S_0 e^{r(T-T)} = S_0 - F_{0,T}. \]
Similar synthetic positions:

\[ \text{Stock} = \text{Forward} + \text{zero-coupon bond} \]
\[ \text{Zero-coupon bond} = \text{Stock} - \text{Forward} \]

All 3 combinations can be shorted also, creating synthetics of the reversed positions.

Above, no-arbitrage implied the price of \( F_{0,T}^P, F_{0,T} \).

If there are transaction costs and bid-ask spreads, we get instead a no-arbitrage interval of possible values of \( F_{0,T}^P, F_{0,T} \) for which arbitrage is not possible.
Futures contracts on stocks and stock indices:
- Traded on exchanges
- Standardized in amounts and expiration dates
- Marked-to-market (profit or loss calculated each day)
- Margin deposit required and may be increased for a losing position (or position will be closed with the loss locked in)
- Initial margin
- Maintenance margin
- Margin call

Use of index futures
- Asset allocation (replicate bonds instead of stocks without selling the stocks by shorting index futures and going long on bond futures)
- Portfolio management
- Cross-hedging
Currency contracts

risk-free interest rate in foreign currency replaces the dividend rate \( s \), otherwise similar formulas

Ex. Change $ to yen. Let \( r_y \) be the risk-free interest rate for yen. Suppose that today, \( \$ x_0 = ¥ 1 \).

For one yen,
prepaid forward price is \( F_0^p = x_0 e^{-r_y T} \)

Forward price is \( F_0^F = x_0 e^{(r_y - r_f) T} \).
Expected price of \( X \) in the future on the basis of the information we have at time \( t = 0 \) is denoted by \( E_0(X) \).

**Prepaid forward price for a unit of a commodity for expiration date \( T \):**

\[
P_{0,T}^P = e^{-\alpha T} E_0(S_T)
\]

\( \alpha \) = discount rate for the commodity

\( S_T \) = price of commodity at time \( T \)

**Forward price of a commodity**

\[
F_{0,T} = e^{rT} P_{0,T}^P = E_0(S_T) e^{(r-\alpha)T}
\]

or

\[
e^{-rT} F_{0,T} = e^{-\alpha T} E_0(S_T)
\]

Stock arbitrage may depend on being able to **short stock** (by borrowing shares) at no cost (except dividend payments). To short a commodity, if possible at all, one may need to pay a fee ("lease rate") to the lender of the commodity. This may reduce or eliminate arbitrage opportunities.
Hence (e.g.) forward prices can remain (essentially) constant for different values of \( T \).

Suppose we buy one unit at \( t=0 \) for \( S_0 \) and lease it until time \( T \). The present value of getting it back at time \( T \) is

\[ e^{-\alpha T} E_0(S_T). \]

If we get continuous lease payments at rate \( \delta \), this becomes

\[ E_0(S_T) e^{-\delta T}. \]

To avoid arbitrage, this should be \( = S_0 \), which gives

\[ \delta = \alpha - \frac{1}{T} \ln \frac{E_0(S_T)}{S_0}. \]

Since \( e^{-\delta T} F_{0,T} = e^{-\alpha T} E_0(S_T) \), we may also write

\[ \delta = r - \frac{1}{T} \ln \frac{F_{0,T}}{S_0}. \]

For stocks, this gives \( \delta = \delta \) (dividend rate).
Storage costs

If there are continuous storage costs at rate \( A \), then we should have with no-arbitrage,

\[
F_{0,T} = S_0 e^{(r+A)T}
\]

If storing the commodity provides a convenience (e.g., for a user of corn who likes to store some corn in case there is a disruption in supply) at rate \( c \), then

\[
F_{0,T} = S_0 e^{(r+c)T}
\]

Ex. Gold. Sometimes \( S_0 < 0 \), perhaps since lender of gold avoids storage costs.

Ex. Corn. Storage outside harvest season but not during them. Prices fall at harvest and rise after harvest.

Ex. Electricity. No storage. There is a maximum to the supply: can supply less than that but not more.

Demand varies by season and by time of day. Higher forward prices at times of higher demand.