The Black-Scholes-Merton equation

Suppose that for a stock price $S$ we use the model

$$\frac{dS}{S} = (r - \delta) dt + \sigma dZ.$$

If we invest the amount $W$ in a risk-free bond with interest rate $r$, then

$$dW = rW dt.$$

Consider a financial instrument (e.g., an option) whose value

$$V(S(t), t)$$

depends on $S(t)$ and $t$.

Suppose that we form a portfolio consisting of one of these options, $N$ shares ($N > 0$ in spite of the notation that may suggest an integer), and the amount $W$ in the bond. Let $I = I(s)$ be the value of the portfolio.

Let $\delta$ be the dividend rate.

By Itô’s lemma,

$$dI = dV + N(dS + \delta S dt) + dW$$

$$= V dt + V S_{\delta} dt + \frac{1}{2} \sigma^2 S^2 V_{SS} dt + N(dS + \delta S dt) + rW dt.$$

To remove risk, we want to remove the $dS$ term.

So we choose

$$N = -V_{S}.$$
We get

\[ dI = \left( V_t + \frac{1}{2} \sigma^2 V_s^2 - V_s \delta S + rW \right) dt. \]

Suppose that we start with zero investment. We spend

\[ V + NS = V - V_s S \]

on the option and stock, so we borrow this amount. Hence

\[ W = -(V - V_s S) = V_s S - V. \]

Hence

\[ dI = (V_t + \frac{1}{2} \sigma^2 V_s^2 - \delta S V_s + r V_s - rV) dt. \]

To avoid arbitrage (since this is now risk-free), the coefficient of \( dt \) must be zero. Thus

\[ V_t + \frac{1}{2} \sigma^2 V_s^2 + (r - \delta) V_s - rV = 0. \]

This is the Black-Scholes equation.

Many functions \( V(S, t) \) satisfy this equation. The boundary conditions vary depending on the situation.
We now go through a number of situations where the Black–Scholes equation is satisfied.

1) If \( V(S,t) = e^{-r(T-t)} \) for a given constant \( T \), then \( V_S = V_{SS} = 0 \), \( V_t = rV \), and

\[
V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + (r-d)SV_S - rV = 0.
\]

This \( V \) is the price of a zero-coupon bond maturing at time \( T \) with a continuous interest calculation. The boundary condition is \( V(S,T) = 1 \).

2) If \( V(S,t) = S(t) e^{-r(T-t)} \), then this \( V \) is the price of a prepaid forward contract, paid at time \( t \), maturing at time \( T \), for one share of stock. The boundary condition is \( V(S,T) = S(T) \). Here

\[
V_S = S(T-t), \quad \text{so} \quad SV_S = V
\]

\[
V_{SS} = 0
\]

\[
V_t = S(T-t) e^{-r(T-t)} = \delta V.
\]

Hence

\[
V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + (r-d)SV_S - rV = e^{S(T-t)} \delta V + (-r)SV - rV = 0.
\]
3) For a European call option with strike price $K$ and expiration at time $t = T$, the boundary condition is

$$V(S(T), T) = \max\{0, S(T) - K\}.$$

We earlier found that the price of the call, now at time $t$, not at time $0$, is

$$S e^{-r(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln(S/K) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}},$$

$$d_2 = d_1 - \sigma \sqrt{T-t}.$$

Let us consider separately the behaviour of the two terms $S e^{-r(T-t)} N(d_1)$ and $K e^{-r(T-t)} N(d_2)$ as $t \uparrow T$. We then have the following:

If $S(T) > K$, we have $\ln(S(T)/K) > 0$, so

$$d_1 \to +\infty, \quad d_2 \to +\infty,$$

hence

$$N(d_1) \to 1 \quad \text{and} \quad N(d_2) \to 1.$$

Thus

$$S e^{-r(T-t)} N(d_1) \to S(T), \quad K e^{-r(T-t)} N(d_2) \to K.$$
If $S(T) < K$, then $\ln(S/K) < 0$, so

$$d_1 \rightarrow -\infty, \quad d_2 \rightarrow -\infty, \quad N(d_1) \rightarrow 0, \quad N(d_2) \rightarrow 0.$$  

Thus

$$S e^{-r(T-t)} N(d_1) \rightarrow 0 \quad \text{and} \quad K e^{-r(T-t)} N(d_2) \rightarrow 0.$$

As far as the boundary condition is concerned, the first term represents an asset-or-nothing option, yielding $S(T)$ (equivalent to one share) if $S(T) > K$ and 0 if $S(T) < K$. The other term represents a $K$ times cash-or-nothing option, yielding $S(T)$ if $S(T) > K$ and 0 if $S(T) < K$.

Each of the two terms satisfies the Black-Scholes equation. We now verify this claim. For this purpose, we first find the following derivatives:

$$\frac{\partial d_1}{\partial S} = \frac{1}{S}, \quad d_1^2 = -\frac{1}{S^2}, \quad \frac{\partial^2 d_1}{\partial S^2} = -\frac{1}{S^2} \quad \frac{\partial d_2}{\partial S} = \frac{1}{\sqrt{T-t}}, \quad d_2^2 = \frac{1}{\sqrt{T-t}} \quad \frac{\partial^2 d_2}{\partial S^2} = \frac{1}{\sqrt{T-t}}$$

$$\frac{\partial d_1}{\partial t} = \frac{1}{2} \ln(S/K) - \frac{1}{2} \left( \frac{S+K}{S} \right) \frac{1}{\sqrt{T-t}} = \frac{d_1}{\sqrt{T-t}} - \frac{S}{2} + \frac{K}{2}$$

$$\frac{\partial d_2}{\partial t} = \frac{d_1}{\partial t} + \frac{1}{\sqrt{T-t}}.$$

Recall $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, so $N''(x) = -x N'(x)$.
\[ \frac{\partial}{\partial S} \left( S e^{-S(T-t)} N(d_1) \right) = S e^{-S(T-t)} \frac{N'(d_1) \partial d_1}{\partial S}, \]
\[ \frac{\partial^2}{\partial S^2} \left( S e^{-S(T-t)} N(d_1) \right) = \frac{S e^{-S(T-t)} N'(d_1) \partial d_1}{\partial S} + \]
\[ \frac{S e^{-S(T-t)} \{ N''(d_1) \left( \frac{\partial d_1}{\partial S} \right)^2 + N'(d_1) \frac{\partial^2 d_1}{\partial S^2} \}}{\partial S^2} \]
\[ \frac{\partial}{\partial t} \left( S e^{-S(T-t)} N(d_1) \right) = S e^{-S(T-t)} \left( S N'(d_1) + N'(d_1) \frac{\partial d_1}{\partial t} \right). \]

So with \( V = S e^{-S(T-t)} N(d_1) \), we get

\[ V_t + \frac{1}{2} \sigma^2 S^2 V_{ss} + (r-S)SV_s - rV = S e^{-S(T-t)} X_1, \]

where

\[ X_1 = S N'(d_1) + N'(d_1) \frac{\partial d_1}{\partial t} + \frac{1}{2} \left( 2SN'(d_1) \frac{\partial d_1}{\partial S} + SN''(d_1) \frac{\partial d_1}{\partial S} + SN'(d_1) \frac{\partial^2 d_1}{\partial S^2} \right) \]
\[ \frac{1}{2} \left( S N''(d_1) \left( \frac{\partial d_1}{\partial S} \right)^2 + N''(d_1) \frac{\partial d_1}{\partial S} \right) \]
\[ + (r-S) \left( N(d_1) + SN'(d_1) \frac{\partial d_1}{\partial S} \right) - rN(d_1). \]

The terms involving \( N(d_1) \) cancel out. What remains is

\[ X_1 = \frac{\sigma^2}{2} S N''(d_1) \left( \frac{\partial d_1}{\partial S} \right)^2 + N'(d_1) X_2, \]

where

\[ X_2 = \frac{\partial d_1}{\partial t} + \frac{1}{2} \sigma^2 (r-S) + \frac{\sigma^2}{2} \frac{\partial^2 d_1}{\partial S^2}. \]
\[
\begin{align*}
\frac{d_1}{2(T-t)} &= \frac{r - \delta + \frac{\sigma^2}{2}}{\sigma \sqrt{T-t}} + \frac{r - \delta + \sigma^2}{\sigma \sqrt{T-t}} \frac{1}{\sigma \sqrt{T-t}} \\
&= \frac{d_1}{2(T-t)}.
\end{align*}
\]

Since \( N''(d_1) = -d_1, N'(d_1) \), we have

\[
\begin{align*}
\frac{X_i}{N'(d_1)} &= \frac{d_1}{2(T-t)} + \frac{\sigma^2}{2} \frac{d_1}{\sigma^2(T-t)} S^2 \frac{1}{\sigma^2(T-t)} = 0.
\end{align*}
\]

We omit the similar calculation for the term \( K e^{-r(T-t)} N(d_2) \).
Thus buying a European call option with strike price $K$ is equivalent to buying one asset-or-nothing option and selling $K$ cash-or-nothing options (each paying $\$1$ if $S(T) > K$, and $0$ if $S(T) < K$).

Here we now see the use of the number $K$ in two roles. Since $d_1$ and $d_2$ involve $K$, the number $K$ appears in both of the quantities $Se^{-r(T-t)}N(d_1)$ and $e^{-r(T-t)}N(d_2)$ through $d_1$ and $d_2$. But $K$ is also the number of $\$1$-cash-or-nothing options sold.

Suppose that choose $K_1 < K$ and do everything as above, except that we sell $K_1$ (and not $K$) cash-or-nothing options. This is a gap option that pays

$$S(T) - K_1 \text{ if } S(T) > K$$

$$0 \text{ if } S(T) < K.$$
Multivariate Black-Scholes equation

Let a derivative depend on stocks $S_1, \ldots, S_n$ and time $t$, with value $V(S_1, \ldots, S_n, t)$, expiration $T$.

Consider a portfolio where we have one derivative, $N_i$ shares of stock $i$, and the amount $W$ in a risk-free bond with interest rate $r$.

The total value of the portfolio is

$$ I = V + \sum_{i=1}^n N_i S_i + W. $$

By the multivariate form of Itô's lemma, we get

$$ dI = V \, dt + \sum_{i=1}^n N_i \, dS_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}^2 \, dW_i \, dW_j + \sum_{i=1}^n N_i \, dS_i + dW + \sum_{i=1}^n N_i S_i \, dt. $$

To eliminate $dS_i$, we set $N_i = -V_{S_i}$. If $S_i, S_j$ have correlation $\rho_{ij}$, then $dS_i \, dS_j = \rho_{ij} \, dt$.

We start with $I = 0$, so $dW = W$ and

$$ W = -V + \sum_{i=1}^n V_{S_i} \, S_i. $$

Now $dI = A \, dt$ for some $A$, and to have no arbitrage, we must have $A = 0$. Calculating what this $A$ is, we get the multivariate Black-Scholes equation

$$ V_{t} + \sum_{i=1}^n (r - \delta_i) S_i V_{S_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}^2 \, S_i S_j V_{S_i S_j} - r V = 0. $$
Changing the numeraire (unit of denomination)

The numeraire is the standard by which values are computed. We have used the dollar. We could use another currency, or the price of a stock, or the price of gold, all of which would have values with respect to $ that vary in time.

Suppose that we have our usual stock price $S_t$ such that

$$\frac{dS}{S} = (\alpha_S - \delta_S) dt + \sigma_S dZ_S(t).$$

Let $Q(t)$ be another such asset price s.t.

$$\frac{dQ}{Q} = (\alpha_Q - \delta_Q) dt + \sigma_Q dZ_Q.$$ Let the correlation between $dZ_S, dZ_Q$ be $\rho$, so $dZ_S dZ_Q = \rho dt$. Suppose $b \in \mathbb{R}$ (to be used as an exponent).

Suppose we have a financial instrument whose payoff at expiration $T$ is

$$V[Q,S,T] = Q(T)^b V[S(T), K, \delta_S, T, \delta_Q]$$

$\forall t = T$

where $V$ represents an option on the stock $S$. Thus the payoff is whatever the option at $S$ would have paid, multiplied by $Q(T)^b$. 
Using the multivariate Black-Scholes equation, it can be proved (see Appendix B, p. 646 in the book) that at a time \( t < T \), the value of this financial instrument is given by

\[
Q(t) = e^{(r - S^*)(T-t)} \sqrt{V(s(t), K, \sigma, r, T-t, \eta)}
\]

where

\[
\eta = S_s - \ln \left( \frac{S_s}{\bar{S}_Q} \right)
\]

\[
S^* = r - b (r - \delta) - \frac{1}{2} b (b-1) \sigma^2.
\]

Here we assume that we know already the value \( V \) of the (hopefully simpler) option based only on the stock \( S_s \). Then in that formula for \( V \), we have replaced \( S_s \) by \( \eta \), and we use, in addition, the "discount factor" \( e^{(r - S^*)(T-t)} \).

The number \( S^* \) can be characterized as the "discount rate" of \( Q \), so that \( Q e^{(r - S^*)(T-t)} \) is the forward price for a claim that pays \( Q e^{\eta} \) at time \( T \). So the price for a claim paying \( Q e^{\eta} V \) at \( T \) is this forward price times \( V \) evaluated using a modified dividend \( (S_s \rightarrow \eta) \). Also \( \eta \) takes care of
the correlation between S and Q.