A lognormal model of stock prices

The return of a stock with price \( S_t \) at time \( t \), from time \( t \) to time \( t+h \) is defined by

\[
R(t, t+h) = \frac{S_{t+h}}{S_t} e^{\frac{\alpha_h t + \sigma_h^2 t}{2}}
\]

We assume that returns over disjoint time periods are independent and that returns over such time periods of equal length are identically distributed.

If \( T = nh \), then the expected value and variance of the return over an interval of length \( h \) are denoted by

\[
\mathbb{E}[R(0, T)] = nh \alpha_h,
\]

\[
\text{Var}[R(0, T)] = nh \sigma_h^2.
\]

The expected value (= mean) and variance are thus proportional to time. We can thus use their normalized values, say \( \hat{\alpha} \) and \( \hat{\sigma}^2 \).
Writing \( E[\mathcal{R}(0,T)] = T \frac{\alpha_k}{k} \), \( \text{Var}[\mathcal{R}(0,T)] = T \frac{\sigma_k^2}{k} \),

we see that both are proportional to \( T \), so we can write

\[
E[\mathcal{R}(0,T)] = \alpha T,
\]

\[
\text{Var}[\mathcal{R}(0,T)] = \sigma^2 T.
\]

Let us now assume that

\[
\ln \frac{S_T}{S_0} \sim N(\mu t, \sigma^2 t).
\]

How should we choose \( \mu^2 \)?

If we denote by \( Z \) a random variable with \( Z \sim N(0,1) \), we may write

\[
\ln \frac{S_T}{S_0} = \mu t + \sigma \sqrt{t} Z,
\]

so

\[
S_T = S_0 e^{\mu t + \sigma \sqrt{t} Z}.
\]

We know that

\[
E[e^{\frac{\sigma \sqrt{t} Z}{\sqrt{t}}}] = e^{\frac{\sigma^2 t}{2}}
\]
So that
\[ E[S_t] = S_0 e^{\mu t + \frac{1}{2} \sigma^2 t} \]

If we would like to have
\[ E[S_t] = S_0 e^{(\alpha - d - \frac{1}{2} \sigma^2) t} \]

we need to choose
\[ \mu = (\alpha - d - \frac{1}{2} \sigma^2) \Omega, \]

Thus
\[ \ln \frac{S_t}{S_0} \sim N \left( (\alpha - d - \frac{1}{2} \sigma^2) t, \sigma^2 t \right) \].

Here \( \delta \) is the continuous dividend rate and \( \alpha \) is the annualized rate of return excluding dividends.

The median stock price (prob. 50% of the price being above or below it) is obtained by setting \( Z = 0 \) so it is
\[ S_0 e^{\mu t} = S_0 e^{(\alpha - d - \frac{1}{2} \sigma^2) t} < E[S_t]. \]

So more than "half the time", \( S_t \) is below its expected value.
Probability calculations

We have

\[ \ln S_t \sim N(\ln S_0 + (\alpha - \delta - \frac{1}{2} \sigma^2) t, \sigma^2 t) \]

\[ Z \sim N(0,1) \text{ where} \]

\[ Z = \frac{\ln S_t - \ln S_0 - (\alpha - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \]

We have

\[ P(S_t < K) = N(x) \text{ where} \]

\[ x = \frac{\ln \frac{K}{S_0} - (\alpha - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \]

Recall the notations

\[ d_1 = \frac{\ln (S/K) + (r - \delta + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \quad \text{(new \ t, \ not \ T)} \]

\[ d_2 = d_1 - \sigma \sqrt{t} = \frac{\ln (S/K) + (r - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \]

With \( S = S_0 \), we have (and with \( \alpha \) instead of \( r \))

\[ x = -d_2 \]

\[ P(S_t < K) = N(-d_2), \quad P(S_t > K) = N(d_2). \]
Prediction intervals

Suppose that $0 < \rho < 1$ and we look for prices $S_t^L$ (lower), $S_t^U$ (upper) such that

$$ P(S_t^L < S_t) = \rho/2 \quad \text{and} \quad P(S_t > S_t^U) = \rho/2, $$

hence with probability $1 - \rho$ (perhaps $\rho$ is small), we have

$$ S_t^L \leq S_t \leq S_t^U. $$

Now $S_t^L$ must satisfy

$$ \frac{\rho}{2} = N(-d_2) $$

where

$$ d_2 = \frac{\ln \frac{S_0}{S_t^L} + (\alpha - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}}. $$

The function $N(x)$ has an inverse function $N^{-1}$, hence

$$ d_2 = -N^{-1}(\rho/2) $$

$$ \ln \frac{S_0}{S_t^L} = (\alpha - \delta - \frac{1}{2} \sigma^2) t - \sigma \sqrt{t} N^{-1}(\rho/2), $$

$$ S_t^L = S_0 e^{(\alpha - \delta - \frac{1}{2} \sigma^2) t + \sigma \sqrt{t} N^{-1}(\rho/2)}. $$
For $S_t^U$, we must have

$$P(S_t > S_t^U) = p/2, \quad \text{so} \quad P(S_t < S_t^U) = 1-p/2,$$

and we proceed as above with $p/2$ replaced by $1-p/2$. This gives

$$S_t^U = S_0 e^{(x-S_0 - \frac{1}{2} \sigma^2) t + \sigma \sqrt{t} N(1-p/2)}.$$
The conditional expected price

Suppose that $B$ is an event ($\omega \in B$) with $P(B) > 0$. The conditional expectation of a random variable $X: \Omega \rightarrow \mathbb{R}$ given $B$ is

$$E[X | B] = \frac{1}{P(B)} \int X \chi_B \, dP = \frac{1}{P(B)} \int_B X \, dP.$$ 

Suppose that $B$ is the event that $S_t > K$ for a certain $K > 0$ (and $t > 0$). So $X = S_t$. Then

$$\int_B X \, dP = \int_B S_t \phi(S_t) \, dS_t$$

where $\phi(S_t)$ is the density function

$$\phi(S_t) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp \left( -\frac{1}{2} \frac{(\ln S_t - (\mu - \frac{1}{2} \sigma^2) t)^2}{\sigma^2 t} \right).$$

This integral is equal to

$$S_0 e^{-\lambda t} \mathcal{N}(td_1)$$

where

$$d_1 = \frac{1}{\sigma \sqrt{t}} \left( \ln \frac{S_0}{K} + (\lambda - \frac{1}{2} \sigma^2) t \right).$$

We have $P(S_t > K) = \mathcal{N}(d_2)$ as seen before, so

$$E[S_t | S_t > K] = S_0 e^{(\lambda - \sigma^2 / 2) t} \mathcal{N}(d_1).$$
The Black-Scholes formula

We take \( \sigma = r \). Then the price of the European call is

\[
C = e^{-rt} E \left[ S_t - K \mid S_t > K \right] \cdot P(S_t > K).
\]

Now

\[
E \left[ K \mid S_t > K \right] = \frac{KP(S_t > K)}{P(S_t > K)} = K.
\]

So

\[
E[S_t - K \mid S_t > K] \cdot P(S_t > K) = S_0 e^{-\frac{rs}{2}} N(d_1) - KN(d_2)
\]

and

\[
C = S_0 e^{-rt} N(d_1) - Ke^{-rt} N(d_2),
\]

the same formula that we have seen before.
Remarks regarding the calculation of certain integrals.

The distribution \( N(\mu, \sigma^2) \) has the probability density function (pdf)

\[
    g(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

and cumulative distribution function (cdf)

\[
    F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, dt.
\]

Suppose now that for a random variable \( Y \), it is \( \ln Y \) and not \( Y \) that has the normal distribution, i.e.,

\( \ln Y \sim N(\mu, \sigma^2) \).

When we look for the cdf for \( Y \), actually we should replace \( t \) by \( \ln y \) in the expression

\[
    F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \, dt.
\]

Since \( t = \ln y \Rightarrow dt = dy/y \), we get

\[
    F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} \, dy.
\]
Suppose now that \( Y = S_t / S_0 \) (since \( \ln S_t \sim N(\mu t, \sigma^2 t) \)), where \( \mu = r - \frac{1}{2} \sigma^2 \). So \( Y = \frac{S_t}{S_0} \).

To calculate the expectation of \( S_t \) when \( S_t > K \), we calculate

\[
\int_{K}^{\infty} \frac{1}{\sqrt{2\pi} S_t} \exp \left( -\frac{1}{2} \left( \frac{\ln S_t - \mu t}{\sigma \sqrt{t}} \right)^2 \right) \, \frac{dS_t}{S_t}.
\]

To evaluate this integral, let us first set

\[
x = \frac{\ln S_t - \ln S_0 - \mu t}{\sigma \sqrt{t}}, \quad dx = \frac{1}{\sigma \sqrt{t} S_t} \, dS_t.
\]

With

\[
K_1 = \frac{1}{\sigma \sqrt{t}} (\ln K - \ln S_0 - \mu t),
\]

this gives

\[
\int_{K_1}^{\infty} \frac{1}{\sigma \sqrt{2\pi} t} \exp \left( -\frac{1}{2} x^2 \right) \, dx = \int_{K_1}^{\infty} \frac{1}{\sigma \sqrt{2\pi} t} \exp \left( -\frac{1}{2} x^2 \right) \, dx
\]

while we still have to express \( S_t \) in terms of \( x \).

We have

\[
\ln S_t = \ln S_0 + \mu t + \sigma \sqrt{t} x,
\]

so

\[
S_t = S_0 e^{\mu t} e^{\sigma \sqrt{t} x}.
\]

Thus the integral is

\[
\frac{S_0 e^{\mu t}}{\sqrt{2\pi}} \int_{K_1}^{\infty} e^{-\frac{1}{2} x^2 - \frac{1}{2} x^2} \, dx.
\]

Now

\[
\sigma \sqrt{t} x - \frac{1}{2} x^2 = -\frac{1}{2} (x - \sigma \sqrt{t})^2 + \frac{1}{2} \sigma^2 t.
\]
So if we act

\[ u = x - \sigma \sqrt{t}, \quad du = dx \]

\[ K_2 = K_1 - \sigma \sqrt{t} \]

we get

\[ S_0 e^{\frac{\mu t}{2} \sigma^2 t} \int_{K_2}^{\infty} e^{-\frac{1}{2} u^2} \frac{1}{\sqrt{2\pi}} du. \]

Now \( \mu t \sigma^2 (x - \delta - \frac{1}{2} \sigma^2) t \sigma^2 t \)

\[ e = e \quad e = e \quad e = e \]

and

\[ \frac{1}{\sqrt{2\pi}} \int_{K_2}^{\infty} e^{-\frac{1}{2} u^2} du = 1 - N(K_2) = N(-K_2). \]

We have

\[ K_2 = \frac{\ln K - \ln S_0 - (x - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} - \sigma \sqrt{t} \]

So that

\[ K_2 = \frac{\ln S_0 K + (x - \delta + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \]

which in previous notation (where we had \( r \) instead of \( \alpha - \delta \)) corresponds to \( d_1 \).

The whole integral is then

\[ S_0 e^{(x - \delta) t} N(d_1) \]

as claimed earlier.
Ch. 19 Monte Carlo Valuation

Getting a sample from a probability distribution

First, we need to be able to draw random numbers that are uniformly distributed in the interval $[0, 1]$. Suppose that this can be done, and let $y_1, y_2, \ldots, y_n$ be numbers in $(0, 1)$ obtained in this way.

Suppose that the probability distribution we are interested in is $N(0, 1)$ (the same ideas work for any distribution). For each $y_j \in (0, 1)$ we find $x_j \in \mathbb{R}$ such that

$$N(x_j) = y_j.$$ 

Then the numbers $x_1, \ldots, x_n$ are a sample from $N(0, 1)$ (here $N(x)$ is the cdf of $N(0, 1)$, and for other distributions we'd use their cdfs).
Simulating a path (e.g., for a stock price)

Suppose that

$$S_t = S_0 \exp \left( (\mu - \frac{1}{2} \sigma^2) t + \sigma \sqrt{t} Z \right)$$

where $Z \sim N(0,1)$. Suppose that numerical values have been given for $\mu$, $\sigma$, $S_0$.

Choosing a value for $t$, and then choosing a sample $x_1, \ldots, x_N$ from $N(0,1)$, and substituting the values $x_1, \ldots, x_N$ for $Z$, we obtain a distribution of stock prices $S_t$. If $N$ is large, these prices should approximate well the continuous distribution for $S_t$.

Choosing one value for $t$ as above is a bit like taking one step in a binomial tree before, except that instead of using only two values for $S_t$, we use $N$ values.

Suppose now that we want to take many steps, which together constitute a path.
This could perhaps be done in many ways, but here is one possibility.

Choose one number $x_1$ as above (and choose a time interval length $h$) and set

$$S_{1h} = S_0 \exp \left( (\mu - \frac{1}{2} \sigma^2) h + \sigma \sqrt{h} \, x_1 \right).$$

Then choose $x_2$ and set

$$S_{2h} = S_1 \exp \left( (\mu - \frac{1}{2} \sigma^2) h + \sigma \sqrt{h} \, x_2 \right).$$

Continue in this way, choosing $x_3, \ldots, x_N$ we get stock prices

$$S_h, S_{2h}, \ldots, S_{Nh},$$

that form a path of stock prices.

Paths like this can then be used in many ways. For example, if we consider an American option or a barrier option, we can find its payoff $V_j$ for this path, say path $j$.

If we do this for $N_0$ paths, then

$$\frac{1}{N_0} \sum_{j=1}^{N_0} V_j.$$
provides an estimate of the expected value of the payoff of the option. The discounted value of this expected payoff can then serve as the option price.

Due to the possibility of early exercise there are difficulties if we use this method for American options. Various methods have been suggested, but we do not discuss them here.

It may be that a very large number of paths are needed for a good accuracy of the option price. Various additional "corrections" can sometimes be used for greater accuracy, but we do not discuss them here.

More details for those who are interested are provided in the book.