Learning Objectives:
5.1 Option Greeks: Black-Scholes valuation formula at time $t$, observed values, model inputs, option characteristics, option Greeks, partial derivatives, sensitivity, change infinitesimally, Delta, Gamma, theta, Vega, rho, Psi, option Greek of portfolio, absolute change, option elasticity, percentage change, option elasticity of portfolio.
5.2 Applications of Option Greeks: risk management via hedging, immunization, adverse market changes, delta-neutrality, delta-hedging, local, re-balance, from-time-to-time, holding profit, delta-hedging, first order, prudent, delta-gamma-hedging, gamma-neutrality, option value approximation, observed values change inevitably, Taylor series expansion, second order approximation, delta-gamma-theta approximation, delta approximation, delta-gamma approximation.

Further Exercises:
SOA Advanced Derivatives Samples Question 9.
5.1 Option Greeks

5.1.1 Consider a continuous-time setting $t \geq 0$. Assume that the Black-Scholes framework holds as in 4.2.2–4.2.4.

5.1.2 The time-0 Black-Scholes European call and put option valuation formula in 4.2.9 and 4.2.10 can be generalized to those at time $t$:

\[
C(t, S; \delta, r, \sigma; K, T) = S e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)
= F_{t,T}^P N(d_1) - PV_{t,T}(K) N(d_2);
\]

\[
P(t, S; \delta, r, \sigma; K, T) = K e^{-r(T-t)} N(-d_2) - S e^{-\delta(T-t)} N(-d_1)
= PV_{t,T}(K) N(-d_2) - F_{t,T}^P N(-d_1);
\]

where $d_1$ and $d_2$ are given by

\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r - \delta + \frac{1}{2} \sigma^2) (T-t)}{\sigma \sqrt{T-t}},
\]

\[
d_2 = \frac{\ln \left( \frac{S}{K} \right) + (r-\delta - \frac{1}{2} \sigma^2) (T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}.
\]

5.1.3 The option value $V$ is a function of observed values, $t$ and $S$, model inputs, $\delta$, $r$, and $\sigma$, as well as option characteristics, $K$ and $T$.

5.1.4 Option Greeks are partial derivatives of the option value $V$ with respect to each observed parameter or model input, $t$, $S$, $\delta$, $r$, $\sigma$, holding other observed values and model inputs constant. Essentially, option Greeks analyze the sensitivity of the option value $V$ when one of the observed values or model inputs is changed infinitesimally, ceteris paribus.

5.1.5 The most common option Greeks are Delta ($\Delta$), Gamma ($\Gamma$), theta ($\theta$), Vega, rho ($\rho$), and Psi ($\Psi$):

\[
\Delta_V = \frac{\partial V}{\partial S}; \quad \Gamma_V = \frac{\partial^2 V}{\partial S^2}; \quad \theta_V = \frac{\partial V}{\partial t}; \quad \text{Vega}_V = \frac{\partial V}{\partial \sigma}; \quad \rho_V = \frac{\partial V}{\partial r}; \quad \Psi_V = \frac{\partial V}{\partial \delta}.
\]

Option Greeks are also functions of the observed values, $t$ and $S$, the model inputs, $\delta$, $r$, and $\sigma$, as well as the option characteristics, $K$ and $T$.

5.1.6 For European call and put options, these six option Greeks are given in the SOA IFM Formula Sheet:

\[
\Delta_C = e^{-\delta(T-t)} N(d_1), \quad \Delta_P = -e^{-\delta(T-t)} N(-d_1);
\]

\[
\Gamma_C = \Gamma_P = \frac{e^{-\delta(T-t)} N'(d_1)}{S \sigma \sqrt{T-t}}.
\]
\[ \theta_C = \delta Se^{-\delta(T-t)} N(d_1) - r Ke^{-r(T-t)} N(d_2) - \frac{Ke^{-r(T-t)} N'(d_2) \sigma}{2 \sqrt{T-t}}, \quad \theta_P = \theta_C + r Ke^{-r(T-t)} - \delta Se^{-\delta(T-t)}; \]

\[ \text{Vega}_C = \text{Vega}_P = Se^{-\delta(T-t)} N'(d_1) \sqrt{T-t}; \]

\[ \rho_C = (T-t) Ke^{-r(T-t)} N(d_2), \quad \rho_P = -(T-t) Ke^{-r(T-t)} N(-d_2); \]

\[ \Psi_C = -(T-t) Se^{-\delta(T-t)} N(d_1), \quad \Psi_P = (T-t) Se^{-\delta(T-t)} N(-d_1). \]

5.1.7 The option Greeks for European call and put options are related by the put-call parity:

\[ C(t, S, \delta; K, r, \sigma, T) - P(t, S, \delta; K, r, \sigma, T) = Se^{-\delta(T-t)} - Ke^{-r(T-t)}. \]

**Example 1** [SOA Advanced Derivatives Sample Question Q8]: You are considering the purchase of a 3-month 41.5-strike American call option on a non-dividend-paying stock. You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 40.

(iii) The stock’s volatility is 30%.

(iv) The current call option delta is 0.5.

Determine the current price of the option.

**Solution:**

**Exercise 1:** For a 6-month European put option on a stock, you are given:

(i) The stock’s price is 50.

(ii) The stock pays continuous dividends proportional to its price, with the dividend yield being 5%.

(iii) The continuously compounded risk-free interest rate is 7%.

(iv) Using the Black-Scholes formula, \( \Delta_P = -0.5 \).

Determine \( \Delta_C \) computed using Black-Scholes formula for an otherwise European call option.

3
5.1.8 An option Greek of a portfolio $V$ of options, with $n_1$ units of $V_1$, $n_2$ units of $V_2$, $\ldots$, $n_N$ units of $V_N$, equals to the sum of the option Greek of all options in the portfolio:

\[
\text{If } V = \sum_{i=1}^{N} n_i V_i, \text{ then } \text{Greek}_V = \sum_{i=1}^{N} n_i \text{Greek}_V.
\]

**Example 2** [SOA Advanced Derivatives Sample Question Q31]: You compute the current delta for a 50-60 bull spread with the following information:

(i) The continuously compounded risk-free rate is 5%.
(ii) The underlying stock pays no dividends.
(iii) The current stock price is $50 per share.
(iv) The stock’s volatility is 20%.
(v) The time to expiration is 3 months.

How much does delta change after 1 month, if the stock price does not change?

**Solution:**

**Exercise 2:** You are considering the purchase of a 6-month 40-70 bear spread. You are given:

(i) The continuously compounded risk-free rate is 10%.
(ii) The stock pays continuous dividends proportional to its price, with the dividend yield being 5%.
(iii) The current stock price is $50 per share.
(iv) The stock’s volatility is 30%.

Compute the delta of the bear spread.
5.1.9 While the option Delta analyzes the absolute change in option value $V$ versus the absolute change in risky asset price $S$, the option elasticity ($\Omega$) studies the percentage change in $V$ versus the percentage change in $S$:

$$\Omega_V = \frac{\partial V}{\partial S} = \frac{S \Delta V}{V}.$$

5.1.10 An option elasticity of a portfolio $V$ of options, with $n_1$ units of $V_1$, $n_2$ units of $V_2$, ..., $n_N$ units of $V_N$, is given by:

$$\text{If } V = \sum_{i=1}^{N} n_i V_i, \text{ then } \Omega_V = \frac{S \Delta V}{V} = \frac{S}{\sum_{i=1}^{N} n_i V_i} \sum_{i=1}^{N} n_i \Delta V_i = \sum_{i=1}^{N} w_i \Omega_{V_i},$$

where $w_i = \frac{n_i V_i}{\sum_{j=1}^{N} n_j V_j}$.

Example 3 [SOA Advanced Derivatives Sample Question Q41]: Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock. You are given:

(i) The time-0 stock price is 45.
(ii) The stock’s volatility is 25%.
(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.
(iv) The continuously compounded risk-free interest rate is 7%.
(v) The time-1 payoff of the contingent claim is as follows:

Calculate the time-0 contingent claim elasticity.

Solution:
Exercise 3 [CAS Exam 3 Fall 2007 Question Q22]: A call option is modeled using the Black-Scholes formula with the following parameters: $S = 25$, $K = 24$, $r = 4\%$, $\delta = 0\%$, $\sigma = 20\%$, $T = 1$. Calculate the call option elasticity, $\Omega$.

Example 4 [SOA Advanced Derivatives Sample Question Q20]: Assume the Black-Scholes framework. Consider a stock, and a European call option and a European put option on the stock. The current stock price, call price, and put price are 45.00, 4.45, and 1.90, respectively. Investor A purchases two calls and one put. Investor B purchases two calls and writes three puts. The current elasticity of Investor A’s portfolio is 5.0. The current delta of Investor B’s portfolio is 3.4. Calculate the current put-option elasticity.

Solution:

Exercise 4: You are given the following information for European call options on a stock:

<table>
<thead>
<tr>
<th>Strike price</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option premium</td>
<td>12</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Option delta</td>
<td>0.8</td>
<td>0.4</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The price of the underlying stock is 60. Determine the elasticity of a 50-60-70 call butterfly spread.
5.2 Applications of Option Greeks

Risk management via hedging

5.2.1 The reason why a market-maker trades options is to make profit through the bid-ask spread, rather than speculate on the market. Therefore, the market-maker should make sure that the value of her portfolio is immunized against any adverse market changes.

5.2.2 Suppose that the market-maker has just purchased an option at time $t$, with its value $V(t, S)$ and delta $\Delta V(t, S)$. If $\Delta V(t, S) > 0$ while the risky asset price $S$ decreases, or if $\Delta V(t, S) < 0$ while the risky asset price $S$ increases, then the option value $V(t, S)$ will be decreased.

5.2.3 On the other hand, suppose that the market-maker has just sold an option at time $t$, with its value $-V(t, S)$ and delta $-\Delta V(t, S)$. If $\Delta V(t, S) > 0$ while the risky asset price $S$ increases, or if $\Delta V(t, S) < 0$ while the risky asset price $S$ decreases, then the option value $-V(t, S)$ will be decreased.

5.2.4 In either cases, the market-maker should combine the option, with value $V$ for long, or $-V$ for short, and $x$ units of underlying risky asset, with value $S$ per unit, in order to achieve delta-neutrality:

$$\Delta \tilde{V} = 0.$$  

Such a procedure of combining the option position with certain units of underlying risky asset is called delta-hedging. For the long position of the option,

$$\Delta_{V+xS} = 0 \Rightarrow \Delta V + x \times 1 = 0 \Rightarrow x = -\Delta V = -\Delta V(t, S).$$

For the short position of the option,

$$\Delta_{-V+xS} = 0 \Rightarrow -\Delta V + x \times 1 = 0 \Rightarrow x = \Delta V = \Delta V(t, S).$$

5.2.5 Since the delta-hedging units $x$ depend on the current time $t$ and the current risky asset price $S$ per unit, such a delta-hedging strategy changes as time elapses and the underlying risky asset price changes, and hence is only local. Therefore, the market-maker has to re-balance her portfolio from-time-to-time.
Example 5 [CAS Exam 8 Spring 2005 Question Q36]: Assume you have purchased European put options for 100,000 shares of a non-dividend-paying stock and you are given the following information:

(i) Price of stock = $49.16.
(ii) Strike price = $50.00.
(iii) Continuously compounded risk-free interest rate = 5% per annum.
(iv) Volatility = 20% per annum.
(v) There are 20 weeks remaining until maturity.
(a) Determine the initial position you should take in the underlying stock to implement a delta hedging strategy.
(b) You now have the following information.

<table>
<thead>
<tr>
<th>$T$ (weeks)</th>
<th>Stock Price</th>
<th>$d_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$49.33</td>
<td>0.10</td>
</tr>
<tr>
<td>2</td>
<td>$49.09</td>
<td>0.05</td>
</tr>
</tbody>
</table>

You decide to re-adjust the delta hedging strategy on a weekly basis. Calculate the cumulative cost, including interest, of the hedge at the end of week 2.

Solution:

Exercise 5: You are given the following information for a delta-hedged portfolio for a European call option that you have written:

<table>
<thead>
<tr>
<th></th>
<th>Stock price</th>
<th>Call premium</th>
<th>Call delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 0</td>
<td>55</td>
<td>6.50</td>
<td>0.4</td>
</tr>
<tr>
<td>Day 1</td>
<td>60</td>
<td>9.50</td>
<td>0.6</td>
</tr>
</tbody>
</table>

The continuously compounded risk-free interest rate is 5%. Calculate the number of shares to buy or sell on day 1 to maintain the delta hedge.
5.2.6 In between the time of re-balancing her delta-hedging strategy, say from time \( t \) to time \( t + h \), the market-maker makes a profit/loss from her portfolio. Consider the case of long position for the option by the market-maker. At time \( t \),

\[
\tilde{V}(t, S_t) = V(t, S_t) - \Delta V(t, S_t) S_t;
\]

At time \( t + h \), before re-balancing her portfolio,

\[
\tilde{V}(t + h, S_{t+h}) = V(t + h, S_{t+h}) - \Delta V(t, S_t) S_{t+h}.
\]

Therefore, the holding profit of her portfolio, with the long position for the option, is given by

\[
\text{Holding profit} = \tilde{V}(t + h, S_{t+h}) - \text{FV}_{tt+h} \left( \tilde{V}(t, S_t) \right)
\]

\[
= V(t + h, S_{t+h}) - \Delta V(t, S_t) S_{t+h} - (V(t, S_t) - \Delta V(t, S_t) S_t) e^{rh}.
\]

5.2.7 Similarly, consider the case of short position for the option by the market-maker.

\[
\tilde{V}(t, S_t) = \Delta V(t, S_t) S_t - V(t, S_t);
\]

\[
\tilde{V}(t + h, S_{t+h}) = \Delta V(t, S_t) S_{t+h} - V(t + h, S_{t+h}).
\]

Therefore, the holding profit of her portfolio, with the short position for the option, is given by

\[
\text{Holding profit} = \tilde{V}(t + h, S_{t+h}) - \text{FV}_{tt+h} \left( \tilde{V}(t, S_t) \right)
\]

\[
= \Delta V(t, S_t) S_{t+h} - V(t + h, S_{t+h}) - (\Delta V(t, S_t) S_t - V(t, S_t)) e^{rh}.
\]

Example 6 [SOA Advanced Derivatives Sample Question Q47]: Several months ago, an investor sold 100 units of a one-year European call option on a non-dividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions. You are given the following information:

(i) The risk-free interest rate is constant.

(ii) The put option in the table belows is a European option on the same stock and with the same strike price and expiration date as the call option.

<table>
<thead>
<tr>
<th></th>
<th>Several months ago</th>
<th>Now</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>$40.00</td>
<td>$50.00</td>
</tr>
<tr>
<td>Call option price</td>
<td>$8.88</td>
<td>$14.42</td>
</tr>
<tr>
<td>Put option price</td>
<td>$1.63</td>
<td>$0.26</td>
</tr>
<tr>
<td>Call option delta</td>
<td>0.794</td>
<td></td>
</tr>
</tbody>
</table>

Calculate her profit.
Solution:

Exercise 6 [SOA Exam MFE Spring 2009 Question Q13]: Assume the Black-Scholes framework. Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 75-strike European call option on a non-dividend-paying stock. At that time, the price of the call option was 8. Today, the stock price is 85. The investor decides to close out all positions. You are given:

(i) The continuously compounded risk-free interest rate is 5%.
(ii) The stock’s volatility is 26%.

Calculate the eight-month holding profit.
5.2.8 **Delta-hedging** simply hedge against the change of the underlying risky asset price in terms of its **first order**. A more **prudent** risk management strategy is the **delta-gamma-hedging**, in order to achieve **delta-neutrality** and **gamma-neutrality**:

\[
\begin{align*}
\Delta \tilde{V} &= 0 \\
\Gamma \tilde{V} &= 0
\end{align*}
\]

5.2.9 Let \(x_1\) and \(x_2\) be units of the first and second hedging instruments \(V_1\) and \(V_2\). For the long position of the option \(V\), delta-gamma-neutrality is achieved by solving \(x_1\) and \(x_2\) from:

\[
\begin{align*}
\Delta_V + x_1 \Delta V_1 + x_2 \Delta V_2 &= 0 \\
\Gamma_V + x_1 \Gamma V_1 + x_2 \Gamma V_2 &= 0
\end{align*}
\]

For the short position of the option \(V\), delta-gamma-neutrality is achieved by solving \(x_1\) and \(x_2\) from:

\[
\begin{align*}
-\Delta_V + x_1 \Delta V_1 + x_2 \Delta V_2 &= 0 \\
-\Gamma_V + x_1 \Gamma V_1 + x_2 \Gamma V_2 &= 0
\end{align*}
\]

5.2.10 Since \(\Gamma_S = 0\), a delta-gamma-neutral portfolio can never be constructed with the underlying risky asset alone.

**Example 7** [SOA Exam MFE Spring 2007 Question Q10]: For two European call options, Call-I and Call-II, on a stock, you are given:

<table>
<thead>
<tr>
<th>Greek</th>
<th>Call-I</th>
<th>Call-II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta</td>
<td>0.5825</td>
<td>0.7773</td>
</tr>
<tr>
<td>Gamma</td>
<td>0.0651</td>
<td>0.0746</td>
</tr>
<tr>
<td>Vega</td>
<td>0.0781</td>
<td>0.0596</td>
</tr>
</tbody>
</table>

Suppose you just sold 1000 units of Call-I. Determine the number of units of Call-II and stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in Call-I.

**Solution:**
Exercise 7 [Modified from CAS Exam 3 Fall 2007 Question Q24]: An investor has a portfolio consisting of 100 put options on stock A, with a strike price of 40, and 5 shares of stock A. The investor can write put options on stock A with a strike price of 35. The deltas and gammas of the options are listed below:

<table>
<thead>
<tr>
<th>Put (Strike = 35)</th>
<th>Put (Strike = 40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta</td>
<td>-0.10</td>
</tr>
<tr>
<td>Gamma</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
</tr>
</tbody>
</table>

Determine the number of units of put option with strike 35 and stock A that the investor should buy or sell in order to delta-gamma-hedge the portfolio.

Option value approximation

5.2.11 Among the observed values, $t$ and $S$, model inputs, $\delta$, $r$, and $\sigma$, as well as option characteristics, $K$ and $T$, the observed values $t$ and $S$ change inevitably as time elapses.

5.2.12 Assume that the model inputs $\delta$, $r$, and $\sigma$ do not change across time. By Taylor-series expansion, for a small enough $h$,

$$V(t + h, S_{t+h}) \approx V(t, S_t) + \Delta V(t, S_t) (S_{t+h} - S_t) + \frac{1}{2} \Gamma V(t, S_t) (S_{t+h} - S_t)^2 + \theta V(t, S_t) h.$$ 

Such a second order approximation is called the delta-gamma-theta approximation.

5.2.13 Similarly, by Taylor-series expansion, for a small enough $h$, delta approximation and delta-gamma approximation are respectively given by:

$$V(t + h, S_{t+h}) \approx V(t, S_t) + \Delta V(t, S_t) (S_{t+h} - S_t);$$

$$V(t + h, S_{t+h}) \approx V(t, S_t) + \Delta V(t, S_t) (S_{t+h} - S_t) + \frac{1}{2} \Gamma V(t, S_t) (S_{t+h} - S_t)^2.$$ 

Example 8: You are given the following information for a European 6-month call option on a stock:

(i) The call premium is 2.5.
(ii) The spot price is 50.
(iii) $\sigma = 25\%$.
(iv) $\Delta = 0.3$.
(v) $\Gamma = 0.05$.
(vi) $\theta = -0.03$.
(vii) $r = 5\%$.

By delta-gamma-theta approximation, determine the approximate value of the call option 1 day later if the stock price decreases by 0.1.
Solution:

Exercise 8 [SOA Exam MFE Spring 2007 Question Q19]: Assume that the Black-Scholes framework holds. The price of a non-dividend-paying stock is $30.00. The price of a put option on this stock is $4.00. You are given:

(i) $\Delta = -0.28$;
(ii) $\Gamma = 0.10$.

Using the delta-gamma approximation, determine the price of the put option if the stock price changes to $31.50.$