Chapter 22: Risk-neutral and martingale pricing

Change of measure

Consider a probability space $(\Omega, F, P)$. Let $w: \Omega \to \mathbb{R}^+$ be a function such that $w(\omega) > 0$ for all $\omega \in \Omega$ and such that $w$ is $F$-measurable, that is, for all $b \in \mathbb{R}$,

$$\{ x \in \Omega : w(x) < b \} \in F,$$

then the integral $\int \omega \, dP$ is defined, but may be $\pm \infty$. In any case, it is $\geq 0$. If

$$0 < \int_\Omega \omega \, dP = a < \infty,$$

then $\int_\Omega \frac{w}{a} \, dP = 1$. Write again $w$ instead of $\frac{w}{a}$, hence assuming that $\int_\Omega w \, dP = 1$.

Then for each $A \in F$, we may define

$$\overline{P}(A) = \int_A w \, dP.$$

Then $\overline{P}$ is a new probability measure on $\Omega$. This is called change of measure.
If the change of measure is performed in an interesting way, relative to some quantity such as the price of a stock or bond or zero-coupon, it may be that at the same time we perform a change of numeraire and start measuring other quantities as multiples of the stock or bond or zero-coupon.
Let \((\Omega, \mathcal{F}, P)\) be a probability space, for each \(t \geq 0\) let \(\mathcal{F}_t\) be a \(\sigma\)-algebra on \(\Omega\) such that
\[
\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}
\]
whenever \(0 \leq s < t\), and let \(X_t\) for \(t \geq 0\) be a stochastic process adapted to the \(\sigma\)-algebras \(\mathcal{F}_t\). Suppose that for all \(t \geq 0\),
\[
\int_{\Omega} |X_t| \, dP < \infty.
\]

We say that the process \(X_t\) is a martingale if, whenever \(0 \leq s < t\) and \(A \in \mathcal{F}_s\), we have
\[
\int_{A} (X_t - X_s) \, dP = 0.
\]
Variable probability measures

We could also consider a more general situation where for each $t \geq 0$ we have a separate probability measure

$$P_t = \mathbb{Q}_t^P$$

where $\mathbb{Q}_t \geq 0$ is a function on $\Omega, \mathcal{F}_t$

$$P_t(A) = \int_A \mathbb{Q}_t^P dP \quad \text{if} \quad A \in \mathcal{F}_t.$$

Here $\mathbb{Q}_t$ is $\mathcal{F}_t$-measurable.

If now, for all $s < t$, $A \in \mathcal{F}_s$, we have

$$0 = \int_A (X_t - X_s) dP_s = \int_A (X_t - X_s) \mathbb{Q}_t^P dP$$

we can call $(X_t)$ a martingale with respect to the measures $P_t$.

Girsanov's theorem for a Brownian motion $Z(t)$

$$d\tilde{Z} = \eta dt + dZ \quad \text{is a martingale w.r.t.}$$

$$\mathbb{Q}_t^P, \quad \mathbb{Q}_t^P = \exp\left(-\eta Z(t) - \frac{1}{2} \eta^2 t\right).$$
Utility functions

Consider an investor ("consumer") who at time \( t \) has available wealth \( W(t) \) to invest ("consume") and of which the amount \( C(t) \) is consumed, leaving \( W(t) - C(t) \) available for future use. The consumer takes the view that consuming the amount \( C(t) \) yields a certain utility \( U(C(t), t) \), and the consumer wants to maximize this utility.

We write

\[
U' = \frac{dU}{dC}, \quad U'' = \frac{d^2U}{dC^2}
\]

and assume that as a function of \( C \), \( U \) is increasing and concave, that is, \( U' \geq 0 \) and \( U'' \leq 0 \).

Suppose that at time \( t \), the consumer can change the allocation in his investment portfolio, which contains \( n \) assets (stocks) with prices \( S_i, 1 \leq i \leq n \) (or \( S_i(t) \)). The prices \( S_i \) are random variables and so is \( U \). The decisions at time \( t \) are based on their expected impact on the situation at some future time \( T \).
Denote by $E_t[X]$ the expected value at time $t$ of the random variable $X$.

It can be shown (we omit the details) that when the utility is maximized, we have

$$E_t \left[ U'(T) \frac{S_i(T)}{S_i(t)} \right] = U'(t) = \text{a known quantity}$$

for $1 \leq i \leq n$. (Also $S_i(t)$ is known, so it can be taken out of $E_t$.)

If we write

$$\frac{1}{1 + \mu_i} = \frac{U'(T)}{U'(t)}$$

this reads

$$E_t \left[ \frac{S_i(T)}{1 + \mu_i} \right] = S_i(t), \quad (\mu_i \text{ is a random variable})$$

we may think of $1 + \mu_i$ being a discount factor.

All of the above can be done in a more refined manner if we introduce further notation and consider the various situations ("states") where we may be at each time as a result of the decisions made, but we omit the details.
Change of measure based on utility functions

First consider a simple situation.

Suppose that we have a discrete probability space with infinitely many elements, say \( \Omega = \{1, 2, \ldots, m\} \) for some positive integer \( m \). Write \( p_i = P(\{i\}) \), so each \( p_i \geq 0 \) and
\[
\sum_{i=1}^{m} p_i = 1.
\]

Let \( X \) be a random variable on \( \Omega \) with values \( X(i) \), \( 1 \leq i \leq m \). Then the expectation of \( X \) is
\[
E[X] = \sum_{i=1}^{m} p_i x_i.
\]

Suppose that it happens to be the case that all \( x_i \geq 0 \) and
\[
\sum_{i=1}^{m} p_i x_i = 1.
\]

Then the numbers \( p_i x_i \) can be used to define a new probability measure (so this is a change of measure), say
\[
p_i^* = p_i x_i, \quad 1 \leq i \leq m.
\]

The "events" \( \{i\} \) for which \( x_i > 1 \) get a larger probability than before, and those with \( x_i < 1 \) get a smaller probability.
Recall that
\[
E \left[ \frac{U'(t) S_i(t)}{U'(t) S_k(t)} \right] = 1,
\]
so we could use this ratio to construct a new probability measure.

If we take the view that as a result of consumption/investment decisions, we could be at any of \( m \) states with labels \( 1, \ldots, m \), and if we indicate this by writing

\[
U'(C(T,j), T) \text{ and so on,}
\]

we get for each \( k \),

\[
(\lambda) = \sum_{j=1}^{m} \frac{U'(C(T,j), T) S_k(T,j)}{\sum_{j=1}^{m} \frac{U'(t)}{S_k(t)} S_k(T,j)},
\]

which can also be written as

\[
S_k(t) = \sum_{j=1}^{m} \frac{U'(C(T,j), T)}{\sum_{j=1}^{m} \frac{U'(t)}{S_k(t)} S_k(T,j)}.
\]

Choosing a particular value for \( k \), say \( k = 1 \), and divide both sides by \( S_1(\cdot) \), and get

\[
S_1(t) = \sum_{j=1}^{m} \frac{U'(C(T,j), T)}{\sum_{j=1}^{m} \frac{U'(t)}{S_1(t)} S_1(T,j)} S_1(T,j),
\]

Here we have performed a change of numeraire: we measure \( S_1(t) \) in terms of (or in units of) \( S_1(t) \), and similarly for \( S_k(T,j) \) in units of \( S_1(T,j) \).
Note that taking \( k = 1 \) in (14), we get
\[
1 = \sum_{j=1}^{m} \frac{p_j}{U'(C(T,j),T) S_j(T,j)} \quad \frac{S_j(T,j)}{S_j(T)} \quad \frac{p_j}{U'(T) S_j(T)}
\]
so that in the end, we have performed both a change of measure and a change of
numeraire. If we write
\[
\frac{S_i(T)}{S_j(T)} = \frac{U'(C(T,i),T) S_i(T,i)}{U'(T) S_i(T)} \quad \frac{U'(C(T,j),T) S_j(T,j)}{U'(T) S_j(T)} \quad \frac{p_j}{S_j(T,j)} \quad \frac{p_i}{S_i(T,i)}
\]
then the above reads
\[
\frac{S_k(T)}{S_i(T)} = \sum_{j=1}^{m} \frac{p_j}{S_j(T,j)} \quad \frac{S_j(T,j)}{S_i(T,i)} \quad \frac{S_j(T,j)}{S_j(T)}
\]
That is, the superscript refers to the new
\[
\frac{S_k(T)}{S_i(T)} = E \left[ \frac{S_k(T)}{S_i(T)} \right]
\]
This means that the ratio \( \frac{S_k}{S_i} \) is a
martingale (with respect to this new
family of probability measures). This is a
general idea in pricing models: the ratio
of any two asset prices should be a
martingale (with respect to an appropriate
probability measure).