We denote by $C(t, S(t), K, T)$ (or $P(t, S(t), K, T)$) the price, at time $t$ (where $t < T$) of a European style call (put) option with strike price $K$ and expiration date $T$, and with the stock price being $S(t)$ at time $t$. We may use our favorite notations if their meaning is clear.

**Choosing options**

Suppose $t < T_1 < T_2$. Buying a choice option at time $t$ entitles one to choose, at time $T_1$, to receive a call or a put for the same stock, for the same strike price $K$, and for the same expiration date $T_2$. So at time $T_1$, the payoff is

$$\max\left\{ C(T_1, S(T_1), K, T_2), P(T_1, S(T_1), K, T_2) \right\}$$

$$= C + (P - C)^+ = P + (C - P)^+$$

By the put-call parity at time $T_1$,

$$C(T_1) - P(T_1) = e^{-(T_2 - T_1)} S(T_1) - e^{-r(T_2 - T_1)} K$$

$$= e^{-r(T_2 - T_1)} (S(T_1) - e^{-(T_2 - T_1)} K).$$
Thus we can write the payoff at time $T_1$ as
\[ \text{P}(T_1, S(T_1), K_1, T_2) + e^{-(r+\delta)(T_2-T_1)}(S(T_1) - e^{-r(T_2-T_1)}K_1)^+ \]

or as
\[ \text{C}(T_1, S(T_1), K_1, T_2) + e^{-(r+\delta)(T_2-T_1)}(e^{-r(T_2-T_1)}K_1 - S(T_1))^+ . \]

The terms in $()^+$ are the payoffs, for one share, of a regular call or put option, expiring at $T_1$, with strike price $e^{-r(T_2-T_1)}K_1$. $-S(T_2-T_1)$ Multiplied by $e^{-(r+\delta)(T_2-T_1)}$ gives the same for, not one share, but $e^{-\delta(T_2-T_1)}$ shares.

Hence the price of the chosen option at time $t$ is
\[ \text{P}(t, S(t), K_1, T_2) + e^{-(r+\delta)(T_2-T_1)}\text{C}(t, S(t), e^{-r(T_2-T_1)}K_1, T_1) \]

or
\[ \text{C}(t, S(t), K_1, T_2) + e^{-(r+\delta)(T_2-T_1)}\text{P}(t, S(t), e^{-r(T_2-T_1)}K_1, T_2) \]

More precise formulas would involve two different parameters of the type $\delta_1$ and $\delta_2$.
Forward start options

Suppose that $t < T_1 < T_2$ and $c > 0$.
If we buy a (long) forward start (put) option at time $t$, it expires at time $T_2$, and its strike price is $c S(T_1)$. So the payoff at time $T_2$ is

$$(S(T_2) - c S(T_1))^+$$

for a call, and

$$(c S(T_1) - S(T_2))^+$$

for a put.

For a call, the price at time $t$ is $e^{-r(T_1-t)}$ times the expectation (at time $t$) of the price of $C(T_1, S(T_1), c S(T_1), T_2)$, i.e.,

$$e^{-r(T_1-t)} E[ C(T_1, S(T_1), c S(T_1), T_2) ] = e^{-r(T_1-t)} E \left[ \frac{S(T_2) - r(T_2 - T_1) S(T_2) + r(T_1 - T_2) S(T_1)}{c S(T_1) - S(T_2)} \right]$$

where

$$d_1 = d_2 + 6 \left( \frac{T_2 - T_1}{S(T_1)} \right) = \frac{-\ln c + (r - \delta + \frac{1}{2} \sigma^2)(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}}.$$

Next, we consider the long forward call price at time $t$ is

$$e^{-r(T_1-t)} S(t) \left( e^{-r(T_1-t)} S(t) \right),$$

where

$$d_1 = d_2 + 6 \left( \frac{T_2 - T_1}{S(T_1)} \right) = \frac{-\ln c + (r - \delta + \frac{1}{2} \sigma^2)(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}}.$$
Similarly, the price at time \( t \) of a long forward put is

\[
-\delta(T_1-t) e^{-r(T_2-T_1)} \left[ c e^{-r(T_2-T_1)} N(-d_2) - e^{-r(T_2-T_1)} N(-d_1) \right]
\]

where \( d_1, d_2 \) are as above.

At time \( t=0 \), these prices are equal to

\[
e^{-rT_1} C(0, S(0), e^{S(0)}, T_2-T_1)
\]
for a call, and

\[
e^{-rT_1} P(0, S(0), e^{S(0)}, T_2-T_1)
\]
for a put.
Lookback options

Suppose we buy an option at time $t=0$, with expiration at $t=T$, and write

$$m_T = \min \left\{ S(t) : 0 \leq t \leq T \right\},$$

$$M_T = \max \left\{ S(t) : 0 \leq t \leq T \right\}.$$

Payoffs at time $T$ (long option, no early exercise):

- Standard lookback call \( S(T) - m_T \)
  
  \[
  \begin{array}{c}
  \text{put} \\
  M_T - S(T)
  \end{array}
  \]

- Extrema lookback call \( (M_T - K)^+ \)
  
  \[
  \begin{array}{c}
  \text{put} \\
  (K - m_T)^+
  \end{array}
  \]
**Shout options**

Buy at time $t$, expiration at time $T$ ($t < T$), additional right to lock in a minimum positive payoff at time $t$ (shouting time) ($t \leq s < T$).

If this right is used, then payoff at time $T$ is

- for a shout call: $\max \{0, S(T) - K, S(s) - K\}$
- for a shout put: $\max \{0, K - S(T), K - S(s)\}$

If not shouted, the option is a regular call or put.

One chooses whether to shout, and if so, when. One can shout at most once, and the choice (having done so) cannot be revoked later.

The terms can be made more flexible by agreement.