Learning Objectives:

4.1 Lognormal Model of Stock Prices: continuous-time, Black-Scholes model, geometric Brownian motion, Brownian motion, lognormal model, lognormal distribution, expectation, first principle, distribution function, survival function, exercise probability, conditional tail expectation, expected payoff.

4.2 The Black-Scholes Formula: continuous-time, two financial assets, risk-free, risky, constant risk-free interest rate, lognormal model, risk-neutral world, risk-neutral valuation formula, Black-Scholes call and put option prices, put-call parity, price bounds.
4.1 Lognormal Model of Stock Prices

4.1.1 Consider a continuous-time setting \( t \geq 0 \), where the current time \( t = 0 \).

4.1.2 Denote the risky asset price by \( S = \{S_t\}_{t \geq 0} \), where the risky asset pays continuous dividend with a continuously compounded yield \( \delta \).

4.1.3 In the Black-Scholes model, the risky asset price stochastic process \( S \) is modeled by a so-called geometric Brownian motion: for any \( t \geq 0 \),

\[
S_t = S_0 e^{(\alpha - \delta - \frac{1}{2} \sigma^2) t + \sigma W_t},
\]

where \( \alpha(> r) \) is the continuously compounded expected rate of return, \( \sigma(> 0) \) is the volatility, and \( W = \{W_t\}_{t \geq 0} \) is a so-called Brownian motion.

4.1.4 A Brownian motion \( W = \{W_t\}_{t \geq 0} \) is a stochastic process which satisfies the following four properties:

(i) \( W_0 = 0 \);
(ii) for any \( 0 \leq u \leq t \), \( W_t - W_u \sim N(0, t - u) \);
(iii) for any \( 0 \leq u \leq t \leq s \leq r \), \( W_t - W_u \) and \( W_r - W_s \) are independent;
(iv) \( W \) has a continuous path.

4.1.5 Using property (ii) of the Brownian motion \( W \), the risky asset price \( S \) is consequently modeled by a lognormal model: for any \( 0 \leq u \leq t \),

\[
S_t = S_u e^{(\alpha - \delta - \frac{1}{2} \sigma^2) (t-u) + \sigma Z_{u,t}},
\]

where \( Z_{u,t} \sim N(0, t-u) \). In particular, when \( u = 0 \), for any \( t \geq 0 \),

\[
S_t = S_0 e^{(\alpha - \delta - \frac{1}{2} \sigma^2) t + \sigma Z_t},
\]

where \( Z_t \sim N(0, t) \). However, the lognormal model is only valid from a distributional perspective, but is not valid from a perspective of stochastic processes.

4.1.6 Below is the reason that the risky asset price model is called lognormal: for any \( t \geq 0 \), the continuously compounded rate of return

\[
\ln \left( \frac{S_t}{S_0} \right) = (\alpha - \delta - \frac{1}{2} \sigma^2) t + \sigma Z_t \sim N \left( \left( \alpha - \delta - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right);
\]

in other words, for any \( t \geq 0 \), \( S_t \) follows a lognormal distribution:

\[
S_t \sim LN(m, v^2),
\]

where \( m \) and \( v^2 \) are the mean and variance of the lognormal distribution, respectively.
where \( m = \ln S_0 + (\alpha - \delta - \frac{1}{2} \sigma^2) t \) and \( v = \sigma \sqrt{t} \).

4.1.7 The density function, distribution function, and \( k \)-th moments of a lognormal random variable \( Y \sim LN (m, v^2) \) are given in the SOA IFM Formula Sheet:

\[
f_Y(y) = \frac{1}{y v \sqrt{2 \pi}} e^{-\frac{1}{2} \left( \frac{\ln(y) - m}{v} \right)^2}, \quad y > 0;
\]

\[
F_Y(y) = N \left( \frac{\ln(y) - m}{v} \right), \quad y > 0,
\]

where \( N (\cdot) \) is the standard normal distribution function; and

\[
\mathbb{E} [Y^k] = e^{km + \frac{k^2}{2}v^2}, \quad k = 1, 2, \ldots
\]

4.1.8 The expectation of \( S_t \) given \( S_0 \) is given by

\[
\mathbb{E} [S_t] = e^{\ln S_0 + (\alpha - \delta - \frac{1}{2} \sigma^2) t + \frac{1}{2} \sigma^2 t} = S_0 e^{(\alpha - \delta) t},
\]

which can also be deduced by the first principle:

\[
\mathbb{E} [S_t] = \mathbb{E} \left[ S_0 e^{(\alpha - \delta - \frac{1}{2} \sigma^2) t + \sigma Z_t} \right] = S_0 e^{(\alpha - \delta - \frac{1}{2} \sigma^2) t} \mathbb{E} \left[ e^{\sigma Z_t} \right] = S_0 e^{(\alpha - \delta - \frac{1}{2} \sigma^2) t} e^{\frac{1}{2} \sigma^2 t} = S_0 e^{(\alpha - \delta) t}.
\]

This also explains why \( \alpha \) is the so-called continuously compounded expected rate of return:

\[
\mathbb{E} [S_t e^{\delta t}] = S_0 e^{\alpha t},
\]

which appeared in 1.2.20 for the non-dividend-paying case.

4.1.9 The distribution function of \( S_t \) given \( S_0 \) is given by, for any \( K > 0 \),

\[
F_{S_t} (K) = \mathbb{P} (S_t \leq K) = N \left( \frac{\ln (K) - (\ln S_0 + (\alpha - \delta - \frac{1}{2} \sigma^2) t)}{\sigma \sqrt{t}} \right) = N \left( -\frac{\ln (\frac{S_0}{K}) + (\alpha - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right) = N \left( -\hat{d}_2 \right),
\]

where \( \hat{d}_2 \) is given by

\[
\hat{d}_2 = \frac{\ln (\frac{S_0}{K}) + (\alpha - \delta - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}}.
\]

The distribution function of \( S_t \) can also deduced by the first principle: for any \( K > 0 \),
In particular, if $K > 0$ is a strike price of an European option with a maturity $T$, then the value $N\left(-\hat{d}_2\right)$, with $t$ being replaced by $T$, is the **probability that the put option will be exercised while the call option will not be exercised**.

4.1.10 **The survival function** of $S_t$ given $S_0$ is given by, for any $K > 0$,

$$S_{S_t}(K) = \mathbb{P}(S_t > K) = \mathbb{P}\left(S_0e^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma Z_t} > K\right) = \mathbb{P}\left(Z > -\hat{d}_2\right) = 1 - N\left(-\hat{d}_2\right) = N\left(\hat{d}_2\right).$$

In particular, if $K > 0$ is a strike price of an European option with a maturity $T$, then the value $N\left(\hat{d}_2\right)$, with $t$ being replaced by $T$, is the **probability that the call option will be exercised while the put option will not be exercised**.

4.1.11 **The conditional right-tail expectation** of $S_t$ is given by, for any $K > 0$,

$$\mathbb{E}[S_t|S_t > K] = \frac{\mathbb{E}[S_t 1_{\{S_t > K\}}]}{\mathbb{P}(S_t > K)} = \frac{S_0e^{(\alpha-\delta)t}N\left(\hat{d}_1\right)}{N\left(\hat{d}_2\right)} = S_0e^{(\alpha-\delta)t}\frac{N\left(\hat{d}_1\right)}{N\left(\hat{d}_2\right)} = \mathbb{E}[S_t] \frac{N\left(\hat{d}_1\right)}{N\left(\hat{d}_2\right)},$$

where $\hat{d}_1$ is given by

$$\hat{d}_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (\alpha - \delta + \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} = \hat{d}_2 + \sigma \sqrt{t}.$$

In particular, if $K > 0$ is a strike price of an European option with a maturity $T$, then the conditional right-tail expectation, with $t$ being replaced by $T$, is the **conditional expectation of $S_T$, when the call option is in-the-money, or when the put option is out-of-the-money, at maturity**.

4.1.12 Using the fact that

$$\mathbb{E}[S_t] = \mathbb{E}[S_t 1_{\{S_t < K\}} + S_t 1_{\{S_t = K\}} + S_t 1_{\{S_t > K\}}] = \mathbb{E}[S_t|S_t < K] \mathbb{P}(S_t < K) + \mathbb{E}[S_t 1_{\{S_t > K\}}].$$

Hence, the **conditional left-tail expectation** of $S_t$ is given by, for any $K > 0$,

$$\mathbb{E}[S_t|S_t < K] = \frac{\mathbb{E}[S_t] - \mathbb{E}[S_t 1_{\{S_t > K\}}]}{\mathbb{P}(S_t < K)} = \mathbb{E}[S_t] \frac{1 - N\left(\hat{d}_1\right)}{N\left(\hat{d}_2\right)} = \mathbb{E}[S_t] \frac{N\left(-\hat{d}_1\right)}{N\left(-\hat{d}_2\right)}.$$
In particular, if $K > 0$ is a strike price of an European option with a maturity $T$, then the conditional left-tail expectation, with $t$ being replaced by $T$, is the conditional expectation of $S_T$, when the put option is in-the-money, or when the call option is out-of-the-money, at maturity.

4.1.13 Consider European options with strike $K$ and maturity $T$. The expected values of payoffs for call and put options are respectively given by:

$$
\mathbb{E} \left[ (S_T - K)_+ \right] = \mathbb{E} \left[ (S_T - K) \mathbb{1}_{\{S_T > K\}} \right] = \mathbb{E} [S_T \mathbb{1}_{\{S_T > K\}}] - K \mathbb{P} (S_T > K)
$$

$$
= S_0 e^{(\alpha - \delta)T} N \left( \hat{d}_1 \right) - K N \left( \hat{d}_2 \right);
$$

$$
\mathbb{E} \left[ (K - S_T)_+ \right] = \mathbb{E} \left[ (K - S_T) \mathbb{1}_{\{S_T < K\}} \right] = K \mathbb{P} (S_T < K) - \mathbb{E} [S_T \mathbb{1}_{\{S_T < K\}}]
$$

$$
= K N \left( -\hat{d}_2 \right) - S_0 e^{(\alpha - \delta)T} N \left( -\hat{d}_1 \right).
$$

Example 1 [Modified from CAS Exam 8 Spring 2000 Q30]: You are given the following information about a European call option.

(i) The current stock price is $35.

(ii) The exercise price is $40.

(iii) The option matures in 6 months.

(iv) The expected annual return on the stock is 18%.

(v) The annual volatility of the stock price is 24%.

(vi) The stock’s price at each future time, given its current price, is lognormally distributed.

(vii) The stock pays no dividends.

Calculate

(a) $\mathbb{E} [S_T]$,

(b) the probability that the call option is in-the-money at maturity,

(c) the probability that the call option is out-of-the-money at maturity,

(d) the expected value of the stock price at maturity if the option is in-the-money,

(e) the expected value of the stock price at maturity if the option is out-of-the-money,

(f) the expected value of the option payoff.

Solution:
Exercise 1 [SOA Exam MFE Spring 2009 Q16]: You are given the following information about a non-dividend-paying stock:

(i) The current stock price is 100.
(ii) Stock prices are lognormally distributed.
(iii) The continuously compounded expected return on the stock is 10%.
(iv) The stock volatility is 30%.

Consider a nine-month 125-strike European call option on the stock. Calculate the probability that the call will be exercised.
Example 2 [SOA Advanced Derivatives Sample Question Q50]: Assume the Black-Scholes framework. You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.

(i) The current stock price is 0.25.
(ii) The stock’s volatility is 0.35.
(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

Calculate the upper limit of the 90% lognormal confidence interval for the price of the stock in 6 months.

Solution:

Exercise 2: A stock’s price follows a lognormal model. You are given:

(i) The initial price is 100.
(ii) $\alpha = 0.2$.
(iii) $\delta = 0.1$.
(iv) $\sigma = 0.6$.

Construct a 95% confidence interval for the price of the stock at the end of three years.
4.2 The Black-Scholes Formula

4.2.1 Consider a continuous-time setting \( t \geq 0 \), where the current time \( t = 0 \).

4.2.2 Again, the financial market consists of **two financial assets**; one is a risk-free asset, while one is a risky asset.

4.2.3 The prices of the risk-free asset is \( B = \{B_t\}_{t \geq 0} \). Assume that the risk-free asset offers a constant continuously compounded risk-free interest rate \( r \). Hence, for any \( t \geq 0 \), \( B_t = B_0 e^{rt} \).

4.2.4 The prices of the risky asset is \( S = \{S_t\}_{t \geq 0} \). Assume that the risky asset price stochastic process \( S \) is modeled by the geometric Brownian motion; in particular, distribution-wise, the risky asset price is given by the lognormal model: for any \( t \geq 0 \),

\[
S_t = S_0 e^{(\alpha - \frac{1}{2} \sigma^2)t + \sigma Z_t},
\]

where \( Z_t \sim N(0, t) \), in the real-world.

4.2.5 Recall that, in the discrete-time binomial model, the financial market is arbitrage-free if and only if

\[
S_t = e^{-rh} \mathbb{E}^Q [S_{t+h} e^{\delta h} | S_t],
\]

in the **risk-neutral world**. In other words, in the risk-neutral world, the risky asset earns the continuously compounded risk-free interest rate \( r \), instead of the continuously compounded expected rate of return \( \alpha \):

\[
\mathbb{E}^Q [S_{t+h} e^{\delta h} | S_t] = S_t e^{rh},
\]

comparing to \( \mathbb{E} [S_{t+h} e^{\delta h} | S_t] = S_t e^{ah} \), in the real-world.

4.2.6 In the continuous-time Black-Scholes model, and in the risk-neutral world, the risky asset also earns the continuously compounded risk-free interest rate \( r \), instead of the continuously compounded expected rate of return \( \alpha \): for any \( t \geq 0 \),

\[
S_t = S_0 e^{(r - \delta - \frac{1}{2} \sigma^2)t + \sigma \tilde{Z}_t},
\]

where \( \tilde{Z}_t \sim N(0, t) \), which is not the random variable \( Z_t \) in the real-world.

4.2.7 In addition, recall that, in the discrete-time binomial model, the **risk-neutral valuation formula** is given by

\[
V_t = e^{-r(T-t)} \mathbb{E}^Q [X_T | S_t],
\]

where \( X_T \) is an European option payoff at the expiration date \( T \).

4.2.8 It turns out that the same **risk-neutral valuation formula** also holds in the continuous-
time Black-Scholes model: for any $t \in [0, T]$,

$$V_t = e^{-r(T-t)}E^Q[X_T|S_t].$$

In particular, when $t = 0$,

$$V_0 = e^{-rT}E^Q[X_T].$$

4.2.9 If the European option is the call option, then, by 4.1.13, the **Black-Scholes call option price** at time 0 is given by

$$C (0, S_0, \delta; K, r; \sigma, T) = e^{-rT}E^Q [(S_T - K)^+]$$

$$= e^{-rT} (S_0 e^{(r-\delta)T} N (d_1) - KN (d_2))$$

$$= S_0 e^{-\delta T} N (d_1) - Ke^{-rT} N (d_2)$$

$$= F^*_{0,T} N (d_1) - PV_{0,T} (K) N (d_2),$$

where $d_1$ and $d_2$ are given by

$$d_1 = \frac{\ln (S_0/K) + (r - \delta + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}},$$

$$d_2 = \frac{\ln (S_0/K) + (r - \delta - \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}.$$

The reason, that all $\alpha$ from 4.1.13 are replaced by $r$, is that the expectation is calculated using the risk-neutral probabilities, rather than the real-world probabilities, and hence, the lognormal model in 4.2.6 should be used for pricing purpose, instead of 4.2.4.

4.2.10 If the European option is the put option, then, by 4.1.13, the **Black-Scholes put option price** at time 0 is given by

$$P (0, S_0, \delta; K, r; \sigma, T) = e^{-rT}E^Q [\left(\frac{K S_T}{K} \right) - S_T]$$

$$= e^{-rT} (KN (-d_2) - S_0 e^{(r-\delta) T} N (-d_1))$$

$$= PV_{0,T} (K) N (-d_2) - F^*_{0,T} N (-d_1).$$

4.2.11 The Black-Scholes call and put option prices obviously satisfy the **put-call parity** and **price bounds** in 2.3.4, 2.3.6, 2.3.7:

$$C (0, S_0, \delta; K, r; \sigma, T) - P (0, S_0, \delta; K, r; \sigma, T) = F^*_{0,T} - PV_{0,T} (K);$$

$$\left(F^*_{0,T} - PV_{0,T} (K)\right)_+ \leq C (0, S_0, \delta; K, r; \sigma, T) \leq F^*_{0,T};$$

$$\left(PV_{0,T} (K) - F^*_{0,T}\right)_+ \leq P (0, S_0, \delta; K, r; \sigma, T) \leq PV_{0,T} (K).$$
After all, the validity of all the (in-)equalities in Chapter 2 are model-independent.

**Example 3** [SOA Advanced Derivatives Sample Question Q6]: You are considering the purchase of 100 units of a 3-month 25-strike European call option on stock. You are given:

(i) The Black-Scholes framework holds.
(ii) The stock is currently selling for 20.
(iii) The stock’s volatility is 24%.
(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.
(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of the block of 100 options.

**Solution:**

**Exercise 3** [SOA Exam MFE Spring 2007 Q3]: You are asked to determine the price of a European put option on a stock. Assuming the Black-Scholes framework holds, you are given:

(i) The stock price is $100.
(ii) The put option will expire in 6 months.
(iii) The strike price is $98.
(iv) The continuously compounded risk-free interest rate is $r = 0.055$.
(v) $\delta = 0.01$.
(vi) $\sigma = 0.50$.

Calculate the price of this put option.
Example 4 [SOA Advanced Derivatives Sample Question Q7]: Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk. You are given the following information:

(i) The deal will be closed 3 months from now.
(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.
(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.
(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.
(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.
(vi) The daily volatility of the yen per dollar exchange rate is 0.261712%.
(vii) 1 year = 365 days; 3 months = \( \frac{1}{4} \) year.

Calculate Company A’s option cost.

Solution:

Exercise 4 [CAS Exam 3 Fall 2007 Q21]: On January 1st, 2007, the following currency information is given:

(i) Spot exchange rate = $0.82/euro
(ii) Dollar interest rate = 5.0% compounded continuously
(iii) Euro interest rate = 2.5% compounded continuously
(iv) Exchange rate volatility = 0.10

What is the price of 850 dollar-denominated euro call options with a strike exchange rate of $0.80/euro that expire on January 1st, 2008?
**Example 5 [SOA Advanced Derivatives Sample Question Q55]**: Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract. You are given:

(i) The continuously compounded risk-free interest rate is 10%.
(ii) The strike price of the option is 20.
(iii) The price of the put option is 1.625.

If three months later the futures price is 17.7, what is the price of the put option at that time?

**Solution:**

---

**Exercise 5**: Assume the Black-Scholes framework. Consider a 1-year at-the-money European call option on a futures contract. You are given:

(i) The continuously compounded risk-free interest rate is 5%.
(ii) The strike price of the option is 10.
(iii) The price of the call option is 1.

If six months later the futures price is 8, what is the price of the call option at that time?