5.1 Continuity

March 24, 2020
Continuity is initially defined at each point separately.

**Definition**

Suppose that $A$ is a non-empty subset of $\mathbb{R}$. Let $f : A \rightarrow \mathbb{R}$ be a function.

Suppose that $c \in A$.

We say that $f$ is **continuous at** $c$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x \in A$ and $|x - c| < \delta$, we have

$$|f(x) - f(c)| < \varepsilon.$$

If $f$ is not continuous at $c$, we say that $f$ is **discontinuous** at $c$. 
1. In the definition of a limit, \( c \) is a cluster point of \( A \), and \( c \) may or may not belong to \( A \). In particular, \( A \) must be an infinite set, which is a necessary but NOT a sufficient condition for \( A \) to have at least one cluster point.

2. In the definition of continuity at \( c \), the point \( c \) must belong to \( A \), while \( c \) may or may not be a cluster point of \( A \). The set \( A \) may be any non-empty subset of \( \mathbb{R} \), and \( A \) is not required to have any cluster points.

**Definition**

Suppose that \( A \subset \mathbb{R} \) and that \( c \in A \). If \( c \) is not a cluster point of \( A \), we say that \( c \) is an **isolated point** of \( A \).
Thus each point of $A$ is either a cluster point of $A$ or an isolated point of $A$.

It is possible for a point $c \in \mathbb{R}$ to be a cluster point of $A$ even if $c \notin A$. However, by definition, each isolated point of $A$ must belong to $A$. 
1. If $c \in A$ is an isolated point of $A$, then every function $f : A \to \mathbb{R}$ is continuous at $c$. Namely, then there is $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $x = c$ and hence $|f(x) - f(c)| = 0 < \varepsilon$ for every $\varepsilon > 0$.

This may sound surprising, but expressed more vaguely, we may say that continuity of $f$ at $c$ means that if $x \in A$ and if $x$ is close to $c$, then $f$ does not “tear apart” $f(x)$ from $f(c)$ (to make $f(x)$ to be far away, at least in relative terms, from $f(c)$).

But if $c$ is an isolated point of $A$, then sufficiently close to $c$ there are no points of $A$ other than $c$, so that there is no point $x \in A \setminus \{c\}$ that $f$ could tear apart from $c$. 
2. If \( c \in A \) is a cluster point of \( A \), then \( f \) is continuous at \( c \) if, and only if, \( \lim_{x \to c} f(x) \) exists and
\[
\lim_{x \to c} f(x) = f(c).
\]

This is easily seen by comparing the definition of a limit and the definition of continuity of \( f \) at \( c \). Note that \( c \) needs to be a cluster point of \( A \) for \( \lim_{x \to c} f(x) \) to make sense.
Continuity on a set

Continuity on a set is defined pointwise, that is, considering each point separately.

**Definition**

Suppose that $A$ is a non-empty subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Suppose that $B \subset A$. We say that $f$ is **continuous on** $B$ if $f$ is continuous at every point $c \in B$.

**Definition**

Suppose that $A$ is a non-empty subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. If $f$ is continuous on $A$, we say that $f$ is **continuous**.
Example regarding continuity on a set

Dirichlet’s function: The characteristic function of the set of rational numbers in \([0, 1]\).

Suppose that \(A = [0, 1]\) and that \(f : A \to \mathbb{R}\) is defined by \(f(x) = 1\) if \(x \in [0, 1] \cap \mathbb{Q}\), and \(f(x) = 0\) if \(x \in [0, 1] \setminus \mathbb{Q}\).

Then \(f\) is discontinuous at each point of \([0, 1]\). If \(B = [0, 1] \cap \mathbb{Q}\), then \(f\) is not continuous on \(B\). For that, it would be sufficient for \(f\) to be discontinuous at one point of \(B\), and here \(f\) is discontinuous at every point of \(B\).

Define \(B = [0, 1] \cap \mathbb{Q}\), and define a function \(g : B \to \mathbb{R}\) to be the restriction of \(f\) to \(B\), that is, for all \(x \in B\) we set

\[
g(x) = f(x) = 1.
\]

Then \(g\) is continuous on \(B\), being a constant function.

This illustrates the difference between continuity properties of a function, and of the restriction of a function to a subset of \(A\).
Since rational and irrational numbers are dense in $[0, 1]$, the graph looks a bit like two line segments, even though each “segment” has a dense set of “holes”.

\[
\text{Dirichlet's function: the graph}
\]

\[
\begin{array}{c}
\text{Out[4]} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{Out[5]} = \\
\end{array}
\]
Sequential Criterion for Continuity

Just like there is a Sequential Criterion for limits of functions (see Section 4.1), there is one for continuity.

**Theorem**

*Sequential Criterion for Continuity*

Suppose that $A$ is a non-empty subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function. Suppose that $c \in A$.

Then $f$ is continuous at $c$ if, and only if, for every sequence $x_n$ of points of $A$ such that $\lim x_n = c$, we have $\lim f(x_n) = f(c)$.

Note that there is always at least one sequence $x_n$ of points of $A$ such that $\lim x_n = c$, namely, the constant sequence where $x_n = c$ for all $n \in \mathbb{N}$.

Note that since $c \in A$, in this case we do have $x_n \in A$.

Unlike in the Sequential Criterion for limits of functions, here we allow the possibility that $x_n = c$. 
1. If \( c \in A \) is an isolated point of \( A \), then every function \( f : A \to \mathbb{R} \) is continuous at \( c \). Also, in this case, if \( x_n \) is a sequence of points of \( A \) such that \( \lim x_n = c \), then there is \( K \in \mathbb{N} \) such that for all \( n \geq K \), we have \( x_n = c \) (why?), and hence also \( f(x_n) = f(c) \). Therefore, we have \( \lim f(x_n) = f(c) \).
2. If $c \in A$ is a cluster point of $A$, then, as we have seen, $f$ is continuous at $c$ if, and only if, \( \lim_{x \to c} f(x) \) exists and \( \lim_{x \to c} f(x) = f(c) \). Now by the Sequential Criterion for limits of functions, we have \( \lim_{x \to c} f(x) = f(c) \) if, and only if, for every sequence $y_n$ of points of $A \setminus \{c\}$ such that \( \lim y_n = c \), we have \( \lim f(y_n) = f(c) \).

Suppose that $x_n$ is a sequence of points of $A$ such that \( \lim x_n = c \). If there are values of $n$ such that $x_n = c$, then for these values of $n$, we have \( f(x_n) = f(c) \). If there are only finitely many other values of $n$, they do not affect the limit of $f(x_n)$, so we may disregard them. Otherwise, these other values of $n$ can be used to form (after re-labeling) a new sequence $y_n$ of points of $A \setminus \{c\}$ such that \( \lim y_n = c \), and by the above, we have \( \lim f(y_n) = f(c) \). All these facts together imply that \( \lim f(x_n) = f(c) \).

In the other direction, if for every sequence $x_n$ of points of $A$ such that \( \lim x_n = c \), we have \( \lim f(x_n) = f(c) \), and if $y_n$ is a sequence of points of $A \setminus \{c\}$ such that \( \lim y_n = c \), then the assumption on $x_n$ implies that \( \lim f(y_n) = f(c) \) (since $y_n$ is a special case of $x_n$ here).
Example: a function continuous at finitely many points

Recall the **Dirichlet function**, now on $\mathbb{R}$. This is the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$  

This $f$ is discontinuous everywhere.

To get a function continuous at finitely many (distinct) points $a_1, \ldots, a_k$ (only), multiply the Dirichlet function by $\prod_{j=1}^{k} (x - a_j)$. If there is only one such point and we take it to be the origin, this gives a **modified Dirichlet function**, say $g : \mathbb{R} \to \mathbb{R}$, such that

$$g(x) = xf(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$  

The function $g$ is continuous at $0$ and not at any other point of $\mathbb{R}$. 
Example: a function continuous at finitely many points

\[ g(x) = x f(x) = \begin{cases} 
  x, & x \in \mathbb{Q} \\
  0, & x \in \mathbb{R} \setminus \mathbb{Q}
\end{cases}. \]

First show by the Sequential Criterion that \( g \) is continuous at 0. If \( x_n \to 0 \), then \( |x_n| \to 0 \), so \( |g(x_n) - g(0)| \leq |x_n| \), hence \( g(x_n) \to g(0) \).

Then fix \( x \in \mathbb{R} \setminus \{0\} \), and show that \( g \) is discontinuous at this \( x \). To this end, find a sequence \( x_n \to x \) such that \( g(x_n) \not\to g(x) \).

**Case 1:** \( x \in \mathbb{R} \setminus \mathbb{Q} \). By the Density Theorem, we can find \( x_1, x_2, \ldots \in \mathbb{Q} \) (e.g., \( x_n \in (x, x + \frac{1}{n}) \)) such that \( x_n \to x \). So \( g(x_n) = x_n \to x \), yet \( g(x) = 0 \neq x \).

**Case 2:** \( x \in \mathbb{Q} \setminus \{0\} \). By denseness of \( \mathbb{R} \setminus \mathbb{Q} \), find \( x_1, x_2, \ldots \in \mathbb{R} \setminus \mathbb{Q} \) such that \( x_n \to x \). Then \( g(x_n) = 0 \) for every \( n \), so \( g(x_n) \to 0 \), yet \( g(x) = x \neq 0 \).

The principles are the same if \( g(x) = \left( \prod_{j=1}^{k} (x - a_j) \right) f(x) \).
Thomae’s function (Example 5.1.6(h)). We define $h : \mathbb{R} \to \mathbb{R}$ as follows. If $x \in \mathbb{R} \setminus \mathbb{Q}$, let $h(x) = 0$. If $x \in \mathbb{Q} \setminus \{0\}$, write $x = \frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $a, b$ are relatively prime. Set $h(x) = \frac{1}{b}$. Let $h(0) = 1$. Then $h$ is continuous at $x$ if, and only if, $x \in \mathbb{R} \setminus \mathbb{Q}$.

(1) If $x \in \mathbb{Q}$, then $h$ is discontinuous at $x$. Find a sequence $x_n \in \mathbb{R} \setminus \mathbb{Q}$, $x_n \to x$. Then $0 = h(x_n) \not\to h(x)$ since $h(x) > 0$.

(2) $x \in \mathbb{R} \setminus \mathbb{Q}$. For $\varepsilon > 0$ find $\delta > 0$ such that $h(y) = |h(x) - h(y)| < \varepsilon$ when $|x - y| < \delta$. Only consider $y \in \mathbb{Q}$.

By the Archimedean Property, find $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. Need to find $\delta > 0$ such that if $|\frac{a}{b} - x| < \delta$, then $b > N$. Indeed, then $f\left(\frac{a}{b}\right) = \frac{1}{b} < \frac{1}{N} < \varepsilon$.

Let $z = (N!) \cdot x$, $\gamma = \min\{z - \lfloor z \rfloor, \lfloor z \rfloor + 1 - z\}$ (distance from $z$ to the nearest integer; $\lfloor z \rfloor = \max\{m \in \mathbb{Z} : m \leq z\}$), $\delta = \frac{\gamma}{N!}$. Suppose, for contradiction, that $|\frac{a}{b} - x| < \delta$, and $b \leq N$. Then $|N! \cdot \frac{a}{b}! - (N!) \cdot x| < N! \cdot \delta = \gamma$. This is impossible, since $N! \cdot \frac{a}{b} \in \mathbb{Z}$. \qed
Let $I$ be an interval containing more than one point, let $f : I \to \mathbb{R}$ be a function, let $c$ be an interior point of $I$, and suppose that $f$ is continuous on $I \setminus \{c\}$ but discontinuous at $c$.

We say that $f$ has a **discontinuity of the first kind** at $c$ if the one-sided limits

$$
\lim_{x \to c^-} f(x) = L_1, \quad \lim_{x \to c^+} f(x) = L_2
$$

both exist but $L_1 \neq L_2$. Then we also say that $f$ has a **jump** at $c$. 

---

Discontinuities at one point on an interval

Aimo Hinkkanen  (University of Illinois)  
Math 444: Elementary real analysis  
March 20, 2020  
16 / 20
The above function has a discontinuity of the first kind at the origin. The vertical line segment at \( x = 0 \) is not part of the graph of the function. Here we define \( f \) on \([-1, 1]\) by \( f(x) = -x \) if \(-1 \leq x \leq 0\) and \( f(x) = x + 1 \) if \( 0 < x \leq 1 \).
Let $I$ be an interval containing more than one point, let $f : I \to \mathbb{R}$ be a function, let $c$ be an interior point of $I$, and suppose that $f$ is continuous on $I \setminus \{c\}$ but discontinuous at $c$.

We say that $f$ has a **discontinuity of the second kind** at $c$ if at least one of the one-sided limits

$$\lim_{x \to c^-} f(x), \quad \lim_{x \to c^+} f(x)$$

does not exist.
The above function has a discontinuity of the second kind at the origin. Here we define $f$ on $[-1, 1]$ by $f(0) = 0$, and $f(x) = \sin(1/x)$ if $0 < |x| \leq 1$. Here neither one of the one-sided limits $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ exists.
The above function has a discontinuity of the second kind at the origin. Here we define $f$ on $[-1, 1]$ by $f(x) = 0$ if $-1 \leq x \leq 0$, and $f(x) = \sin(1/x)$ if $0 < x \leq 1$. Here the left hand limit exists but the right hand limit does not exist at the origin.