Learning Objectives:

2.1 Option Contracts: underlying asset, long, exercise, short, exercise date, expiration date, strike price, right, obligation, call, put, European, American, Bermudan, early exercise, position, exercise style, payoff, payoff diagram, moneyness, in-the-money, at-the-money, out-of-the-money, option price, profit, profit diagram.

2.2 Option Strategies: insurance, floor, cap, written covered call, written covered put, synthetic forward, genuine forward, initial payment, put-call parity, bull spread, bear spread, box spread, ratio spread, collar, collar width, zero-cost collar, collared stock, straddle, strangle, butterfly spread.

Further Exercises:

SOA Introductory Derivatives Samples Questions 2, 9, 14, 16, 35, 40, 41, 42, 46, 48, 50, 59, 62, 65, 74, and 75.

SOA Advanced Derivatives Samples Question 40.
2.1 Option Contracts

What is an option contract?

2.1.1 An option contract on an underlying asset is an agreement between two parties such that, the party who longs the contract will choose to exercise the contract for long or short the underlying asset, while the party who shorts the contract will have to short or long the underlying asset, if the contract is exercised by the long party, at an exercise date by an expiration date, and at a strike price. The long party of the option contract holds the right while the short party of the contract bears the obligation, once the contract is signed.

2.1.2 The option contract is called call option if the long position of the contract will choose to exercise the contract for long the underlying asset, while the option contract is called put option if the long position of the contract will choose to exercise the contract for short the underlying asset.

2.1.3 The option contract is called European-style if the available exercise date equals only the expiration date; the option contract is called American-style if the available exercise date spans from the contract-signing date to the expiration date; the option contract is called Bermudan-style if the available exercise date spans across some time between the contract-signing date and the expiration date. American and Bermudan options provide opportunities to early exercise.

2.1.4 The six key terms describe the right/obligation on an option contract for a party.

- Underlying asset.
- Expiration date.
- Strike price.
- Position: long/short.
- Call/Put.
- Exercise style: European/American/Bermudan.

2.1.5 Suppose that the time when the option contract is signed is \( t = 0 \). Denote the price of the underlying asset by \( S_t \), for \( t \geq 0 \), with \( S_0 \) known as the spot price; denote the expiration date by \( T \geq 0 \); denote the exercise date by \( \tau \in [0, T] \); denote the strike price by \( K \).

2.1.6 Recall the example in 1.1.3: Alfred and John sign an option contract that, after one week, John will choose to pay Alfred 70 USD for purchasing a barrel of crude oil. The underlying asset is the barrel of crude oil; the exercise date and the expiration date \( \tau = T = 1 \) week, the strike price is \( K = 70 \) USD; John longs the option, while Alfred shorts it; the contract is an European call option.
Payoff, payoff diagram, and moneyness.

2.1.7 For the long party of call option, his/her payoff of the option contract at the exercise date is given by:

\[
\text{Payoff of long call option} = (S_t - K)_+ = \max\{S_t - K, 0\} = \begin{cases} 
0 & \text{if } S_t \leq K \\
S_t - K & \text{if } S_t > K.
\end{cases}
\]

For the short party of call option, his/her payoff of the option contract at the exercise date is given by:

\[
\text{Payoff of short call option} = -(S_t - K)_+ = \begin{cases} 
0 & \text{if } S_t \leq K \\
K - S_t & \text{if } S_t > K.
\end{cases}
\]

2.1.8 For the long party of put option, his/her payoff of the option contract at the exercise date is given by:

\[
\text{Payoff of long put option} = (K - S_t)_+ = \begin{cases} 
K - S_t & \text{if } S_t \leq K \\
0 & \text{if } S_t > K.
\end{cases}
\]

For the short party of put option, his/her payoff of the option contract at the exercise date is given by:

\[
\text{Payoff of short put option} = -(K - S_t)_+ = \begin{cases} 
S_t - K & \text{if } S_t \leq K \\
0 & \text{if } S_t > K.
\end{cases}
\]
2.1.9 Signing the call/put option contract is a zero-sum game for these two parties, since the sum of payoffs of long and short positions is zero.

2.1.10 Option contracts can be compared by moneyness. The call option is called **out-of-the-money** if $S_0 < K$, **at-the-money** if $S_0 = K$, and **in-the-money** if $S_0 > K$; the put option is called **in-the-money** if $S_0 < K$, **at-the-money** if $S_0 = K$, and **out-of-the-money** if $S_0 > K$.

**Example 1** [SOA Introductory Derivatives Sample Question Q61]: An investor purchased Option A and Option B for a certain stock today, with strike prices 70 and 80, respectively. Both options are European one-year put options. Determine which statement is true about the moneyness of these options, based on a particular stock price.

(A) If Option A is in-the-money, then Option B is in-the-money.
(B) If Option A is at-the-money, then Option B is out-of-the-money.
(C) If Option A is in-the-money, then Option B is out-of-the-money.
(D) If Option A is out-of-the-money, then Option B is in-the-money.
(E) If Option A is out-of-the-money, then Option B is out-of-the-money.

**Solution:**

**Exercise 1** [SOA Introductory Derivatives Sample Question Q66]: The current price of a stock is 80. Both call and put options on this stock are available for purchase at a strike price of 65. Determine which of the following statements about these options is true.

(A) Both the call and put options are at-the-money.
(B) Both the call and put options are in-the-money.
(C) Both the call and put options are out-of-the-money.
(D) The call option is in-the-money, but the put option is out-of-the-money.
(E) The call option is out-of-the-money, but the put option is in-the-money.
Example 2 [SOA Introductory Derivatives Sample Question Q44]: You are given the following information about two options, A and B:

(i) Option A is a one-year European put with exercise price 45.
(ii) Option B is a one-year American call with exercise price 55.
(iii) Both options are based on the same underlying asset, a stock that pays no dividends.
(iv) Both options go into effect at the same time and expire at \( t = 1 \).

You are also given the following information about the stock price:

(i) The initial stock price is 50.
(ii) The stock price at expiration is also 50.
(iii) The minimum stock price (from \( t = 0 \) to \( t = 1 \)) is 46.
(iv) The maximum stock price (from \( t = 0 \) to \( t = 1 \)) is 58.

Determine which of the following statements is true.

(A) Both options A and B are “at-the-money” at expiration.
(B) Both options A and B are “in-the-money” at expiration.
(C) Both options A and B are “out-of-the-money” throughout each option’s term.
(D) Only option A is ever “in-the-money” at some time during its term.
(E) Only option B is ever “in-the-money” at some time during its term.

Solution:

Exercise 2 [SOA Introductory Derivatives Sample Question Q33]: Several years ago, John bought three separate 6-month options on the same stock.

- Option I was an American-style put with strike price 20.
- Option II was a Bermuda-style call with strike price 25, where exercise was allowed at any time following an initial 3-month period of call protection.
- Option III was a European-style put with strike price 30.

When the options were bought, the stock price was 20. When the options expired, the stock price was 26. The table below gives the maximum and minimum stock price during the 6 month period:

<table>
<thead>
<tr>
<th>Time period:</th>
<th>1st 3 months of Option Term</th>
<th>2nd 3 months of Option Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Stock Price</td>
<td>24</td>
<td>28</td>
</tr>
<tr>
<td>Minimum Stock Price</td>
<td>18</td>
<td>22</td>
</tr>
</tbody>
</table>

John exercised each option at the optimal time. Rank the three options, from highest to lowest payoff.

(A) I > II > III
(B) I > III > II
(C) II > I > III
(D) III > I > II
(E) III > II > I
Is there any initial payment for an option contract?

2.1.11 Recall that a forward contract requires no initial payment or premium as long as the forward price is given by the no-arbitrage one. However, given the underlying asset and the expiration date of the option contract, regardless of call/put, European/American/Bermudan, and the value of strike price, the **long party** of the option contract always make a **non-negative payoff** at the exercise date, while the **short party** of the option contract always make a **non-positive payoff** at the exercise date. Therefore, the **short party** of the option contract **demands** an initial payment or premium, which is known as the **option price** or option premium, from the long party. Denote the option price by $C$ for call and by $P$ for put.

*Profit and profit diagram.*

2.1.12 Recall that the **profit** of a derivative contract is given by:

\[
\text{Profit of contract} = \text{Payoff of contract} - FV_{0,\tau} \left( \text{Cost of contract} \right).
\]

2.1.13 The **profits** for the **long/short** party of the **call/put** option contract **at the exercise date** are given by:

\[
\begin{align*}
\text{Profit of long call option} &= (S_{\tau} - K)_+ - FV_{0,\tau} (C), \\
\text{Profit of short call option} &= FV_{0,\tau} (C) - (S_{\tau} - K)_+.
\end{align*}
\]

\[
\begin{align*}
\text{Profit of long put option} &= (K - S_{\tau})_+ - FV_{0,\tau} (P), \\
\text{Profit of short put option} &= FV_{0,\tau} (P) - (K - S_{\tau})_+.
\end{align*}
\]
Example 3 [SOA Introductory Derivatives Sample Question Q26]: Determine which, if any, of the following positions has or have an unlimited loss potential from adverse price movement in the underlying asset, regardless of the initial premium received.

(I) Short 1 forward contract
(II) Short 1 call option
(III) Short 1 put option

(A) None
(B) I and II only
(C) I and III only
(D) II and III only
(E) The correct answer is not given by (A), (B), (C), or (D)

Solution:

Exercise 3 [SOA Introductory Derivatives Sample Question Q49]: The market price of Stock A is 50. A customer buys a 50-strike put contract on Stock A for 500. The put contract is for 100 shares of A. Calculate the customer’s maximum possible loss.

Example 4 [SOA Introductory Derivatives Sample Question Q11]: Stock XYZ has the following characteristics:

• The current price is 40.
• The price of a 35-strike 1-year European call option is 9.12.
• The price of a 40-strike 1-year European call option is 6.22.
• The price of a 45-strike 1-year European call option is 4.08.

The annual effective risk-free interest rate is 8%. Let $S$ be the price of the stock one year from now. All call positions being compared are long. Determine the range for $S$ such that the 45-strike call produce a higher profit than the 40-strike call, but a lower profit than the 35-strike call.

Solution:

Exercise 4 [SOA Introductory Derivatives Sample Question Q12]: Consider a European put option on a stock index without dividends, with 6 months to expiration and a strike price of 1,000. Suppose that the effective six-month interest rate is 2%, and that the put costs 74.20 today. Calculate the price that the index must be in 6 months so that being long in the put would produce the same profit as being short in the put.
2.2 Option Strategies

2.2.1 Using zero-coupon bonds, underlying assets, forward contracts, and option contracts, a wide variety of new derivative contracts are constructed for an investor to better manage its risk or speculate on price movements.

2.2.2 Option contracts in this section are assumed to be European with expiration date $T$.

*Floor, cap, written covered call, and written covered put.*

2.2.3 The investor who longs the underlying asset by the outright purchase will experience a loss if $S_T < FV_{0,T}(S_0)$, and a gain if $S_T > FV_{0,T}(S_0)$. To *insure* against the loss, the investor longs the put option with strike $K$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = S_T + (K - S_T)_+ = \max\{S_T, K\} = \begin{cases} K & \text{if } S_T \leq K \\ S_T & \text{if } S_T > K \end{cases}.$$  

The long put option is called a *floor* for the long position of the underlying asset.

2.2.4 The investor who shorts the underlying asset by the outright sale will experience a gain if $S_T < FV_{0,T}(S_0)$, and a loss if $S_T > FV_{0,T}(S_0)$. To *insure* against the loss, the investor longs the call option with strike $K$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = -S_T + (S_T - K)_+ = \max\{-S_T, -K\} = \begin{cases} -S_T & \text{if } S_T \leq K \\ -K & \text{if } S_T > K \end{cases}.$$  

The long call option is called a *cap* for the short position of the underlying asset.
For the investor who longs the underlying asset, if he/she shorts the call option with strike \( K \), then the payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = S_T - (S_T - K)_+ = \min\{S_T, K\} = \begin{cases} 
S_T & \text{if } S_T \leq K \\
K & \text{if } S_T > K.
\end{cases}
\]

To compensate the sacrifice by the investor for the potential payoff beyond the strike \( K \) from the underlying asset, he/she actually acquires the call option price at time 0. The short call option is called a **written covered call** by the long position of the underlying asset.

For the investor who shorts the underlying asset, if he/she shorts the put option with strike \( K \), then the payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = -S_T - (K - S_T)_+ = \min\{-S_T, -K\} = \begin{cases} 
-K & \text{if } S_T \leq K \\
-S_T & \text{if } S_T > K.
\end{cases}
\]

To compensate the sacrifice by the investor for the potential payoff below the strike \( K \) from the underlying asset, he/she actually acquires the put option price at time 0. The short put option is called a **written covered put** by the short position of the underlying asset.

**Example 5** [SOA Introductory Derivatives Sample Question Q13]: A trader shorts one share of a stock index for 50 and buys a 60-strike European call option on that stock that expires in 2 years for 10. Assume the annual effective risk-free interest rate is 3%. The stock index increases to 75 after 2 years. Calculate the profit on your combined position, and determine an alternative name for this combined position.

**Solution:**
Exercise 5 [SOA Introductory Derivatives Sample Question Q47]: An investor has written a covered call. Determine which of the following represents the investor’s position.

(A) Short the call and short the stock
(B) Short the call and long the stock
(C) Short the call and no position on the stock
(D) Long the call and short the stock
(E) Long the call and long the stock

Synthetic forwards.

2.2.7 In Chapter 1, synthetic long and short forwards are constructed using zero-coupon bonds and underlying assets, in which the synthetic forward price equals to the no-arbitrage genuine forward price \( F_{0,T} \).

2.2.8 The synthetic long forward with forward price \( K \) can be constructed by, long 1 call option and short 1 put option, both with strike \( K \), with the payoff of the combined position at the expiration date given by:

\[
\text{Payoff} = (S_T - K)_+ - (K - S_T)_+ = S_T - K.
\]

2.2.9 The synthetic short forward with forward price \( K \) can be constructed by, short 1 call option and long 1 put option, both with strike \( K \), with the payoff of the combined position at the expiration date given by:

\[
\text{Payoff} = -(S_T - K)_+ + (K - S_T)_+ = K - S_T.
\]

2.2.10 However, since the forward price \( K \) of the synthetic long or short forward does not necessarily equal to the no-arbitrage genuine forward price \( F_{0,T} \), there could be an initial
payment or premium at time 0 for long or short the synthetic forward. Denote the call price by \( C(K, T) \) and the put price by \( P(K, T) \). Then the initial payment for synthetic forward is given by:

\[
\text{Initial payment} = \begin{cases} 
    C(K, T) - P(K, T) & \text{for long} \\
    P(K, T) - C(K, T) & \text{for short}
\end{cases}
\]

**Put-call parity.**

2.2.11 If the forward price of synthetic forward \( K = F_{0,T} \), the no-arbitrage genuine forward price, then there should be zero initial payment or premium at time 0. Therefore,

\[
C(F_{0,T}, T) = P(F_{0,T}, T).
\]

2.2.12 A natural extension is the relationship between \( C(K, T) \) and \( P(K, T) \), where \( K \) is not necessarily \( F_{0,T} \). This is given by the **put-call parity**:

\[
C(K, T) - P(K, T) = F^P_{0,T} - PV_{0,T}(K).
\]

Indeed, construct two investment strategies \( \pi^1 \) and \( \pi^2 \) at time 0 and hold till time \( T \). The \( \pi^1 \) consists of long 1 call and short 1 put, both with strike \( K \); the \( \pi^2 \) consists of long 1 prepaid forward on the same underlying asset and short 1 bond with face-value \( K \) matured at \( T \). Then,

\[
V^\pi_1_T = (S_T - K)^+ - (K - S_T)^+ = S_T - K = V^\pi_2_T.
\]

By the law of one-price, \( C(K, T) - P(K, T) = V^\pi_1_0 = V^\pi_2_0 = F^P_{0,T} - PV_{0,T}(K) \).

**Example 6** [SOA Introductory Derivatives Sample Question Q5]: The PS index has the following characteristics:

- One share of the PS index currently sells for 1,000.
- The PS index does not pay dividends.

Same wants to lock in the ability to buy this index in one year for a price of 1,025. He can do this by buying or selling European put and call options with a strike price of 1,025. The annual effective risk-free interest rate is 5%. Determine which of the following gives the hedging strategy that will achieve Sam’s objective and also gives the cost today of establishing this position.

(A) Buy the put and sell the call, receive 23.81
(B) Buy the put and sell the call, spend 23.81
(C) Buy the put and sell the call, no cost
(D) Buy the call and sell the put, receive 23.81
(E) Buy the call and sell the put, spend 23.81

**Solution:**
Exercise 6 [SOA Advanced Derivatives Sample Question Q1]: Consider a European call option and a European put option on a non-dividend-paying stock. You are given:

(i) The current price of the stock is 60.
(ii) The call option currently sells for 0.15 more than the put option.
(iii) Both the call option and put option will expire in 4 years.
(iv) Both the call option and put option have a strike price of 70.

Calculate the continuously compounded risk-free interest rate.

Example 7 [SOA Introductory Derivatives Sample Question Q72]: CornGrower is going to sell corn in one year. In order to lock in a fixed selling price, CornGrower buys a put option and sells a call option on each bushel, each with the same strike price and the same one-year expiration date. The current price of corn is 3.59 per bushel, and the net premium that CornGrower pays now to lock in the future price is 0.10 per bushel. The continuously compounded risk-free interest rate is 4%. Calculate the fixed selling price per bushel one year from now.

Solution:

Exercise 7 [SOA Advanced Derivatives Sample Question Q53]: For each ton of a certain type of rice commodity, the four-year forward price is 300. A four-year 400-strike European call option costs 110. The continuously compounded risk-free interest rate is 6.5%. Calculate the cost of a four-year 400-strike European put option for this rice commodity.

Spreads.

2.2.13 The investor longs 1 call with strike $K_1$ and shorts 1 call with strike $K_2$, where $K_1 < K_2$. The payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = (S_T - K_1)_+ - (S_T - K_2)_+ = \begin{cases} 
0 & \text{if } S_T \leq K_1 \\
S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\
K_2 - K_1 & \text{if } S_T > K_2
\end{cases}
\]

This combined position is called a $K_1$-$K_2$ call bull spread.
2.2.14 The investor longs 1 put with strike $K_1$ and shorts 1 put with strike $K_2$, where $K_1 < K_2$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = (K_1 - S_T)_+ - (K_2 - S_T)_+ = \begin{cases} -(K_2 - K_1) & \text{if } S_T \leq K_1 \\ S_T - K_2 & \text{if } K_1 < S_T \leq K_2 \\ 0 & \text{if } S_T > K_2 \end{cases}.$$ 

This combined position is called a $K_1$-$K_2$ put bull spread.

2.2.15 The investor shorts 1 call with strike $K_1$ and longs 1 call with strike $K_2$, where $K_1 < K_2$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = -(S_T - K_1)_+ + (S_T - K_2)_+ = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ K_1 - S_T & \text{if } K_1 < S_T \leq K_2 \\ -(K_2 - K_1) & \text{if } S_T > K_2 \end{cases}.$$ 

This combined position is called a $K_1$-$K_2$ call bear spread.

2.2.16 The investor shorts 1 put with strike $K_1$ and longs 1 put with strike $K_2$, where $K_1 < K_2$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = -(K_1 - S_T)_+ + (K_2 - S_T)_+ = \begin{cases} K_2 - K_1 & \text{if } S_T \leq K_1 \\ K_2 - S_T & \text{if } K_1 < S_T \leq K_2 \\ 0 & \text{if } S_T > K_2 \end{cases}.$$ 

This combined position is called a $K_1$-$K_2$ put bear spread.
2.2.17 The investor longs 1 synthetic forward with forward price $K_1$ and shorts 1 synthetic forward with forward price $K_2$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = (S_T - K_1) + (K_2 - S_T) = K_2 - K_1.$$ 

This combined position is called a **box spread**. Assuming $K_1 < K_2$, such a box spread can also be constructed by long 1 $K_1$-$K_2$ call bull spread and long 1 $K_1$-$K_2$ put bear spread.

2.2.18 The investor longs $m$ call (resp. put) with strike $K_1$ and shorts $n$ call (resp. put) with strike $K_2$, where $m \neq n$ and $K_1 \neq K_2$. This combined position is called a **ratio spread**. For example, if $K_1 < K_2$, $m = 1$, and $n = 3$, then the combined position at the expiration date is given by:

$$\text{Payoff} = (S_T - K_1)_+ - 3(S_T - K_2)_+ = \begin{cases} 
0 & \text{if } S_T \leq K_1 \\
S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\
-2S_T + 3K_2 - K_1 & \text{if } S_T > K_2
\end{cases}.$$ 

---

**Example 8** [SOA Introductory Derivatives Sample Question Q15]: The current price of a non-dividend paying stock is 40 and the continuously compounded risk-free interest rate is 8%. You enter into a short position on 3 call options, each with 3 months to maturity, a strike price of 35, and an option premium of 6.13. Simultaneously, you enter into a long position on 5 call options, each with 3 months to maturity, a strike price of 40, and an option premium of 2.78. All 8 options are held until maturity. Calculate the maximum possible profit and the maximum possible loss for the entire option portfolio.

**Solution:**
Exercise 8 [SOA Advanced Derivatives Sample Question Q39]: Determine which of the following strategies creates a ratio spread, assuming all options are European.

(A) Buy a one-year call, and sell a three-year call with the same strike price.
(B) Buy a one-year call, and sell a three-year call with a different strike price.
(C) Buy a one-year call, and buy three one-year calls with a different strike price.
(D) Buy a one-year call, and sell three one-year puts with a different strike price.
(E) Buy a one-year call, and sell three one-year calls with a different strike price.

Example 9 [SOA Introductory Derivatives Sample Question Q17]: The current price for a stock index is 1,000. The following premiums exist for various options to buy or sell the stock index six months from now:

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Call Premium</th>
<th>Put Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>950</td>
<td>120.41</td>
<td>51.78</td>
</tr>
<tr>
<td>1,000</td>
<td>93.81</td>
<td>74.20</td>
</tr>
<tr>
<td>1,050</td>
<td>71.80</td>
<td>101.21</td>
</tr>
</tbody>
</table>

Strategy I is to buy the 1,050-strike call and to sell the 950-strike call. Strategy II is to buy the 1,050-strike put and to sell the 950-strike put. Strategy III is to buy the 950-strike call, sell the 1,000-strike call, sell the 950-strike put, and buy the 1,000-strike put. Assume that the price of the stock index in 6 months will be between 950 and 1,050. Determine which, if any, of the three strategies will have greater payoffs in six months for lower prices of the stock index than for relatively higher prices.

(A) None
(B) I and II only
(C) I and III only
(D) II and III only
(E) The correct answer is not given by (A), (B), (C), or (D)

Solution:
Exercise 9 [SOA Introductory Derivatives Sample Question Q55]: Box spreads are used to guarantee a fixed cash flow in the future. Thus, they are purely a means of borrowing or lending money, and have no stock price risk. Consider a box spread based on two distinct strike price \((K, L)\) that is used to lend money, so that there is a positive cost to this transaction up front, but a guaranteed positive payoff at expiration. Determine which of the following sets of transactions is equivalent to this type of box spread.

(A) A long position in a \((K, L)\) bull spread using calls and a long position in a \((K, L)\) bear spread using puts.

(B) A long position in a \((K, L)\) bull spread using calls and a short position in a \((K, L)\) bear spread using puts.

(C) A long position in a \((K, L)\) bull spread using calls and a long position in a \((K, L)\) bull spread using puts.

(D) A short position in a \((K, L)\) bull spread using calls and a short position in a \((K, L)\) bear spread using puts.

(E) A short position in a \((K, L)\) bull spread using calls and a short position in a \((K, L)\) bull spread using puts.

**Collar.**

2.2.19 The investor longs 1 put with strike \(K_1\) and shorts 1 call with strike \(K_2\), where \(K_1 \leq K_2\). The payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = (K_1 - S_T)_+ - (S_T - K_2)_+ = \begin{cases} 
K_1 - S_T & \text{if } S_T \leq K_1 \\
0 & \text{if } K_1 < S_T \leq K_2 \\
K_2 - S_T & \text{if } S_T > K_2
\end{cases}
\]

This combined position is called a \(K_1\)-\(K_2\) collar. The difference between the strike prices \(K_2 - K_1\) is called the collar width.

2.2.20 The initial payment at time 0 for the \(K_1\)-\(K_2\) collar is given by \(P(K_1, T) - C(K_2, T)\). The collar is called zero-cost collar if \(P(K_1, T) = C(K_2, T)\).

2.2.21 For the investor who longs the underlying asset, if he/she longs the \(K_1\)-\(K_2\) collar, then the payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = S_T + (K_1 - S_T)_+ - (S_T - K_2)_+ = \begin{cases} 
K_1 & \text{if } S_T \leq K_1 \\
S_T & \text{if } K_1 < S_T \leq K_2 \\
K_2 & \text{if } S_T > K_2
\end{cases}
\]
This combined position is called a $K_1$-$K_2$ collared stock.

**Example 10** [SOA Introductory Derivatives Sample Question Q43]: You are given:

(i) An investor short-sells a non-dividend paying stock that has a current price of 44 per share.

(ii) This investor also writes a collar on this stock consisting of a 40-strike European put option and a 50-strike European call option. Both options expire in one year.

(iii) The prices of the options on this stock are:

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Call option</th>
<th>Put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>8.42</td>
<td>2.47</td>
</tr>
<tr>
<td>50</td>
<td>3.86</td>
<td>7.42</td>
</tr>
</tbody>
</table>

(iv) The continuously compounded risk-free interest rate is 5%.

(v) Assume there are no transaction costs.

Calculate the maximum profit for the overall position at expiration.

**Solution:**

---

**Exercise 10** [SOA Introductory Derivatives Sample Question Q3]: Happy Jalapenos, LLC has an exclusive contract to supply jalapeno peppers to the organizers of the annual jalapeno eating contest. The contract states that the contest organizers will take delivery of 10,000 jalapenos in one year at the market price. It will cost Happy Jalapenos 1,457,000 to provide 10,000 jalapenos and today’s market price is 0.12 per jalapeno. The continuously compounded risk-free interest rate is 6%. Happy Jalapenos has decided to hedge as follows: Buy 10,000 0.12-strike put options for 84.30 and sell 10,000 0.14-strike call options for 74.80. Both options are one-year European. Happy Jalapenos believes the market price in one year will be somewhere between 0.10 and 0.15 per jalapeno. Determine which of the following intervals represents the range of possible profit one year from now for Happy Jalapenos.

(A) −200 to 100
(B) −110 to 190
(C) −100 to 200
(D) 190 to 390
(E) 200 to 400
**Example 11** [SOA Introductory Derivatives Sample Question Q1]: Determine which statement about zero-cost purchased collars is FALSE:

(A) A zero-width, zero-cost collar can be created by setting both the put and call strike prices at the forward price.

(B) There are an infinite number of zero-cost collars.

(C) The put option can be at-the-money.

(D) The call option can be at-the-money.

(E) The strike price on the put option must be at or below the forward price.

**Solution:**

**Exercise 11** [SOA Introductory Derivatives Sample Question Q60]: Farmer Brown grows wheat, and will be selling his crop in 6 months. The current price of wheat is 8.50 per bushel. To reduce the risk of fluctuation in price, Brown wants to use derivatives with a 6-month expiration date to sell wheat between 8.60 and 8.80 per bushel. Brown also wants to minimize the cost of using derivatives. The continuously compounded risk-free interest rate is 2%. Which of the following strategies fulfills Farmer Brown’s objectives?

(A) Short a forward contract

(B) Long a call with strike 8.70 and short a put with strike 8.70

(C) Long a call with strike 8.80 and short a put with strike 8.60

(D) Long a put with strike 8.60

(E) Long a put with strike 8.60 and short a call with strike 8.80

**Straddle, strangle, and butterfly spread.**

2.2.22 The investor longs 1 call and 1 put, both with strike $K$. The payoff of the combined position at the expiration date is given by:

$$\text{Payoff} = (S_T - K)_+ + (K - S_T)_+ = |S_T - K| = \begin{cases} 
K - S_T & \text{if } S_T \leq K \\
S_T - K & \text{if } S_T > K 
\end{cases}$$
This combined position is called a **straddle** with strike \( K \).

**2.2.23** The investor longs 1 call with strike \( K_2 \) and 1 put with strike \( K_1 \), where \( K_1 \leq K_2 \). The payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = (S_T - K_2)_+ + (K_1 - S_T)_+ = \begin{cases} 
K_1 - S_T & \text{if } S_T \leq K_1 \\
0 & \text{if } K_1 < S_T \leq K_2 \\
S_T - K_2 & \text{if } S_T > K_2
\end{cases}.
\]

This combined position is called a \( K_1-K_2 \) **strangle**.

**2.2.24** The investor shorts 1 straddle with strike \( K_2 \) and longs 1 \( K_1-K_3 \) strangle, where \( K_1 \leq K_2 \leq K_3 \). The payoff of the combined position at the expiration date is given by:

\[
\text{Payoff} = -|S_T - K_2| + (S_T - K_3)_+ + (K_1 - S_T)_+ = \begin{cases} 
-(K_2 - K_1) & \text{if } S_T \leq K_1 \\
S_T - K_2 & \text{if } K_1 < S_T \leq K_2 \\
-S_T + K_2 & \text{if } K_2 < S_T \leq K_3 \\
-(K_2 - K_1) & \text{if } S_T > K_3
\end{cases}.
\]

This combined position is called a \( K_1-K_2-K_3 \) **butterfly spread**. There are other means of constructing the \( K_1-K_2-K_3 \) butterfly spread:

(i) long 1 call with strike \( K_1 \), short 2 calls with strike \( K_2 \), and long 1 call with strike \( K_3 \);
(ii) long 1 put with strike \( K_1 \), short 2 puts with strike \( K_2 \), and long 1 put with strike \( K_3 \);
(iii) long 1 \( K_1-K_2 \) call bull spread and long 1 \( K_2-K_3 \) call bear spread;
(iv) long 1 \( K_1-K_2 \) put bull spread and long 1 \( K_2-K_3 \) put bear spread.
Example 12 [SOA Introductory Derivatives Sample Question Q67]: Consider the following investment strategy involving put options on a stock with the same expiration date.

(i) Buy one 25-strike put
(ii) Sell two 30-strike puts
(iii) Buy one 35-strike put

Calculate the payoffs of this strategy assuming stock prices (i.e., at the time the put options expire) of 27 and 37, respectively.

Solution:

Exercise 12 [SOA Introductory Derivatives Sample Question Q8]: Joe believes that the volatility of a stock is higher than indicated by market prices for options on that stock. He wants to speculate on that belief by buying or selling at-the-money options. Determine which of the following strategies would achieve Joe’s goal.

(A) Buy a strangle
(B) Buy a straddle
(C) Sell a straddle
(D) Sell a butterfly spread
(E) Sell a butterfly spread
Learning Objectives:
2.3 Option Comparisons and Price Bounds: moneyness, put-call parity, at any time, European call price bound, European put price bound, American worths more than European, non-dividend-paying, early exercise call sub-optimal, zero risk-free rate, early exercise put sub-optimal, American call price bound, American put price bound, monotonicity, Lipschitz property, convexity, term-to-maturity.
2.3 Option Comparisons and Price Bounds

Comparing options with respect to price of underlying asset.

2.3.1 In 2.1.10, moneyness of options were defined only at the issuance time 0. Similar concepts can be defined also at any time \( t \in [0, T] \), which is not necessarily the exercise time \( \tau \), i.e. options are not necessarily exercisable at time \( t \). See also Example 2.

2.3.2 A call option is called out-of-the-money at time \( t \) if \( S_t < K \), at-the-money at time \( t \) if \( S_t = K \), and in-the-money at time \( t \) if \( S_t > K \); a put option is called in-the-money at time \( t \) if \( S_t < K \), at-the-money at time \( t \) if \( S_t = K \), and out-of-the-money at time \( t \) if \( S_t > K \).

Put-call parity.

2.3.3 With the strike price \( K \) and the expiration date \( T \), denote the European call option price at time \( t \in [0, T] \) by \( C_E(t, S_t, K, T) \) and the European put option price at time \( t \in [0, T] \) by \( P_E(t, S_t, K, T) \).

2.3.4 In 2.2.12, the put-call parity, which depicts the relationship between the European call option price at time 0 and the European put option price at time 0, with the same strike price \( K \) and the expiration date \( T \), should be more precisely written as:

\[
C_E(0, S_0, K, T) - P_E(0, S_0, K, T) = F_{0,T}^{P} - PV_{0,T}(K).
\]

2.3.5 The put-call parity at time 0 can be easily generalized to that at time \( t \):

\[
C_E(t, S_t, K, T) - P_E(t, S_t, K, T) = F_{t,T}^{P} - PV_{t,T}(K),
\]

where \( F_{t,T}^{P} \) is the prepaid forward price at time \( t \) of the prepaid forward contract, which obviously depends on \( S_t \). Indeed, construct two investment strategies \( \pi^1 \) and \( \pi^2 \) at time \( t \) and hold till time \( T \). The \( \pi^1 \) consists of long 1 European call and short 1 European put, both with strike \( K \) and expiration date \( T \); the \( \pi^2 \) consists of long 1 prepaid forward on the same underlying asset with expiration date \( T \), and short 1 bond with face-value \( K \) matured at \( T \). Then,

\[
V_{T}^\pi^1 = (S_T - K)_{+} - (K - S_T)_{+} = S_T - K = V_{T}^\pi^2.
\]

By the law of one-price, \( C_E(t, S_t, K, T) - P_E(t, S_t, K, T) = V_{t}^\pi^1 = V_{t}^\pi^2 = F_{t,T}^{P} - PV_{t,T}(K) \).
Bounds on European option prices.

2.3.6 The following provides bounds on the European call option price at time $t$ with the strike price $K$ and the expiration date $T$:

\[
(F_{t,T}^C - PV_{t,T}(K))_+ \leq C_E(t, S_t, K, T) \leq F_{t,T}^C.
\]

Indeed, assume to the contrary that, $F_{t,T}^C - PV_{t,T}(K) > C_E(t, S_t, K, T)$. Then, construct an investment strategy $\pi^1$ at time $t$ and hold till $T$, consisting of short 1 prepaid forward, long 1 European call, and long 1 bond with face-value $FV_{t,T} (F_{t,T}^C - C_E(t, S_t, K, T))$. It is easy to check that

\[
V_{t}^{\pi^1} = -F_{t,T}^C + C_E(t, S_t, K, T) + (F_{t,T}^C - C_E(t, S_t, K, T)) = 0,
\]

\[
V_{T}^{\pi^1} = -S_T + (S_T - K)_+ + FV_{t,T} (F_{t,T}^C - C_E(t, S_t, K, T))
\geq -K + FV_{t,T} (F_{t,T}^C - C_E(t, S_t, K, T))
= FV_{t,T} (F_{t,T}^C - PV_{t,T}(K) - C_E(t, S_t, K, T)) > 0.
\]

Hence, $\pi^1$ is an arbitrage strategy. By no-arbitrage assumption, $F_{t,T}^C - PV_{t,T}(K) \leq C_E(t, S_t, K, T)$.

Finally, since a price must be non-negative, $(F_{t,T}^C - PV_{t,T}(K))_+ \leq C_E(t, S_t, K, T)$.

Also, assume to the contrary that, $C_E(t, S_t, K, T) > F_{t,T}^P$. Then, construct an investment strategy $\pi^2$ at time $t$ and hold till $T$, consisting of short 1 European call, long 1 prepaid forward, and long 1 bond with face-value $FV_{t,T} (C_E(t, S_t, K, T) - F_{t,T}^P)$. It is easy to check that

\[
V_{t}^{\pi^2} = -C_E(t, S_t, K, T) + F_{t,T}^P + (C_E(t, S_t, K, T) - F_{t,T}^P) = 0,
\]

\[
V_{T}^{\pi^2} = -(S_T - K)_+ + S_T + FV_{t,T} (C_E(t, S_t, K, T) - F_{t,T}^P) > 0.
\]

Hence, $\pi^2$ is an arbitrage strategy. By no-arbitrage assumption, $C_E(t, S_t, K, T) \leq F_{t,T}^P$.

2.3.7 Using similar no-arbitrage arguments, the following provides bounds on the European put option price at time $t$ with the strike price $K$ and the expiration date $T$:

\[
(PV_{t,T}(K) - F_{t,T}^P)_+ \leq P_E(t, S_t, K, T) \leq PV_{t,T}(K).
\]
**Example 13:** You are given:

(i) The spot price of a stock is 35.

(ii) The stock pays continuous dividends proportional to its price at an annual rate of 4%.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) A one-year European call option on the stock has a strike price of 32.

If the observed time-0 price of the call option is 2, construct an arbitrage strategy; if the observed time-0 price of the call option is 40, construct another arbitrage strategy.

**Solution:**

**Exercise 13:** Prove the bounds on the European put option price:

\[ (PV_{t,T}(K) - F^P_{t,T})_+ \leq P_E(t, S_t, K, T) \leq PV_{t,T}(K). \]
Comparing European and American options.

2.3.8 With the strike price $K$ and the expiration date $T$, denote the American call option price at time $t \in [0, T]$ by $C_A(t, S_t, K, T)$ and the American put option price at time $t \in [0, T]$ by $P_A(t, S_t, K, T)$.

2.3.9 Comparing to the European counterparts, American options provide opportunities to early exercise at any time $\tau \in [0, T]$. Hence, an American option should intuitively worth at least as much as the European counterpart:

$$C_A(t, S_t, K, T) \geq C_E(t, S_t, K, T);$$
$$P_A(t, S_t, K, T) \geq P_E(t, S_t, K, T).$$

Indeed, assume to the contrary that, $C_A(t, S_t, K, T) < C_E(t, S_t, K, T)$. Then, construct an investment strategy $\pi$ at time $t$, consisting of short 1 European call, long 1 American call, and long 1 bond with face-value $FV_{t,T}(C_E(t, S_t, K, T) - C_A(t, S_t, K, T) \right)$, do not exercise the American call at any time $\tau \in [t, T)$, and hold till $T$. It is easy to check that

$$V_\pi^t = -C_E(t, S_t, K, T) + C_A(t, S_t, K, T) + (C_E(t, S_t, K, T) - C_A(t, S_t, K, T)) = 0,$$

$$V_\pi^T = -(S_T - K)_+ + (S_T - K)_+ + FV_{t,T}(C_E(t, S_t, K, T) - C_A(t, S_t, K, T))$$
$$= FV_{t,T}(C_E(t, S_t, K, T) - C_A(t, S_t, K, T)) > 0.$$

Hence, $\pi$ is an arbitrage strategy. By no-arbitrage assumption, $C_A(t, S_t, K, T) \geq C_E(t, S_t, K, T)$. The inequality for American and European put options can be proved by similar no-arbitrage arguments.

2.3.10 For a non-dividend-paying underlying asset, early exercising an American call option is never optimal, and hence

$$C_A(t, S_t, K, T) = C_E(t, S_t, K, T).$$

Indeed, since the American call option always worths at least as much as the European call option, together with the lower-bound of European call option price in 2.3.6,

$$C_A(t, S_t, K, T) \geq C_E(t, S_t, K, T) \geq F^p_{t,T} - PV_{t,T}(K) = S_t - PV_{t,T}(K) \geq S_t - K.$$

Therefore, for any time $t \in [0, T]$, selling the American call with price $C_A(t, S_t, K, T)$ always yields a higher or equal profit than early exercising the American call.

2.3.11 For a zero risk-free interest rate, early exercising an American put option is never optimal, and hence
Indeed, since the American put option always worths at least as much as the European put option, together with the lower-bound of European put option price in 2.3.7,

\[ P_A(t, S_t, K, T) \geq P_E(t, S_t, K, T) \geq PV_{t,T}(K) - F_{t,T}^P = K - F_{t,T}^P \geq K - S_t. \]

Therefore, for any time \( t \in [0, T] \), selling the American put with price \( P_A(t, S_t, K, T) \) always yields a higher or equal profit than early exercising the American put.

**Example 14:** You are given:
- (i) The spot price of a stock is 35.
- (ii) The stock pays continuous dividends proportional to its price at an annual rate of 4%.
- (iii) The continuously compounded risk-free interest rate is 6%.
- (iv) A one-year European call option on the stock has a strike price of 32.
- (v) A one-year American call option on the stock also has a strike price of 32.

If the observed time-0 price of the European call option is 30, while that of the American call option is 20, construct an arbitrage strategy.

**Solution:**

**Exercise 14:** Prove that the American put option worths at least as much as the European put option counterpart:

\[ P_A(t, S_t, K, T) \geq P_E(t, S_t, K, T). \]
2.3.12 The following provides **bounds on the American call option price** at time $t$ with the strike price $K$ and the expiration date $T$:

$$\max \left\{ (S_t - K)_+, \left( F_{t,T}^p - PV_{t,T}(K) \right)_+ \right\} \leq C_A(t, S_t, K, T) \leq S_t.$$ 

Since the American call option always worths at least as much as the European call counterpart, together with the lower-bound of European call option price in 2.3.6,

$$\left( F_{t,T}^p - PV_{t,T}(K) \right)_+ \leq C_E(t, S_t, K, T) \leq C_A(t, S_t, K, T).$$

Assume to the contrary that, $S_t - K > C_A(t, S_t, K, T)$. Then, construct an investment strategy $\pi_1$ at time $t$, consisting of long 1 bond with face-value $FV_{t,T}(S_t - K - C_A(t, S_t, K, T))$ and long 1 American call, exercise the American call immediately at time $t$, and hold the bond till $T$. It is easy to check that

$$V_{t}^{\pi_1} = (S_t - K - C_A(t, S_t, K, T)) + C_A(t, S_t, K, T) - (S_t - K) = 0,$$

$$V_{T}^{\pi_1} = FV_{t,T}(S_t - K - C_A(t, S_t, K, T)) > 0.$$ 

Hence, $\pi_1$ is an arbitrage strategy. By no-arbitrage assumption, $S_t - K \leq C_A(t, S_t, K, T)$.

Finally, since a price must be non-negative, $(S_t - K)_+ \leq C_A(t, S_t, K, T)$.

Also, assume to the contrary that, $C_A(t, S_t, K, T) > S_t$. Then, construct an investment strategy $\pi_2$ at time $t$, consisting of short 1 American call, long 1 underlying asset, and long 1 bond with face-value $FV_{t,T}(C_A(t, S_t, K, T) - S_t)$. It is easy to check that

$$V_{t}^{\pi_2} = -C_A(t, S_t, K, T) + S_t + (C_A(t, S_t, K, T) - S_t) = 0.$$

If the long party exercises the American call at a time $\tau \in [t, T]$, then

$$V_{\tau}^{\pi_2} = (S_\tau - K) + S_\tau + FV_{t,\tau}(\text{Div}) + FV_{t,\tau} \left( C_A(t, S_t, K, T) - S_t \right)$$

$$= K + FV_{t,\tau}(\text{Div}_{t,\tau}) + FV_{t,\tau} \left( C_A(t, S_t, K, T) - S_t \right),$$

while $V_{T}^{\pi_2} = FV_{t,T}(K) + FV_{t,T}(\text{Div}_{t,T}) + FV_{t,T} \left( C_A(t, S_t, K, T) - S_t \right) > 0$. If the long party does not exercise the American call by the expiration date $T$, then

$$V_{T}^{\pi_2} = S_T + FV_{t,T}(\text{Div}) + FV_{t,T} \left( C_A(t, S_t, K, T) - S_t \right) > 0.$$ 

In both cases, $\pi_2$ is an arbitrage strategy. By no-arbitrage assumption, $C_A(t, S_t, K, T) \leq S_t$.

2.3.13 Using similar no-arbitrage arguments, the following provides **bounds on the American**...
put option price at time $t$ with the strike price $K$ and the expiration date $T$:

$$\max \left\{ (K - S_t)_+, (PV_{t,T}(K) - F_{t,T}^P)_+ \right\} \leq P_A(t, S_t, K, T) \leq K.$$

Example 15 [SOA Advanced Derivatives Sample Question Q26]: Consider European and American options on a non-dividend-paying stock. You are given:

(i) All options have the same strike price 100.
(ii) All options expire in six months.
(iii) The continuously compounded risk-free interest rate is 10%.

You are interested in the graph for the price of an option as a function of the current stock price. In each of the following four charts I–IV, the horizontal axis, $S$, represents the current stock price, and the vertical axis, $\pi$, represents the price of an option.

Match the option with the shaded region with its graph lies. If there are two or more possibilities, choose the chart with the smallest shaded region.

<table>
<thead>
<tr>
<th>European Call</th>
<th>American Call</th>
<th>European Put</th>
<th>American Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) I</td>
<td>I</td>
<td>III</td>
<td>III</td>
</tr>
<tr>
<td>(B) II</td>
<td>I</td>
<td>IV</td>
<td>III</td>
</tr>
<tr>
<td>(C) II</td>
<td>I</td>
<td>III</td>
<td>III</td>
</tr>
<tr>
<td>(D) II</td>
<td>II</td>
<td>IV</td>
<td>III</td>
</tr>
<tr>
<td>(E) II</td>
<td>II</td>
<td>IV</td>
<td>IV</td>
</tr>
</tbody>
</table>

Solution:
Exercise 15: Prove the bounds on the American put option price:
\[
\max \left\{ (K - S_t)_+ + PV_{t,T}(K) - P^P_{t,T} \right\} \leq P_A(t, S_t, K, T) \leq K.
\]

Comparing options with respect to strike price.

2.3.14 With the same expiration date \(T\), the European call option price at time \(t\) is a **non-increasing** function in the strike price: for any \(K_1 \leq K_2\),

\[
C_E(t, S_t, K_1, T) \geq C_E(t, S_t, K_2, T).
\]

Assume to the contrary that, \(C_E(t, S_t, K_1, T) < C_E(t, S_t, K_2, T)\). Then, construct an investment strategy \(\pi\) at time \(t\) and hold till \(T\), consisting of short 1 \(K_2\)-strike European call, long 1 \(K_1\)-strike European call, and long 1 bond with face-value \(FV_{t,T}(C_E(t, S_t, K_2, T) - C_E(t, S_t, K_1, T))\).

It is easy to check that

\[
V_\pi^t = -C_E(t, S_t, K_2, T) + C_E(t, S_t, K_1, T) + (C_E(t, S_t, K_2, T) - C_E(t, S_t, K_1, T)) = 0,
\]

\[
V_\pi^T = -(S_T - K_2)_+ + (S_T - K_1)_+ + FV_{t,T}(C_E(t, S_t, K_2, T) - C_E(t, S_t, K_1, T)) > 0.
\]

Hence, \(\pi\) is an arbitrage strategy. By no-arbitrage assumption, \(C_E(t, S_t, K_1, T) \geq C_E(t, S_t, K_2, T)\).

2.3.15 Using similar no-arbitrage arguments, with the same expiration date \(T\), the European put option price at time \(t\) is a **non-decreasing** function in the strike price: for any \(K_1 \leq K_2\),

\[
P_E(t, S_t, K_1, T) \leq P_E(t, S_t, K_2, T).
\]

2.3.16 Similarly, with the same expiration date \(T\), the American call (resp. put) option price at time \(t\) is a **non-increasing** (resp. **non-decreasing**) function in the strike price: for any \(K_1 \leq K_2\),

\[
C_A(t, S_t, K_1, T) \geq C_A(t, S_t, K_2, T);
\]

\[
P_A(t, S_t, K_1, T) \leq P_A(t, S_t, K_2, T).
\]
2.3.17 With the same expiration date $T$, the European (resp. American) call (resp. put) option price at time $t$ is a \textbf{Lipschitz} function in the strike price: for any $K_1 \leq K_2$,

\[
C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T) \leq PV_{t,T}(K_2 - K_1);
\]

\[
P_E(t, S_t, K_2, T) - P_E(t, S_t, K_1, T) \leq PV_{t,T}(K_2 - K_1);
\]

\[
C_A(t, S_t, K_1, T) - C_A(t, S_t, K_2, T) \leq K_2 - K_1;
\]

\[
P_A(t, S_t, K_2, T) - P_A(t, S_t, K_1, T) \leq K_2 - K_1.
\]

Assume to the contrary that, $C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T) > PV_{t,T}(K_2 - K_1)$. Then, construct an investment strategy $\pi$ at time $t$ and hold till $T$, consisting of short 1 $K_1$-strike European call, long 1 $K_2$-strike European call, and long 1 bond with face-value $FV_{t,T}(C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T))$. It is easy to check that

\[
V_t^\pi = -C_E(t, S_t, K_1, T) + C_E(t, S_t, K_2, T) + (C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T)) = 0,
\]

\[
V_T^\pi = -(S_T - K_1) + (S_T - K_2) + PV_{t,T}(C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T))
\]

\[\geq -(K_2 - K_1) + PV_{t,T}(C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T))
\]

\[= PV_{t,T}(C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T) - PV_{t,T}(K_2 - K_1)) > 0.
\]

Hence, $\pi$ is an arbitrage strategy. By no-arbitrage assumption, $C_E(t, S_t, K_1, T) - C_E(t, S_t, K_2, T) \leq PV_{t,T}(K_2 - K_1)$. The other three inequalities can be proved by similar no-arbitrage arguments.

2.3.18 With the same expiration date $T$, the European (resp. American) call (resp. put) option price at time $t$ is a \textbf{convex} function in the strike price: for any $K_1, K_2, \lambda \in [0, 1]$,

\[
C_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) \leq \lambda C_E(t, S_t, K_1, T) + (1 - \lambda)C_E(t, S_t, K_2, T);
\]

\[
P_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) \leq \lambda P_E(t, S_t, K_1, T) + (1 - \lambda)P_E(t, S_t, K_2, T);
\]

\[
C_A(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) \leq \lambda C_A(t, S_t, K_1, T) + (1 - \lambda)C_A(t, S_t, K_2, T);
\]

\[
P_A(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) \leq \lambda P_A(t, S_t, K_1, T) + (1 - \lambda)P_A(t, S_t, K_2, T).
\]

Assume to the contrary that, $C_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) > \lambda C_E(t, S_t, K_1, T) + (1 - \lambda)C_E(t, S_t, K_2, T)$. Then, construct an investment strategy $\pi$ at time $t$ and hold till $T$, consisting of short 1 $(\lambda K_1 + (1 - \lambda)K_2)$-strike European call, long $\lambda$ $K_1$-strike European call, long $(1 - \lambda)$ $K_2$-strike European call, and long 1 bond with face-value $FV_{t,T}(C_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) - \lambda C_E(t, S_t, K_1, T) - (1 - \lambda)C_E(t, S_t, K_2, T))$. It is easy to check that

\[
V_t^\pi = -C_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) + \lambda C_E(t, S_t, K_1, T) + (1 - \lambda)C_E(t, S_t, K_2, T)
\]

\[+ (C_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) - \lambda C_E(t, S_t, K_1, T) - (1 - \lambda)C_E(t, S_t, K_2, T)) = 0,
\]
\[ V_T^\pi = -(S_T - (\lambda K_1 + (1 - \lambda)K_2)_+ + \lambda (S_T - K_1)_+ + (1 - \lambda)(S_T - K_2)_+ \\
+ FV_{t,T}(C_E(t,S_t,\lambda K_1 + (1 - \lambda)K_2, T) - \lambda C_E(t,S_t,K_1, T) - (1 - \lambda)C_E(t,S_t,K_2, T)) > 0. \]

Hence, \( \pi \) is an arbitrage strategy. By no-arbitrage assumption, 
\[ C_E(t,S_t,\lambda K_1 + (1 - \lambda)K_2, T) \leq \lambda C_E(t,S_t,K_1, T) + (1 - \lambda)C_E(t,S_t,K_2, T). \]

The other three inequalities can be proved by similar no-arbitrage arguments.

2.3.19 The following graphs summarize the **monotonicity**, **Lipschitz property**, and **convexity** of option prices with respect to the strike price.

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**Example 16** [SOA Advanced Derivatives Sample Question Q2]: Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Call Price</th>
<th>Put Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$40</td>
<td>$11</td>
<td>$3</td>
</tr>
<tr>
<td>$50</td>
<td>$6</td>
<td>$8</td>
</tr>
<tr>
<td>$55</td>
<td>$3</td>
<td>$11</td>
</tr>
</tbody>
</table>

All six options have the same expiration date.

After reviewing the information above, John tells Mary and Peter that no arbitrage opportunities can arise from these prices.

Mary disagrees with John. She argues that one could use the following portfolio to obtain arbitrage profit: Long one call option with strike price 40; short three call options with strike price 50; lend $1; and long some calls with strike price 55.

Peter also disagrees with John. He claims that the following portfolio, which is different from Mary’s, can product arbitrage profit: Long 2 calls and short 2 puts with strike price 55; long 1 call and short 1 put with strike price 40; lend $2; and short some calls and long the same number of puts with strike price 50.

Which of the following statements is true?

(A) Only John is correct.
(B) Only Mary is correct.
(C) Only Peter is correct.
(D) Both Mary and Peter are correct.
(E) None of them is correct.

---

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Exercise 16: Prove all of the following inequalities: for any $K_1 \leq K_2$, $\lambda \in [0, 1]$,

\[
\begin{align*}
P_E(t, S_t, K_1, T) &\leq P_E(t, S_t, K_2, T); \\
P_A(t, S_t, K_1, T) &\leq P_A(t, S_t, K_2, T); \\
P_E(t, S_t, K_2, T) - P_E(t, S_t, K_1, T) &\leq PV_{t,T}(K_2 - K_1); \\
P_A(t, S_t, K_2, T) - P_A(t, S_t, K_1, T) &\leq K_2 - K_1; \\
P_E(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) &\leq \lambda P_E(t, S_t, K_1, T) + (1 - \lambda)P_E(t, S_t, K_2, T); \\
P_A(t, S_t, \lambda K_1 + (1 - \lambda)K_2, T) &\leq \lambda P_A(t, S_t, K_1, T) + (1 - \lambda)P_A(t, S_t, K_2, T).
\end{align*}
\]
Comparing options with respect to term-to-maturity

2.3.20 For an option with expiration date $T$, its term-to-maturity at time $t \in [0, T]$ is defined by $T - t$.

2.3.21 With the same strike price $K$, the American call (resp. put) option price at time $t$ is intuitively a non-decreasing function in the term-to-maturity: for any $T_1 \leq T_2$,

\[
C_A(t, S_t, K, T_1) \leq C_A(t, S_t, K, T_2);
\]

\[
P_A(t, S_t, K, T_1) \leq P_A(t, S_t, K, T_2).
\]

Assume to the contrary that, $C_A(t, S_t, K, T_1) > C_A(t, S_t, K, T_2)$. Then, construct an investment strategy $\pi$ at time $t$, consisting of short 1 American call, with term-to-maturity $T_1 - t$, long 1 American call, with term-to-maturity $T_2 - t$, and long 1 bond with face-value $FV_{t,T_2}(C_A(t, S_t, K, T_1) - C_A(t, S_t, K, T_2))$. It is easy to check that

\[
V_\pi^t = -C_A(t, S_t, K, T_1) + C_A(t, S_t, K, T_2) + (C_A(t, S_t, K, T_1) - C_A(t, S_t, K, T_2)) = 0.
\]

If the long party exercises the American call, with term-to-maturity $T_1 - t$, at a time $\tau \in [t, T_1]$, then exercise the American call, with term-to-maturity $T_2 - t$, immediately at time $\tau$, and hence

\[
V_\pi^\tau = -(S_\tau - K) + (S_\tau - K) + FV_{t,\tau} (C_A(t, S_t, K, T_1) - C_A(t, S_t, K, T_2))
\]

\[= FV_{t,\tau} (C_A(t, S_t, K, T_1) - C_A(t, S_t, K, T_2)),
\]

while $V_\pi^{T_2} = FV_{t,T_2}(C_A(t, S_t, K, T_1) - C_A(t, S_t, K, T_2)) > 0$. If the long party does not exercise the American call, with term-to-maturity $T_1 - t$, by the expiration date $T_1$, then do not exercise the American call, with term-to-maturity $T_2 - t$, at any time $\tau \in [t, T_2]$, and hence

\[
V_\pi^{T_2} = (S_{T_2} - K)_+ + FV_{t,T_2}(C_A(t, S_t, K, T_1) - C_A(t, S_t, K, T_2)) > 0.
\]

In both cases, $\pi$ is an arbitrage strategy. By no-arbitrage assumption, $C_A(t, S_t, K, T_1) \leq C_A(t, S_t, K, T_2)$. The inequality for American put options can be proved by similar no-arbitrage arguments.

2.3.22 For a non-dividend-paying underlying asset, with the same strike price $K$, the European call option price at time $t$ is a non-decreasing function in the term-to-maturity: for any $T_1 \leq T_2$,

\[
C_E(t, S_t, K, T_1) \leq C_E(t, S_t, K, T_2).
\]

Indeed, by the fact that early exercising an American call option is never optimal (c.f. 2.3.10),

\[
C_E(t, S_t, K, T_1) = C_A(t, S_t, K, T_1) \leq C_A(t, S_t, K, T_2) = C_E(t, S_t, K, T_2).
\]
2.3.23 For a zero risk-free interest rate, with the same strike price $K$, the European put option price at time $t$ is a non-decreasing function in the term-to-maturity: for any $T_1 \leq T_2$,

$$P_E(t, S_t, K, T_1) \leq P_E(t, S_t, K, T_2).$$

Indeed, by the fact that early exercising an American put option is never optimal (c.f. 2.3.11),

$$P_E(t, S_t, K, T_1) = P_A(t, S_t, K, T_1) \leq P_A(t, S_t, K, T_2) = P_E(t, S_t, K, T_2).$$

**Example 17** [CAS Exam 3 Fall 2007 Q13]: Given the following chart about call options on a particular dividend-paying stock, which option has the highest value?

<table>
<thead>
<tr>
<th>Option</th>
<th>Option Style</th>
<th>Time Until Expiration</th>
<th>Strike Price</th>
<th>Stock Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>European</td>
<td>1 year</td>
<td>50</td>
<td>42</td>
</tr>
<tr>
<td>B</td>
<td>American</td>
<td>1 year</td>
<td>50</td>
<td>42</td>
</tr>
<tr>
<td>C</td>
<td>European</td>
<td>2 year</td>
<td>50</td>
<td>42</td>
</tr>
<tr>
<td>D</td>
<td>American</td>
<td>2 year</td>
<td>50</td>
<td>42</td>
</tr>
<tr>
<td>E</td>
<td>American</td>
<td>2 year</td>
<td>55</td>
<td>42</td>
</tr>
</tbody>
</table>

**Solution:**

**Exercise 17**: Prove that the American put option price is a non-decreasing function in the term-to-maturity: for any $T_1 \leq T_2$,

$$P_A(t, S_t, K, T_1) \leq P_A(t, S_t, K, T_2).$$