LOCATION OF THE CRITICAL POINTS OF CERTAIN POLYNOMIALS

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ABSTRACT. Let \( \mathbb{D} \) denote the unit disk \( \{ z : |z| < 1 \} \) in the complex plane \( \mathbb{C} \). In this paper, we study a family of polynomials \( P \) with only one zero lying outside \( \overline{\mathbb{D}} \). We establish criteria for \( P \) to have exactly one critical point outside \( \overline{\mathbb{D}} \).

1. INTRODUCTION

Let \( P \) be a polynomial in the complex plane \( \mathbb{C} \). We denote the degree of \( P \) by \( \text{deg} \, P \). We say that \( \alpha \) is a critical point of \( P \) if \( P'(\alpha) = 0 \). There are several known results involving the critical points of polynomials. The most classical one is the Gauss–Lucas Theorem, [8, p. 25].

Gauss–Lucas Theorem. Let \( P \) be a polynomial of degree \( n \) with zeros \( z_1, z_2, \ldots, z_n \), not necessarily distinct. The zeros of the derivative \( P' \) lie in the convex hull of the set \( \{ z_1, z_2, \ldots, z_n \} \).

If \( P \) has a zero lying outside the closed unit disk \( \overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \} \), by the Gauss–Lucas Theorem, it follows that the zeros of its derivative are in the convex hull of the zeros of \( P \), which includes a region outside \( \overline{\mathbb{D}} \). But we do not know how many zeros of \( P' \) are outside \( \overline{\mathbb{D}} \). We may ask the question of under what conditions does \( P \) have only one critical point, counting multiplicities, outside the closed unit disk? Our purpose here is to give some answers to that question. Throughout this paper, if not otherwise stated, when we talk about the number of zeros of a polynomial in a domain, we mean the number of zeros counting multiplicities. As the critical points of \( P \) are the zeros of \( P' \), this applies also to the number of critical points.

Theorem 1.1. Let \( Q(z) = c \prod_{k=1}^{n}(z - \alpha_k) \) be a polynomial of degree \( n \geq 2 \), where \( c \neq 0 \). Suppose that \( \alpha_k \notin \overline{\mathbb{D}} \) for \( 1 \leq k \leq m \), and that the remaining points \( \alpha_k \) are in \( \overline{\mathbb{D}} \). If we have

\[
\sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} > \frac{1}{2},
\]

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then \( Q \) has exactly \( m \) critical points outside \( \overline{D} \), counting multiplicities. If, in addition, all the points \( \alpha_k \) lying on the unit circle are simple zeros of \( Q \), then \( Q' \) has no zeros on the unit circle.

**Corollary 1.2.** Let \( Q(z) = c \prod_{k=1}^{n} (z - \alpha_k) \) be a polynomial of degree \( n \geq 3 \), where \( c \neq 0 \). Suppose that \( |\alpha_1| > \frac{n}{n-2} \), and that all the remaining points \( \alpha_k \) are in \( \overline{D} \). Then \( Q \) has exactly one critical point outside \( \overline{D} \), counting multiplicities. If, in addition, all the points \( \alpha_k \) that are on the unit circle are simple zeros of \( Q \), then \( Q \) has exactly \( n-2 \) critical points in \( D \), counting multiplicities.

By using Lemma 2.2 and the same argument as in the proof of Theorem 1.1, we immediately derive **Corollary 1.3.**

**Corollary 1.3.** Let \( Q(z) = c \prod_{k=1}^{n} (z - \alpha_k) \) be a polynomial of degree \( n \geq 2 \), \( c \neq 0 \). Suppose that \( \alpha_1 = \alpha, \alpha_2 = \alpha^{-1} \), where \( \alpha \) is real and \( |\alpha| > 1 \), and all the remaining points \( \alpha_k \), if any, are in \( \overline{D} \). Then \( Q \) has exactly one critical point outside \( \overline{D} \), counting multiplicities. If, in addition, all the points \( \alpha_k \) that are on the unit circle are simple zeros of \( Q \), then \( Q \) has exactly \( n-2 \) critical points in \( D \), counting multiplicities.

A polynomial \( P \) is said to be **anti-reciprocal** if \( P(z) = -z^{\deg P} P(z^{-1}) \). Note that if \( P \) is anti-reciprocal, then 1 is a zero of \( P \), we have \( P(0) \neq 0 \), and for \( \alpha \neq 0 \), we have \( P(\alpha) = 0 \) if, and only if, \( P(\alpha^{-1}) = 0 \). If \( P \) is an anti-reciprocal polynomial with exactly one zero, counting multiplicities, lying outside \( \overline{D} \), then \( P \) satisfies the assumptions of Corollary 1.3, and so \( P \) has only one critical point outside \( \overline{D} \). Indeed, it \( P \) is anti-reciprocal with exactly one zero, say \( \alpha \), which is furthermore simple, outside \( \overline{D} \), then \( P \) has exactly one zero (namely, \( 1/\alpha \)) in \( D \), and all the other zeros of \( P \) must lie on \( \partial D \). In Theorem 1.4, we prove that if \( P \) satisfies certain additional conditions then not only does \( P' \) have only one zero outside \( \overline{D} \) but the same is also true for \( P'' \).

**Theorem 1.4.** Let \( Q \) be an anti-reciprocal polynomial with real coefficients of degree \( n \geq 3 \). Suppose that the zeros of \( Q \) are simple and that \( \alpha > 1 \) is the only zero of \( Q \) lying outside \( \overline{D} \). Then each of the polynomials \( Q' \) and \( Q'' \) has exactly one zero outside \( \overline{D} \), counting multiplicities.

We can construct a family of anti-reciprocal polynomials satisfying Theorem 1.4. Let \( P \) be a polynomial with real coefficients, and set \( P^*(z) := z^{\deg P} P(z^{-1}) \). Suppose that \( P \) has a real zero greater than 1, that the remaining zeros of \( P \) are in \( D \) (so \( P(1) \neq 0 \)), and that \( P^* \neq P \). Boyd [1, p. 320] showed that the polynomial

\[
Q(z) = z^n P(z) - P^*(z)
\]

(1)
satisfies the assumptions of Theorem 1.4 provided that \( n > \deg P - 2 \frac{P'(1)}{P(1)} \) and that all zeros of \( P \) are simple. The polynomial in (1) was originally introduced by R. Salem [6, Theorem IV, p. 166], [7, p. 30]. Therefore, this gives the following corollary.

**Corollary 1.5.** Let \( P \) be a polynomial with real coefficients such that \( P^* \neq P \). For \( n > \deg P - 2 \frac{P'(1)}{P(1)} \), let \( Q \) be defined as in (1). Suppose that \( P \) has a real zero greater than 1, that the remaining zeros of \( P \) are in \( \mathbb{D} \), and that all zeros of \( P \) are simple. Then each of \( Q, Q' \) and \( Q'' \) has exactly one zero outside \( \mathbb{D} \), counting multiplicities.

2. Proof of Theorem 1.1

**Lemma 2.1.** Let \( Q(z) = c \prod_{k=1}^{n} (z - \alpha_k) \) be a polynomial of degree \( n \geq 2 \), where \( c \neq 0 \). Suppose that \( \alpha_k \notin \mathbb{D} \) for \( 1 \leq k \leq m \), and that the remaining points \( \alpha_k \) are in \( \mathbb{D} \). If we have
\[
\sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} > \frac{1}{2},
\]
then there is a positive \( \delta \) such that for any \( r \in (1, 1 + \delta) \), we have
\[
\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > \frac{1}{2} \text{ on } |z| = r.
\]

Hence, we have \( \text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} \geq \frac{1}{2} \) whenever \( |z| = 1 \) and \( Q(z) \neq 0 \).

**Proof.** By an elementary calculation, we can show that if \( |z| > 1 \) and \( \alpha_k \neq 0 \), then \( \text{Re} \left\{ \frac{z}{z-\alpha_k} \right\} > \frac{1}{1 + |\alpha_k|} \) for \( m + 1 \leq k \leq n \), the two sides being equal if \( \alpha_k = 0 \). Also, if \( |z| = 1 \) then \( \text{Re} \left\{ \frac{z}{z-\alpha_k} \right\} \geq \frac{1}{1 - |\alpha_k|} \) for \( 1 \leq k \leq m \).

Define \( \varepsilon > 0 \) by
\[
\sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} = \frac{1}{2} + \varepsilon.
\]

Since \( \text{Re} \left\{ \frac{z}{z-\alpha_k} \right\} \) is a continuous function except at \( z = \alpha_k \) and since \( |\alpha_k| > 1 \) for \( 1 \leq k \leq m \), there exists a positive constant \( \delta \) with \( 1 + \delta < \min\{|\alpha_k| : 1 \leq k \leq m\} \) such that
\[
\sum_{k=1}^{m} \text{Re} \left\{ \frac{z}{z-\alpha_k} \right\} > \sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} - \frac{\varepsilon}{2}
\]
on \( |z| = r \), for all \( r \in (1, 1 + \delta) \). Therefore, if \( r \in (1, 1 + \delta) \) and \( |z| = r \), we have
\[
\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \sum_{k=1}^{n} \text{Re} \left\{ \frac{z}{z-\alpha_k} \right\} > \sum_{k=1}^{m} \frac{1}{1 - |\alpha_k|} - \frac{\varepsilon}{2} + \sum_{k=m+1}^{n} \frac{1}{1 + |\alpha_k|} = \frac{1}{2} + \varepsilon.
\]

This proves Lemma 2.1. \( \Box \)
Lemma 2.2. Let \( Q(z) = c \prod_{k=1}^{n}(z - \alpha_k) \) be a polynomial of degree \( n \geq 2 \), where \( c \neq 0 \). Suppose that \( \alpha_1 = \alpha, \alpha_2 = \alpha^{-1} \), where \( \alpha \) is real and \( |\alpha| > 1 \), and all the remaining points \( \alpha_k \), if any, are in \( \overline{D} \). Then there is a positive \( \delta \) such that for any \( r \in (1, 1 + \delta) \), we have
\[
\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > \frac{1}{2} \text{ on } |z| = r.
\]
Furthermore, we have \( \text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} \geq \frac{1}{2} \) whenever \( |z| = 1 \) and \( Q(z) \neq 0 \).

Proof. Note that \( \text{Re} \left\{ \frac{z}{z - \alpha} + \frac{z}{z - \alpha^{-1}} \right\} = 1 \) for all \( z \) with \( |z| = 1 \). Applying the same argument as in the proof of Lemma 2.1, we derive the required result. \( \square \)

Now we are ready to present a proof of Theorem 1.1.

Proof of Theorem 1.1. We are to show that \( zQ'(z) \) and \( Q(z) \) have the same number of zeros lying in \( \overline{D} \). By Lemma 2.1, there is \( \delta > 0 \) such that, for all \( r \in (1, 1 + \delta) \), we have \( \text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} > \frac{1}{2} \) on \( |z| = r \). So, for each fixed \( r \in (1, 1 + \delta) \), we have
\[
\left| \frac{zQ'(z)}{Q(z)} - 1 \right| < \left| \frac{zQ'(z)}{Q(z)} \right|,
\]
hence \( |zQ'(z) - Q(z)| < |zQ'(z)| \), on \( |z| = r \). Then, by Rouché’s theorem, \( zQ'(z) \) and \( Q(z) \) must have the same number of zeros lying in \( \{ z : |z| \leq r \} \) for all \( r \in (1, 1 + \delta) \). This proves the first part of the theorem.

Next suppose that all the zeros \( \alpha_k \) that are on the unit circle, if any, are simple. If \( Q' \) has a zero \( \gamma \) on the unit circle then \( \text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = 0 \), which contradicts the fact that \( \text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} \geq \frac{1}{2} \) when \( |z| = 1 \) outside the zeros of \( Q \). Hence \( Q' \) has no zeros on \( \partial\overline{D} \). The proof of Theorem 1.1 is now complete.

3. Preliminaries for Theorem 1.4

To prove Theorem 1.4, we need the following lemmas.

Lemma 3.1. If \( x > 1 \) and \( y \in [-1, 1) \) then
\[
\frac{1 + x^4 - 2x(1 + x^2)y + 2x^2(2y^2 - 1)}{(x^2 - 2xy + 1)^2} - \frac{y}{2(1 - y)} < 2.
\]
Proof. This can be proved by using only elementary calculus ([3], Lemma 5.10, p. 54). \( \square \)

Lemma 3.2. If \( Q \) is an anti-reciprocal polynomial of degree \( n \geq 2 \) with real coefficients, then
\[
\text{Re} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2} \text{ and } \text{Im} \left\{ \frac{z^2Q''(z)}{(n - 1)Q(z)} \right\} = \text{Im} \left\{ \frac{zQ'(z)}{Q(z)} \right\}
\]
whenever \( |z| = 1 \) and \( Q(z) \neq 0 \).
Proof. We give a proof that yields the entire statement of this lemma, but we note that the first equality in (2) has been proved in [8], (7.5), p. 229, for reciprocal polynomials \( Q \).

Now, since \( Q \) is anti-reciprocal, we have \( Q(z) = -z^n Q \left( \frac{1}{z} \right) \). Taking the derivative and multiplying both sides by \( z \), we get

\[
zQ'(z) = -nz^n Q \left( \frac{1}{z} \right) + z^{n-1} Q' \left( \frac{1}{z} \right) = nQ(z) + z^{n-1} Q' \left( \frac{1}{z} \right).
\]

So, we have

\[
z^{n-1} Q' \left( \frac{1}{z} \right) = zQ'(z) - nQ(z).
\]

After taking the derivative both sides of this equation, and then multiplying both sides by \( z \) and applying the identity (3), we obtain

\[
-z^{n-2} Q'' \left( \frac{1}{z} \right) = z^2 Q''(z) + 2(1-n)zQ'(z) + n(n-1)Q(z).
\]

Let \( z \in \partial \mathbb{D} \) with \( Q(z) \neq 0 \). Next dividing both sides of (4) by \( n(n-1)Q(z) \), we get

\[
-\frac{z^{n-2} Q'' \left( \frac{1}{z} \right)}{n(n-1)Q(z)} = \frac{z^2 Q''(z)}{n(n-1)Q(z)} - \frac{2zQ'(z)}{nQ(z)} + \frac{1}{n(n-1)Q(z)}.
\]

By replacing \( Q(z) \) on the left hand side of (5) by \( -z^n Q \left( \frac{1}{z} \right) \), the left hand side becomes

\[
\frac{z^{n-2} Q'' \left( \frac{1}{z} \right)}{n(n-1)z^n Q \left( \frac{1}{z} \right)} = \frac{z^2 Q'' \left( \frac{1}{z} \right)}{n(n-1)Q \left( \frac{1}{z} \right)} = \frac{z^2 Q''(z)}{n(n-1)Q(z)}.
\]

Here we have used the fact that since \(|z| = 1\) and \( Q \) has real coefficients, we have \( Q(1/z) = Q(\bar{z}) = \overline{Q(z)} \), and similarly for \( Q'' \) instead of \( Q \). Then from (5) we derive

\[
\frac{z^2 Q''(z)}{n(n-1)Q(z)} - \frac{z^2 Q''(z)}{n(n-1)Q(z)} = 1 - \frac{2zQ'(z)}{nQ(z)},
\]

which gives

\[
2i \text{Im} \left\{ \frac{z^2 Q''(z)}{n(n-1)Q(z)} \right\} = \frac{2zQ'(z)}{nQ(z)} - 1.
\]

This implies that \( \text{Re} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2} \) and \( \text{Im} \left\{ \frac{z^2 Q''(z)}{n(n-1)Q(z)} \right\} = \text{Im} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} \), as desired. \( \square \)

Lemma 3.3. Let \( Q(z) = \prod_{k=1}^{n} (z - \alpha_k) \) be an anti-reciprocal polynomial of degree \( n \geq 3 \). Suppose that \( \alpha_1 = \tau > 1 \), \( \alpha_2 = \tau^{-1} \), \( \alpha_3 = 1 \), and \( |\alpha_k| = 1 \) for \( k > 3 \). For \( |z| = 1 \) with \( Q(z) \neq 0 \), if \( \frac{z^2 Q''(z)}{Q(z)} \) is a real number then it is positive. In particular, then \( Q''(z) \neq 0 \).

Proof. The assumptions imply that \( Q \) has real coefficients. Let \( z \) be a point on the unit circle with \( Q(z) \neq 0 \). We have

\[
\frac{z^2 Q''(z)}{Q(z)} = z^2 \left( \left( \frac{Q'}{Q} \right)'(z) + \left( \left( \frac{Q'}{Q} \right)(z) \right)^2 \right) = \left( \frac{zQ'(z)}{Q(z)} \right)^2 - \sum_{k=1}^{n} \frac{z^2}{(z - \alpha_k)^2}.
\]
Suppose that \( \frac{2Q''(z)}{Q(z)} \) is a real number. Thus, by Lemma 3.2, \( \frac{2Q''(z)}{nQ(z)} \) is real as well, and so is also \( \sum_{k=1}^{n} \frac{z^2}{(z-\alpha_k)^2} \). Since \( \text{Re} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2} \) on \( |z| = 1 \) when \( Q(z) \neq 0 \), we have

\[
\frac{z^2Q''(z)}{Q(z)} = \frac{n^2}{4} - \sum_{k=1}^{n} \frac{z^2}{(z-\alpha_k)^2}.
\]

Next we want to find an upper bound for the real part of \( \sum_{k=1}^{n} \frac{z^2}{(z-\alpha_k)^2} \) on the unit circle. Let \( z = e^{i\theta} \), where \( \theta \in (0, 2\pi) \) (note that \( z \neq 1 \) since \( Q(1) = 0 \)). If \( \alpha \) is real, we have

\[
\text{Re} \left\{ \frac{z^2}{(z-\alpha)^2} \right\} = \frac{1 - 2\alpha \cos \theta + \alpha^2(2 \cos^2 \theta - 1)}{(1 + \alpha^2 - 2\alpha \cos \theta)^2}.
\]

For \( k \geq 3 \), by letting \( \alpha_k = e^{i\theta_k}, \theta_k \in [0, 2\pi) \), we have \( \text{Re} \left\{ \frac{z^2}{(z-\alpha_k)^2} \right\} = \frac{-\cos \beta_k}{2 - 2\cos \beta_k} \), where \( \beta_k = \theta - \theta_k \). Therefore,

\[
\text{Re} \left\{ \sum_{k=1}^{n} \frac{z^2}{(z-\alpha_k)^2} \right\} = \frac{1 + \tau^4 - 2\tau(1 + \tau^2)\cos \theta + 2\tau^2(2 \cos^2 \theta - 1)}{(1 + \tau^2 - 2\tau \cos \theta)^2} - \sum_{k=3}^{n} \frac{\cos \beta_k}{2 - 2\cos \beta_k}.
\]

Taking \( x = \tau \) and \( y = \cos \theta \) in Lemma 3.1, we see that

\[
\frac{1 + \tau^4 - 2\tau(1 + \tau^2)\cos \theta + 2\tau^2(2 \cos^2 \theta - 1)}{(1 + \tau^2 - 2\tau \cos \theta)^2} - \frac{\cos \theta}{2 - 2\cos \theta} < 2.
\]

It is easy to see that \( \frac{-\cos \omega}{2 - 2\cos \omega} \leq \frac{1}{4} \) for all \( \omega \in (0, 2\pi) \). So, we obtain

\[
\text{Re} \left\{ \sum_{k=1}^{n} \frac{z^2}{(z-\alpha_k)^2} \right\} < 2 + \frac{1}{4}(n-3) = \frac{n+5}{4}.
\]

Hence, from (6), we derive

\[
\frac{z^2Q''(z)}{Q(z)} = \frac{n^2}{4} - \sum_{k=1}^{n} \frac{z^2}{(z-\alpha_k)^2} > \frac{n^2}{4} - \frac{n+5}{4} > 0
\]

if \( n \geq 3 \), as desired. This proves Lemma 3.3.

4. PROOF OF THEOREM 1.4

Let the assumptions of Theorem 1.4 be satisfied. By Corollary 1.3 we know that \( Q' \) has only one zero outside \( \overline{D} \) and has no zeros on \( \partial \mathbb{D} \). Let \( G(z) = -z^{n-2}Q'' \left( \frac{1}{z} \right) \) and \( T(z) = z^{n-1}Q' \left( \frac{1}{z} \right) \). In order to prove that \( Q'' \) has exactly one zero outside \( \overline{D} \), it is equivalent to show that \( G \) has only one zero in \( \mathbb{D} \). Since \( Q' \) has only one zero outside \( \overline{D} \) and has no zeros on \( \partial \mathbb{D} \), \( T \) has exactly one zero in \( \mathbb{D} \) and has no zeros on \( \partial \mathbb{D} \). If we have

\[
|G(z) + 2(n-1)T(z)| < |G(z)| + 2(n-1)|T(z)|
\]

on \( \partial \mathbb{D} \), then, by a form of Rouché’s Theorem ([4], Theorem 3.6, p. 341), both \( G \) and \( T \) have the same number of zeros inside \( \mathbb{D} \). This will prove the theorem. From (3) and (4), we have

\[
G(z) + 2(n-1)T(z) = z^2Q''(z) - n(n-1)Q(z).
\]
Let \( z \in \partial \mathbb{D} \). It is easy to see that if \( Q(z) = 0 \), then (7) holds. Now, for \( Q(z) \neq 0 \), write \( \frac{z^2Q''(z)}{(n-1)Q(z)} = a + ib \), where \( a, b \in \mathbb{R} \). So \( G(z) + 2(n-1)T(z) = (a-n+ib)(n-1)Q(z) \). Since, by Lemma 3.2, \( \text{Im} \left\{ \frac{z^2Q''(z)}{(n-1)Q(z)} \right\} = \text{Im} \left\{ \frac{zQ'(z)}{Q(z)} \right\} \) and \( \text{Re} \left\{ \frac{zQ'(z)}{nQ(z)} \right\} = \frac{1}{2} \), we have \( zQ'(z) = (\frac{1}{2} + ib)Q(z) \). We also have \( |G(z)| = |z^2Q''(z)| = (n-1)|a+ib||Q(z)| \) and, by (3),

\[
2|T(z)| = 2|zQ'(z) - nQ(z)| = |n - 2ib||Q(z)|.
\]

Thus, the inequality (7) is equivalent to

\[
|a - n + ib| < |a + ib| + |n + 2ib|
\]

which is clearly true if \( b \neq 0 \). If \( b = 0 \), then by Lemma 3.3, we have \( a > 0 \) and so the inequality above is true. Therefore, the inequality (7) holds on \( \partial \mathbb{D} \), as desired. The proof of Theorem 1.4 is now complete.

5. DISTANCE BETWEEN THE ZEROS OF \( P \) AND \( P' \)

In this section we give a possible range of the distances between the zeros of \( P \) and \( P' \) that are outside a circular domain \( C \). We are to show that if \( P \) has only one zero \( \alpha \) outside \( C \), and \( \beta \) is a zero of \( P' \) lying outside \( C \), then \( \beta \) gets closer to \( \alpha \) when the degree of \( P \) increases.

**Lemma 5.1** (G. Pólya). Let \( f \) be a polynomial of degree \( n \) and let \( C \) denote a closed disk containing all the zeros of \( f \). For any point \( z \), we define

\[
\zeta_z = z - \frac{nf(z)}{f'(z)}.
\]

Then \( z \) and \( \zeta_z \) cannot both lie outside \( C \).

**Proof.** The proof can be found on pages 54–55, 231 in [5], see Problem 107 and Problem 111. \( \square \)

Another version of Lemma 5.1 can be found in [8], (5.89), p. 187. We now obtain the following result.

**Theorem 5.2.** Let \( P \) be a polynomial of degree \( n \geq 2 \) and let \( C = \{z \in \mathbb{C} : |z - w| \leq r \} \) be a closed disk. Suppose that \( P \) has only one zero, say \( \alpha \), of multiplicity \( m \), outside \( C \) and that \( \beta \) is a zero of \( P' \) lying outside \( C \) with \( \beta \neq \alpha \). We then have

\[
\frac{m}{n} (|w - \alpha| - r) \leq |\alpha - \beta| \leq \frac{m}{n} (|w - \alpha| + r).
\]

**Proof.** Let \( f(z) \equiv \frac{P(z)}{(z-\alpha)^m} \). So \( f \) is a polynomial of degree \( n - m \) and all zeros of \( f \) are in \( C \), and

\[
P'(z) = (z - \alpha)^m f'(z) + m(z - \alpha)^{m-1} f(z).
\]
Since \( P'(\beta) = 0 \) and \( \beta \neq \alpha \), from (10) we get \( \frac{f(\beta)}{f'(\beta)} = \frac{\alpha - \beta}{m} \).

Let \( \zeta_{\beta} := \beta - (n - m) \frac{f(\beta)}{f'(\beta)} \) be defined as in Lemma 5.1. That is, \( \zeta_{\beta} = \beta - (n - m) \frac{\alpha - \beta}{m} = \alpha - n \frac{\alpha - \beta}{m} \). By Lemma 5.1, \( \zeta_{\beta} \) must lie in \( C \) and hence we have

\[
|w - \alpha| - r \leq |\zeta_{\beta} - \alpha| \leq |w - \alpha| + r.
\]

Putting \( \zeta_{\beta} = \alpha - n \frac{\alpha - \beta}{m} \), we immediately obtain the inequalities (9), as desired. \( \square \)

In Theorem 5.2, if \( C \) is \( \overline{D} \), we get the following corollary:

**Corollary 5.3.** Let \( P \) be a polynomial of degree \( n \geq 2 \). Suppose that \( P \) has only one zero, say \( \alpha \), of multiplicity \( m \), outside \( \overline{D} \) and that \( \beta \) is a zero of \( P' \) lying outside \( \overline{D} \) with \( \beta \neq \alpha \). Then we have

\[
\frac{m}{n} \left( |\alpha| - 1 \right) \leq |\alpha - \beta| \leq \frac{m}{n} \left( |\alpha| + 1 \right).
\]

In Theorem 1.4, we have \( m = 1 \). If \( m = 1 \) and \( \alpha > 1 \), then we obtain

\[
1 < \left( 1 - \frac{1}{n} \right) \alpha + \frac{1}{n} < \beta < \left( 1 + \frac{1}{n} \right) \alpha + \frac{1}{n}
\]

for the only zero \( \beta \) of \( P' \) outside \( \overline{D} \) (in this case, \( \beta \) is necessarily real with \( 1 < \beta < \alpha \)).

**REFERENCES**


