Learning Objectives:

6.1 Introduction to Exotic Options: plan vanilla options, non-standard option characteristics, exotic options, gap, exchange, maxima and minima, chooser, forward start, Asian (arithmetic and geometric), barrier, lookback, compound, shout, path-dependency, path-independent, path-dependent, closed-form valuation formula, binomial model.

6.2 Exotic Options I: payment trigger, strike price, European gap call and put, obligation, negative payoff, decomposition, Greeks, deterministic cash, short risky asset, exchange option, European maximum and minimum contingent claim, decomposition, chooser option, choose, call or put, choosing decision, decomposition, forward start call and put option, strike price determined at intermediate date observed risky asset price, not observed, price.

6.3 Exotic Options II: Asian option, path-dependent payoff, arithmetic average, geometric average, average price, average strike, barrier option, hitting barrier, from above, down, from below, up, knock-in, knock-out, running maximum, running minimum, parity relation, lookback option, standard lookback, extrema lookback, shout option, additional right, locking minimum positive payoff, guaranteed payoff, in-the-money, optimal, shouting time, compound option, option on option.
6.1 Introduction to Exotic Options

6.1.1 The options considered in this course so far are called plain vanilla options. Options with non-standard option characteristics differed from plain vanilla options are called exotic options, which are actively traded OTC.

6.1.2 Gap, exchange, maxima and minima, chooser, forward start, Asian (arithmetic and geometric), barrier, lookback, compound, and shout options are the exotic options appeared in the SOA IFM examination syllabus.

6.1.3 Exotic options can be categorized by path-dependency. Path-independent exotic options are those whose payoffs at the exercise date depend only on the underlying risky asset price at the exercise date. Path-dependent exotic options are those whose payoffs at the exercise date depend on the underlying risky asset price at some intermediate date(s), between the issuance date and the exercise date. Gap, exchange, maxima and minima, and compound options are path-independent; chooser, forward start, Asian, barrier, lookback, and shout options are path-dependent.

6.1.4 Pricing and hedging exotic options are not easy tasks at all. The first batch of exotic options discussed below are those with simple closed-form valuation formula under the continuous-time Black-Scholes framework; the second batch of exotic options discussed below are those without simple closed-form valuation formula under the continuous-time Black-Scholes framework, and hence the binomial model must be resorted.

6.2 Exotic Options I: Gap, Exchange, Maxima and Minima, Chooser, and Forward Start

6.2.1 Recall that a long plain vanilla European call option, with strike price $K$ and expiration date $T$, has the payoff at the expiration date $T$:

\[ (S_T - K)_+ = (S_T - K) \mathbb{1}_{\{S_T > K\}}. \]

The $K$ appeared in the indicator function is in fact more precisely called the payment trigger; the other $K$ is really the strike price of the option, which is the price that the long position will have to pay at time $T$ if he/she chooses to exercise the option.

6.2.2 If these two $K$’s are different, say strike price $K_1$ and payment trigger $K_2$, with $K_1 \neq K_2$, this option is called a European gap call option:

\[
\text{Payoff of long gap call option at } T = (S_T - K_1) \mathbb{1}_{\{S_T > K_2\}}.
\]

Similarly, the long payoff of a European gap put option is given by:

\[
\text{Payoff of long gap put option at } T = (K_1 - S_T) \mathbb{1}_{\{S_T < K_2\}}.
\]
6.2.3 Though gap call/put options are called options, the long position of these options actually holds his/her obligation instead of right. Therefore, the payoffs of long gap call/put options could be positive or negative:

\[ C_{\text{gap}} (t, S; \delta, r, \sigma; K_1, K_2, T) = S e^{-\delta(T-t)} N (d_1) - K_1 e^{-r(T-t)} N (d_2) \]
\[ = F_{t,T}^P (S) N (d_1) - \text{PV}_{t,T} (K_1) N (d_2); \]

\[ P_{\text{gap}} (t, S; \delta, r, \sigma; K_1, K_2, T) = K_1 e^{-r(T-t)} N (-d_2) - S e^{-\delta(T-t)} N (-d_1) \]
\[ = \text{PV}_{t,T} (K_1) N (-d_2) - F_{t,T}^P (S) N (-d_1), \]

where \( d_1 \) and \( d_2 \) are given by

\[ d_1 = \frac{\ln \left( \frac{S}{K_2} \right) + \left( r - \delta + \frac{1}{2} \sigma^2 \right) (T-t)}{\sigma \sqrt{T-t}} = d_2 + \sigma \sqrt{T-t}, \]

\[ d_2 = \frac{\ln \left( \frac{S}{K_2} \right) + \left( r - \delta - \frac{1}{2} \sigma^2 \right) (T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}. \]

6.2.4 Thanks to the most general risk-neutral valuation formula in 4.2.8, together with similar arguments as in 4.1.13 (but under \( Q \), and thus using \( r \)),

6.2.5 A brutal\(^1\) way to memorize these formula is to remember that the payment trigger \( K_2 \) appears in the \( d_1 \) and \( d_2 \) terms, while the strike price \( K_1 \) appears outside the standard normal distribution function \( N (\cdot) \).

6.2.6 Another equivalent forms of the valuation formula are given by:

\[ C_{\text{gap}} (t, S; \delta, r, \sigma; K_1, K_2, T) = C (t, S; \delta, r, \sigma; K_2, T) + (K_2 - K_1) e^{-r(T-t)} N (d_2); \]

\[ P_{\text{gap}} (t, S; \delta, r, \sigma; K_1, K_2, T) = P (t, S; \delta, r, \sigma; K_2, T) + (K_1 - K_2) e^{-r(T-t)} N (-d_2). \]

\(^1\)Now, you should be able to understand why I always emphasized that we should learn proofs. More complicated formula are on their ways :(
Indeed, the payoff of long gap call option can be decomposed as:

\[(S_T - K_1) \mathbb{1}_{\{S_T > K_2\}} = (S_T - K_2)_+ + (K_2 - K_1) \mathbb{1}_{\{S_T > K_2\}},\]

and hence the most general risk-neutral valuation formula in 4.2.8, together with similar arguments as in 4.1.10 (but under \(Q\), and thus using \(r\)), can be applied. Similarly, the payoff of long gap put option can be decomposed as:

\[(K_1 - S_T) \mathbb{1}_{\{S_T < K_2\}} = (K_2 - S_T)_+ + (K_1 - K_2) \mathbb{1}_{\{S_T < K_2\}},\]

and hence, 4.2.8 and similar arguments as in 4.1.9 (but under \(Q\), and thus using \(r\)), can be applied.

**6.2.7** The equivalent forms in 6.2.6 are handy in calculating the Greeks of gap call/put options. For instance,

\[
\Delta_{C_{\text{gap}}} = \Delta_C + (K_2 - K_1) e^{-r(T-t)} \frac{\partial N(d_2)}{\partial S}
\]

\[= e^{-\delta(T-t)} N(d_1) + \left( K_2 - K_1 \right) \frac{e^{-r(T-t)}}{S\sigma\sqrt{T-t}} N'(d_2) ;\]

\[
\Delta_{P_{\text{gap}}} = \Delta_P + (K_1 - K_2) e^{-r(T-t)} \frac{\partial N(-d_2)}{\partial S}
\]

\[= -e^{-\delta(T-t)} N(-d_1) - \left( K_1 - K_2 \right) \frac{e^{-r(T-t)}}{S\sigma\sqrt{T-t}} N'(-d_2) .\]

**Example 1** [SOA Advanced Derivatives Sample Question Q18]: A market-maker sells 1,000 1-year European gap call options, and delta-hedges the position with shares. You are given:

(i) Each gap call option is written on 1 share of a non-dividend-paying stock.

(ii) The current price of the stock is 100.

(iii) The stock’s volatility is 100%.

(iv) Each gap call option has a strike price of 130.

(v) Each gap call option has a payment trigger of 100.

(vi) The risk-free interest rate is 0%.

Under the Black-Scholes framework, determine the initial number of shares in the delta-hedge.

**Solution:**
Exercise 1 [SOA Exam MFE Spring 2007 Question Q17]: Let $S_t$ denote the price at time $t$ of a stock that pays dividends continuously at a rate proportional to its price. Consider a European gap option with expiration date $T$, $T > 0$. If the stock price at time $T$ is greater than $100$, the payoff is $S_T - 90$; otherwise, the payoff is zero. You are given:

(i) $S_0 = $80.

(ii) The price of a European call option with expiration date $T$ and strike price $100$ is $4$.

(iii) The delta of the call option in (ii) is 0.2.

Calculate the price of the gap option.

6.2.8 Recall that the long position of a plain vanilla European call option, with strike price $K$ and expiration date $T$, pays deterministic cash $K$ and longs the underlying risky asset at $T$, if he/she chooses to exercise the option.

6.2.9 If, instead of deterministic cash, the long position shorts the other risky asset and longs the underlying risky asset at time $T$ when he/she chooses to exercise the option, then this option is called exchange option.

6.2.10 More precisely, let $S^1 = \{S^1_t\}_{t \geq 0}$ and $S^2 = \{S^2_t\}_{t \geq 0}$ be prices of the risky asset 1 and the risky asset 2. The payoff of a long $T$-year European exchange option, such that the long position holds the right to exchange $S^2$ (short) for $S^1$ (long), is given by:

\[
\text{Payoff of long exchange option at } T = (S^1_T - S^2_T)_+. 
\]

6.2.11 By symmetry, the difference between call and put exchange options is minimal, as long as the right of short $S^2$ and long $S^1$ is clearly defined. However, the difference between long and short positions of an exchange option still matters.

6.2.12 A simple closed-form valuation formula is available under the continuous-time Black-Scholes framework:

\[
V_{ex} \left( t, S^1, S^2; \delta_1, \delta_2, \sigma_1, \sigma_2, \rho; T \right) = S^1 e^{-\delta_1 (T-t)} N (d_1) - S^2 e^{-\delta_2 (T-t)} N (d_2) = F^P_{t,T} (S^1) N (d_1) - F^P_{t,T} (S^2) N (d_2),
\]

where $d_1$, $d_2$, and $\sigma$ are given by

\[
d_1 = \frac{\ln \left( \frac{S^1}{S^2} \right) + (\delta_2 - \delta_1 + \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}} = d_2 + \sigma \sqrt{T - t},
\]

\(^2\)In fact, we have to generalize the Black-Scholes model to include two risky assets and two Brownian motions, which induces a dependence structure between two risky asset prices. Hence, the proof of this formula is indeed beyond the scope of this course.
\[ d_2 = \frac{\ln \left( \frac{S_1}{S_T} \right) + (\delta_2 - \delta_1 - \frac{1}{2} \sigma^2) (T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}, \]

\[ \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}. \]

6.2.13 A brutal way to memorize this formula is to remember that the deterministic cash \( K \) is replaced by the other risky asset price \( S^2 \) while the risk-free interest rate \( r \) for \( K \) is replaced by the dividend yield \( \delta_2 \) for \( S^2 \).

6.2.14 The payoff of a long \( T \)-year European maximum contingent claim written on \( S^1 \) and \( S^2 \) is given by:

\[
\text{Payoff of long maximum contingent claim at } T = \max \{ S^1_T, S^2_T \}.
\]

6.2.15 Since this payoff of long maximum contingent claim can be decomposed as:

\[
\max \{ S^1_T, S^2_T \} = S^1_T + (S^2_T - S^1_T)_+ = S^2_T + (S^1_T - S^2_T)_+,
\]

together with 4.2.8 and 6.2.12,

\[
V_{\text{max}} \left( t, S^1, S^2; \delta_1, \delta_2, \sigma_1, \sigma_2, \rho; T \right) = S^1 e^{-\delta_1(T-t)} + V_{\text{ex}} \left( t, S^2, S^1; \delta_2, \delta_1, \sigma_2, \sigma_1, \rho; T \right)
\]
\[
= S^2 e^{-\delta_2(T-t)} + V_{\text{ex}} \left( t, S^1, S^2; \delta_1, \delta_2, \sigma_1, \sigma_2, \rho; T \right).
\]

6.2.16 The payoff of a long \( T \)-year European minimum contingent claim written on \( S^1 \) and \( S^2 \) is given by:

\[
\text{Payoff of long minimum contingent claim at } T = \min \{ S^1_T, S^2_T \}.
\]

6.2.17 Since this payoff of long minimum contingent claim can be decomposed as:

\[
\min \{ S^1_T, S^2_T \} = S^1_T - (S^1_T - S^2_T)_+ = S^2_T - (S^2_T - S^1_T)_+,
\]

together with 4.2.8 and 6.2.12,

\[
V_{\text{min}} \left( t, S^1, S^2; \delta_1, \delta_2, \sigma_1, \sigma_2, \rho; T \right) = S^1 e^{-\delta_1(T-t)} - V_{\text{ex}} \left( t, S^1, S^2; \delta_1, \delta_2, \sigma_1, \sigma_2, \rho; T \right)
\]
\[
= S^2 e^{-\delta_2(T-t)} - V_{\text{ex}} \left( t, S^2, S^1; \delta_2, \delta_1, \sigma_2, \sigma_1, \rho; T \right).
\]
Example 2 [SOA Advanced Derivatives Sample Question Q54]: Assume the Black-Scholes framework. Consider two non-dividend-paying stocks whose time-t prices are denoted by $S^1_t$ and $S^2_t$, respectively. You are given:

(i) $S^1_0 = 10$ and $S^2_0 = 20$.
(ii) Stock 1’s volatility is 0.18.
(iii) Stock 2’s volatility is 0.25.
(iv) The correlation between the continuously compounded returns of the two stocks is $-0.40$.
(v) The continuously compounded risk-free interest rate is 5%.
(vi) A one-year European option with payoff $\max\{\min\{2S^1_1, S^2_1\} - 17, 0\}$ has a current (time-0) price of 1.632.

Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now. Calculate the current (time-0) price of this option.

Solution:
Exercise 2 [SOA Exam MFE Spring 2007 Question Q6]: Consider a model with two stocks. Each stock pays dividends continuously at a rate proportional to its price. $S^j_t$ denotes the price of one share of stock $j$ at time $t$. Consider a claim maturing at time 3. The payoff of the claim is $\max\{S^1_3, S^2_3\}$. You are given:

(i) $S^1_0 = $100.
(ii) $S^2_0 = $200.
(iii) Stock 1 pays dividends of amount $0.05S^1_t dt$ between time $t$ and time $t + dt$.
(iv) Stock 2 pays dividends of amount $0.1S^2_t dt$ between time $t$ and time $t + dt$.
(v) The price of a European option to exchange Stock 2 for Stock 1 at time 3 is $10. Calculate the price of the claim.

Example 3 [SOA Advanced Derivatives Sample Question Q3]: An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-$t$ value of one unit of which is denoted by $S_t$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount $\pi$ is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, $T$, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \max\left\{\frac{S_T}{S_0}, (1 + g\%)^T\right\}.$$ 

You are given the following information:

(i) The contract will mature in one year.
(ii) The minimum guarantee rate of return, $g\%$, is 3%.
(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividend reinvested.
(iv) $S_0 = 100$.
(v) The price of a one-year European put option, with strike price of $103$, on the stock index is $15.21$.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.

Solution:
Exercise 3: The pound/dollar exchange rate is assumed to follow the Black-Scholes framework. You are given:

(i) The spot exchange rate is USD2/GBP.
(ii) The continuously compounded risk-free interest rate for pounds is 4%.
(iii) The continuously compounded risk-free interest rate for dollars is 3%.
(iv) The volatility of the currency exchange rate is 10%.

An agreement will pay you the maximum of USD100 and GBP50 at the end of one year. Calculate the value in dollars of this agreement to you.
6.2.18 A chooser option gives the right to the long position of the option to choose, at time $T_1$, whether the long option is a plain vanilla European call or put option with an expiration date $T_2 (\geq T_1)$ and a strike price $K$. Though the choosing decision happens at time $T_1$, the expiration date $T_2$ and the strike price $K$ of the call or put option are set at the issuance date 0 of the chooser option.

6.2.19 The payoff of the long chooser option at time $T_1$ (not $T_2$!) is given by:

\[
\text{Payoff of long chooser option at } T_1 = \max \left\{ C (T_1, S; \delta, r, \sigma; K, T_2), P (T_1, S; \delta, r, \sigma; K, T_2) \right\}.
\]

6.2.20 Using similar arguments as in 6.2.15, the payoff of the long chooser option at time $T_1$ can be decomposed as:

\[
\max \{ C (T_1, S; \delta, r, \sigma; K, T_2), P (T_1, S; \delta, r, \sigma; K, T_2) \}
= C (T_1, S; \delta, r, \sigma; K, T_2) + (P (T_1, S; \delta, r, \sigma; K, T_2) - C (T_1, S; \delta, r, \sigma; K, T_2))_+
= P (T_1, S; \delta, r, \sigma; K, T_2) + (C (T_1, S; \delta, r, \sigma; K, T_2) - P (T_1, S; \delta, r, \sigma; K, T_2))_+.
\]

Furthermore, by put-call parity, which holds from time $T_1$ to $T_2$,

\[
\max \{ C (T_1, S; \delta, r, \sigma; K, T_2), P (T_1, S; \delta, r, \sigma; K, T_2) \}
= C (T_1, S; \delta, r, \sigma; K, T_2) + e^{-\delta(T_2-T_1)} (K e^{-(r-\delta)(T_2-T_1)} - S_{T_1})_+
= P (T_1, S; \delta, r, \sigma; K, T_2) + e^{-\delta(T_2-T_1)} (S_{T_1} - K e^{-(r-\delta)(T_2-T_1)})_+.
\]

Finally, making use of 4.2.8 (again!), for any $t \in [0, T_1]$,

\[
V_{\text{chooser}} (t, S; \delta, r, \sigma; T_1, K, T_2)
= C (t, S; \delta, r, \sigma; K, T_2) + e^{-\delta(T_2-T_1)} P (t, S; \delta, r, \sigma; K e^{-(r-\delta)(T_2-T_1)}, T_1)
= P (t, S; \delta, r, \sigma; K, T_2) + e^{-\delta(T_2-T_1)} C (t, S; \delta, r, \sigma; K e^{-(r-\delta)(T_2-T_1)}, T_1).
\]
**Example 4** [SOA Advanced Derivatives Sample Question Q25]: Consider a chooser option (also known as an as-you-like-it option) on a non-dividend-paying stock. At time 1, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of $100. The chooser option price is $20 at time 0. The stock price is $95 at time $t = 0$. Let $C_T$ denote the price of a European call option at time $t = 0$ on the stock expiring at time $T$, $T > 0$, with a strike price of $100. You are given:

(i) The risk-free interest rate is 0.
(ii) $C_1 = $4.

Determine $C_3$.

**Solution:**

**Exercise 4**: For a non-dividend-paying stock, you are given:

(i) The stock’s price is 50.
(ii) The stock’s volatility is 10%.
(iii) The continuously compounded risk-free interest rate is 7%.

An option allows you to choose, at the end of 1 month, between at-the-money European call and put options expiring at the end of 3 months from now. Using the Black-Scholes framework, determine the value of this chooser option.

6.2.21 A $T_2$-expire **forward start option** intimates plain vanilla European options, except that the strike price is not deterministically fixed at the issuance date 0. Instead, the **strike price is determined at an intermediate date** $T_1$ $(\leq T_2)$, between the issuance date 0 and the expiration date $T_2$. 


6.2.22 The payoff of the **long forward start call option** at time $T_2$ (not $T_1$!) is given by:

\[
\text{Payoff of long forward start call option at } T_2 = (S_{T_2} - cS_{T_1})_+,\]

for some $c > 0$. Similarly, the payoff of the **long forward start put option** at time $T_2$ (not $T_1$!) is given by:

\[
\text{Payoff of long forward start put option at } T_2 = (cS_{T_1} - S_{T_2})_+.
\]

6.2.23 For any $t \in [T_1, T_2]$, the **valuation formula** are simply given by those for the plain vanilla European options. This is because, at any time $t \in [T_1, T_2]$, $S_{T_1}$ has been **observed**, and hence the strike price is actually deterministic at $t$:

\[
C_{\text{forward}}(t, S; \delta, r, \sigma; T_1, T_2) = C(t, S; \delta, r, \sigma; cS_{T_1}, T_2).
\]

\[
P_{\text{forward}}(t, S; \delta, r, \sigma; T_1, T_2) = P(t, S; \delta, r, \sigma; cS_{T_1}, T_2).
\]

6.2.24 For any $t \in [0, T_1)$, $S_{T_1}$ has **not** been **observed** yet. However, by 4.2.8 (!) and tower property of expectation, together with 6.2.23 ($t = T_1$),

\[
C_{\text{forward}}(t, S; \delta, r, \sigma; T_1, T_2) = e^{-r(T_2-t)}E^Q\left[(S_{T_2} - cS_{T_1})_+ | S \right] = e^{-r(T_2-t)}E^Q\left[ E^Q\left[ (S_{T_2} - cS_{T_1})_+ | S_{T_1} \right] | S \right] = e^{-r(T_2-t)}E^Q\left[ e^{r(T_2-T_1)}C(T_1, S_{T_1}; \delta, r, \sigma; cS_{T_1}, T_2) | S \right] = e^{-r(T_1-t)}E^Q\left[ S_{T_1} e^{-\delta(T_2-T_1)} N(d_1) - cS_{T_1} e^{-r(T_2-T_1)} N(d_2) | S \right] = e^{-r(T_1-t)}E^Q\left[ S_{T_1} | S \right] \left( e^{-\delta(T_2-T_1)} N(d_1) - e^{-r(T_2-T_1)} N(d_2) \right),
\]

where $d_1$ and $d_2$ are independent of $S_{T_1}$:

\[
d_1 = \frac{-\ln c + (r - \delta + \frac{1}{2} \sigma^2) (T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} = d_2 + \sigma \sqrt{T_2 - T_1},
\]

\[
d_2 = \frac{-\ln c + (r - \delta - \frac{1}{2} \sigma^2) (T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} = d_1 - \sigma \sqrt{T_2 - T_1}.
\]

Therefore, the **valuation formula** of the long forward start call option is substantially simplified: for any $t \in [0, T_1)$,

\[
C_{\text{forward}}(t, S; \delta, r, \sigma; T_1, T_2) = e^{-\delta(T_1-t)} S \left( e^{-\delta(T_2-T_1)} N(d_1) - c e^{-r(T_2-T_1)} N(d_2) \right).
\]

In particular, at time 0, the **price** of the long forward start call option is given by:

\[
C_{\text{forward}}(0, S; \delta, r, \sigma; T_1, T_2) = e^{-\delta T_1} C(0, S; \delta, r, \sigma; cS, T_2 - T_1).
\]
6.2.25 Similarly, the valuation formula of the long forward start put option is given by: for any $t \in [0, T_1)$,

$$P_{\text{forward}}(t, S; \delta, r, \sigma; T_1, T_2) = e^{-\delta(T_1-t)} S \left( e^{r(T_2-T_1)} N (-d_2) - e^{-\delta(T_2-T_1)} N (-d_1) \right).$$

In particular, at time 0, the price of the long forward start put option is given by:

$$P_{\text{forward}}(0, S; \delta, r, \sigma; T_1, T_2) = e^{-\delta T_1} P(0, S; \delta, r, \sigma; cS, T_2 - T_1).$$

**Example 5** [SOA Advanced Derivatives Sample Question Q19]: Consider a forward start option which, 1 year from today, will give its owner a 1-year European call option with a strike price equal to the stock price at that time. You are given:

(i) The European call option is on a stock that pays no dividends.

(ii) The stock’s volatility is 30%.

(iii) The forward price for delivery of 1 share of the stock 1 year from today is 100.

(iv) The continuously compounded risk-free interest rate is 8%.

Under the Black-Scholes framework, determine the price today of the forward start option.

**Solution:**

**Exercise 5** [SOA Advanced Derivatives Sample Question Q33]: You own one share of a non-dividend-paying stock. Because you worry that its price may drop over the next year, you decide to employ a rolling insurance strategy, which entails obtaining one 3-month European put option on the stock every three months, with the first one being bought immediately. You are given:

(i) The continuously compounded risk-free interest rate is 8%.

(ii) The stock’s volatility is 30%.

(iii) The current stock price is 45.

(iv) The strike price for each option is 90% of the then-current stock price.

Your broker will sell you the four options but will charge you for their total cost now. Under the Black-Scholes framework, how much do you now pay your broker?
6.3 Exotic Options II: Asian (Arithmetic and Geometric), Barrier, Lookback, Shout, and Compound

6.3.1 Recall that, for a long position of a plain vanilla European option, with strike price $K$ and expiration date $T$, the payoff is given by the positive part difference between the underlying risky asset price $S_T$ at expiration $T$ and the strike price $K$:

$$(S_T - K)_+ \text{ for call; } (K - S_T)_+ \text{ for put.}$$

6.3.2 An Asian option has a path-dependent payoff, which considers an arithmetic average $A_T$ or a geometric average $G_T$ of the underlying risky asset price $S_t$, $t \in [0, T]$, either as the role of $S_T$ or $K$:

- Payoff of arithmetic average price $K$-strike call = $(A_T - K)_+$;
- Payoff of geometric average price $K$-strike call = $(G_T - K)_+$;
- Payoff of arithmetic average strike call = $(S_T - A_T)_+$;
- Payoff of geometric average strike call = $(S_T - G_T)_+$;
- Payoff of arithmetic average price $K$-strike put = $(K - A_T)_+$;
- Payoff of geometric average price $K$-strike put = $(K - G_T)_+$;
- Payoff of arithmetic average strike put = $(S_T - A_T)_+$;
- Payoff of geometric average strike put = $(G_T - S_T)_+$,

where the arithmetic average $A_T$ and the geometric average $G_T$ are respectively given by

$$A_T = \frac{1}{T+1} \sum_{t=0}^{T} S_t, \quad G_T = \left( \prod_{t=0}^{T} S_t \right)^{\frac{1}{T+1}},$$

in a discrete-time setting (binomial model), and

$$A_T = \frac{1}{T} \int_0^T S_t dt, \quad G_T = e^{\int_0^T \ln(S_t) dt},$$

in a continuous-time setting (Black-Scholes framework).

6.3.3 The trick to write down a correct payoff for an Asian option is to (i) identify call or put, and write down plain vanilla European option payoff, (ii) identify arithmetic or geometric average, (iii) identify average price or average strike; for average price, replace $S_T$ by $A_T/G_T$; for average strike, replace $K$ by $A_T/G_T$, and finally (iv) identify time setting (discrete or continuous).
Example 6 [CAS Exam 3 Fall 2007 Question Q27]: At the beginning of the year, a speculator purchases a six-month geometric average price call option on a company’s stock. The strike price is 3.5. The payoff is based on an evaluation of the stock price at each month’s end.

<table>
<thead>
<tr>
<th>Date</th>
<th>Stock Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 31</td>
<td>1.27</td>
</tr>
<tr>
<td>February 28</td>
<td>4.11</td>
</tr>
<tr>
<td>March 31</td>
<td>5.10</td>
</tr>
<tr>
<td>April 30</td>
<td>5.50</td>
</tr>
<tr>
<td>May 31</td>
<td>5.13</td>
</tr>
<tr>
<td>June 30</td>
<td>4.70</td>
</tr>
</tbody>
</table>

Based on the above stock prices, calculate the payoff of the option.

Solution:

6.3.4 For Asian options, in the SOA IFM examination, almost surely, only pricing in the binomial model will be examined, i.e. in a discrete-time setting.

Example 7: For a two-period binomial model, you are given:

(i) The duration of the period is 6 months.
(ii) The current price for a continuous-dividend-paying stock is 20, with dividend yield 3%.
(iii) \(u = 1.3\).
(iv) \(d = 0.8\).
(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of a 1-year at-the-money arithmetic average price Asian call option.

Solution:
6.3.5 A barrier option has a path-dependent payoff, which depends on whether the underlying risky asset price $S_t$, $t \in [0,T]$, hits a specified level $B$, called the barrier.

6.3.6 The barrier option could be call or put; the barrier could be hit from above (necessarily, $S_0 > B$, also called down) or from below (necessarily, $S_0 < B$, also called up); the option could be knock-in or knock-out right after the barrier being hit.

6.3.7 The trick to write down an explicit payoff for a barrier option is to make use of running maximum and minimum of underlying risky asset price from time 0 to time $T$:

$$M_T = \max_{t \in [0,T]} S_t, \quad m_T = \min_{t \in [0,T]} S_t.$$ 

6.3.8 Payoffs of barrier options with different characteristics are given by:

| Payoff of up-and-in European call | $$(S_T - K)_+ \mathbf{1}_{\{M_T > B\}}$$; |
| Payoff of up-and-out European call | $$(S_T - K)_+ \mathbf{1}_{\{M_T < B\}}$$; |
| Payoff of down-and-in European call | $$(S_T - K)_+ \mathbf{1}_{\{m_T < B\}}$$; |
| Payoff of down-and-out European call | $$(S_T - K)_+ \mathbf{1}_{\{m_T > B\}}$$; |
| Payoff of up-and-in European put | $$(K - S_T)_+ \mathbf{1}_{\{M_T > B\}}$$; |
| Payoff of up-and-out European put | $$(K - S_T)_+ \mathbf{1}_{\{M_T < B\}}$$; |
| Payoff of down-and-in European put | $$(K - S_T)_+ \mathbf{1}_{\{m_T < B\}}$$; |
| Payoff of down-and-out European put | $$(K - S_T)_+ \mathbf{1}_{\{m_T > B\}}.$$ |

6.3.9 Due to the fact that, for two identical up/down European call/put barrier options, their knock-in and knock-out counterparts payoffs sum up to the plain vanilla European option payoff, for example:

$$(S_T - K)_+ \mathbf{1}_{\{M_T > B\}} + (S_T - K)_+ \mathbf{1}_{\{M_T < B\}} = (S_T - K)_+.$$
the following parity relation hold for barrier options:

\[ V(\text{up/down, in, call/put}) + V(\text{up/down, out, call/put}) = V(\text{call/put}) \].

**Example 8 [SOA Advanced Derivatives Sample Question Q42]**: Prices for 6-month 60-strike European up-and-out call options on a stock \( S \) are available. Below is a table of option prices with respect to various \( B \), the level of the barrier. Here, \( S_0 = 50 \).

<table>
<thead>
<tr>
<th>( B )</th>
<th>Price of up-and-out call</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>0.1294</td>
</tr>
<tr>
<td>80</td>
<td>0.7583</td>
</tr>
<tr>
<td>90</td>
<td>1.6616</td>
</tr>
<tr>
<td>( \infty )</td>
<td>4.0861</td>
</tr>
</tbody>
</table>

Consider a special 6-month 60-strike European “knock-in, partial knock-out” call option that knocks in at \( B_1 = 70 \), and “partially” knocks out at \( B_2 = 80 \). The strike price of the option is 60. The following table summarizes the payoff at the exercise date:

<table>
<thead>
<tr>
<th>( B_1 ) Not Hit</th>
<th>( B_1 ) Hit and ( B_2 ) Not Hit</th>
<th>( B_1 ) Hit and ( B_2 ) Hit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 2 \times \max{S_{0.5} - 60, 0} )</td>
<td>( \max{S_{0.5} - 60, 0} )</td>
</tr>
</tbody>
</table>

**Solution:**
6.3.10 For barrier options, in the SOA IFM examination, again, almost surely, only pricing in the binomial model will be examined, i.e. in a discrete-time setting.

Example 9: For a two-period binomial model, you are given:

(i) The duration of the period is 6 months.
(ii) The current price for a continuous-dividend-paying stock is 20, with dividend yield 3%.
(iii) \( u = 1.3 \).
(iv) \( d = 0.8 \).
(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of

(i) a 1-year at-the-money down-and-out European call option with \( B = 18 \);
(ii) a 1-year at-the-money down-and-in European call option with \( B = 18 \);
(iii) a 1-year at-the-money European call option.

Solution:
6.3.11 A lookback option has a path-dependent payoff, which considers the running maximum $M_T$ or minimum $m_T$ of underlying risky asset price from time 0 to time $T$, either as the role of $S_T$ or $K$:

- Payoff of **standard lookback** call = $S_T - m_T$;
- Payoff of **standard lookback** put = $M_T - S_T$;
- Payoff of **extrema lookback** call = $(M_T - K)_+$;
- Payoff of **extrema lookback** put = $(K - m_T)_+$.

**Example 10** [SOA Finance and Investment Sample Question Q40]: Several lookback options are written on the same underlying index. They all expire in 3 years. Let $S_t$ denote the value at time $t$ of the index on which the option is written. The initial index price, $S_0$, is 150. The index price when the option expires, $S_3$, is 200. The maximum index price over the 3-year period is 210. The minimum index price over the 3-year period is 120. Calculate the sum of the payoffs for the following three lookback options:
- Standard lookback call
- Extrema lookback call with a strike price of 100
- Extrema lookback put with a strike price of 100

**Solution:**

6.3.12 For lookback options, in the SOA IFM examination, again, almost surely, only pricing in the binomial model will be examined, i.e. in a discrete-time setting.

**Example 11**: For a two-period binomial model, you are given:

(i) The duration of the period is 6 months.
(ii) The current price for a continuous-dividend-paying stock is 20, with dividend yield 3%.
(iii) $u = 1.3$.
(iv) $d = 0.8$.
(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of a 1-year European standard lookback put option.

**Solution:**
6.3.13 A **shout option** has a path-dependent payoff, which gives an **additional right** to the long position of the option in **locking**, at a **shouting time** $\tau$ (not exercise time!), a **minimum positive payoff**, at the expiration date:

| Payoff of shout call at $T$ = | $\max \{S_T - K, S_\tau - K, 0\}$ if $\tau \leq T$ | $\max \{S_T - K, 0\}$ if $\tau > T$ |
|-----------------------------|-------------------------------------------------|---------------------------------
| Payoff of shout put at $T$ =| $\max \{K - S_T, K - S_\tau, 0\}$ if $\tau \leq T$ | $\max \{K - S_T, 0\}$ if $\tau > T$ |

6.3.14 Since shouting aims at placing a **positive guaranteed payoff** at the expiration date, if there exists a time $\tau$ such that the option is **in-the-money**, then such a time $\tau$ is one of the possible choices to shout the option. The long position has to determine an **optimal shouting time** among these choices.

6.3.15 For shout options, in the SOA IFM examination, again, almost surely, only pricing in the binomial model will be examined, i.e. in a discrete-time setting.

*Example 12:* For a two-period binomial model, you are given:

- (i) The duration of the period is 6 months.
- (ii) The current price for a continuous-dividend-paying stock is 20, with dividend yield 3%.
- (iii) $u = 1.3$.
- (iv) $d = 0.8$.
- (v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of a 1-year 20.5-strike shout put option.

*Solution:*
The final exotic option without a simple closed-form valuation formula under the continuous time Black-Scholes framework is a compound option: option on option. For compound options, in the SOA IFM examination, again, almost surely, only pricing in the binomial model will be examined, i.e. in a discrete-time setting.

Example 13 [Modified from CAS Exam 3 Fall 2007 Question Q28]: You are given the following information on a compound CallOnPut option:

- The continuously compounded risk-free rate is 5%.
- The strike price of the underlying option is 43.
- The strike price of the compound option is 3.
- The compound option expires in 6 months.
- The underlying option expires 6 months after the compound option.
- The underlying option is American.
- The underlying option is written on the underlying non-dividend-paying stock, with 
  \[ u = 1.27, \quad d = 0.83, \quad S_0 = 40. \]

Calculate the value of the compound option.

Solution: