Learning Objectives:

1.1 Derivative Markets: derivative, underlying asset, forward/futures, option, long, short, obligation, exercise, right, regulated exchanges, OTC, hedging, speculation, transaction costs, regulatory arbitrage.

1.2 Forward Contracts: forward, underlying asset, expiration date, forward price, long, short, position, obligation, spot price, payoff and profit, diagrams, no initial payment, fair forward price, no-arbitrage principle, investment strategy, value process, arbitrage strategy, cash flow, no-cost, arbitrage opportunity, free lunch, no-arbitrage pricing principle, arbitrage-free, law of one-price, buy-low-sell-high argument, replicating portfolio, perfect hedge, forward prices, non-dividend-paying stock, discrete dividends, continuous dividend, synthetic long/short forward, expected rate of return, expected stock price greater.

1.3 Variations on Forwards: purchase method, payment time, asset delivery time, payment amount, outright purchase, spot price, forward contract, forward price, fully leveraged purchase, full loan, prepaid forward contract, prepaid forward price, futures contract, exchange-based forward, mark-to-market, daily settlement, liquidity, offset, opposite position, standardization, credit risk minimized, price limit, temporary halt trigger, margin account, initial margin, notional value, multiplier, earn interest, debit/credit, maintenance margin, margin call, additional deposit, close position.

Further Exercises:
SOA Introductory Derivatives Samples Questions 7, 10, 38, 51, and 52.
1.1 Derivatives Markets

What is a derivative security?

1.1.1 A derivative is a financial instrument which has a value determined by the price of an underlying asset, which could be an equity, an index, a commodity, a currency, or even another derivative. Forwards/Futures and options are important examples of derivative contracts.

1.1.2 Suppose that Alfred and John enter an agreement that, after one week, Alfred will have to pay John 70 USD for buying a barrel of crude oil. This is an example of forwards/futures with the barrel of crude oil as the underlying asset. Consider the following two scenarios.

- The oil price per barrel falls below 70 USD, say 65 USD. Then Alfred will have to pay 70 USD to John, while John will deliver a barrel of crude oil to Alfred. In this case, Alfred experiences a loss of 5 USD, which is the amount of a gain by John.

- The oil price per barrel rises above 70 USD, say 75 USD. Then Alfred will have to pay 70 USD to John, while John will deliver a barrel of crude oil to Alfred. In this case, Alfred experiences a gain of 5 USD, which is the amount of a loss by John.

Alfred and John are respectively the buyer and the seller of this forward/futures contract. Alfred longs the contract, while John shorts it. Both Alfred and John bear their obligations once the contract is signed.

1.1.3 Suppose that Alfred and John enter a different agreement that, after a week, John will choose to pay Alfred 70 USD for purchasing a barrel of crude oil. This is an example of options with the barrel of crude oil as the underlying asset. Consider the same two scenarios.

- The oil price per barrel falls below 70 USD, say 65 USD. Then John will choose to not exercise the option, while Alfred will not deliver a barrel of crude oil to John. In this case, both Alfred and John experience zero gain or loss.

- The oil price per barrel rises above 70 USD, say 75 USD. Then John will choose to exercise the option and pay 70 USD to Alfred, while Alfred will deliver a barrel of crude oil to John. In this case, John experiences a gain of 5 USD, which is the amount of a loss by Alfred.

Alfred and John are respectively the seller and the buyer of this option contract. Alfred is in a short position, while John is in a long position. John holds his right of exercising the contract once it is signed, while Alfred bears his passive obligation to John.

Where are derivative securities traded?

1.1.4 Derivative contracts are mostly traded in regulated exchanges, such as the Chicago Mercantile Exchange (CME), the Chicago Board Options Exchange (CBOE), and the CBOE Futures Exchange (CFE):
1.1.5 Derivative contracts can also be traded over-the-counter (OTC), which bypasses the regulated exchanges. Firstly, OTC is easier to trade a large quantity directly between two parties. Secondly, custom financial claims are available in OTC but not in exchanges. Finally, OTC allows to trade a number of different financial claims at once. The two examples in 1.1.2
and 1.1.3 for forwards/futures and options are OTC.

**Why are derivative securities traded?**

1.1.6 Derivatives provide an alternative to a simple sale or purchase the underlying asset, and thus increase the range of possibilities for an investor or manager seeking to accomplish some goal, which includes the followings.

- **Risk management/Hedging/Insurance**: Derivatives are a tool for an individual or a company to share the risk with another entity. In 1.1.2, if Alfred foresees that a barrel of crude oil will be a necessity to him after a week, then such a forward/futures contract with John allows Alfred to lock the price at the agreed amount 70 USD. Essentially, a random payment for the barrel of crude oil is transformed to the constant agreed price.

- **Speculation**: Derivatives can serve as investment vehicles. In 1.1.3, if John anticipates that the oil price per barrel will increase, then such an option contract with Alfred allows John to make a bet on his belief with a little initial investment.

- **Reduced transaction costs**: Sometimes derivatives provide a lower-cost way to undertake a particular financial transaction.

- **Regulatory arbitrage**: It is sometimes possible to circumvent regulatory restrictions, taxes, and accounting rules by trading derivatives.

**Example 1** [SOA Introductory Derivatives Sample Question 24]: Determine which of the following statements is NOT a typical reason for why derivative securities are used to manage financial risk.

(A) Derivatives are used as a means of hedging.

(B) Derivatives are used to reduce the likelihood of bankruptcy.

(C) Derivatives are used to reduce transaction costs.

(D) Derivatives are used to satisfy regulatory, tax, and accounting constraints.

(E) Derivatives are used as a form of insurance.

**Exercise 1**: Suppose that Daniel and Klara enter an agreement that, after one year, Klara will have to deliver Daniel a piece of gold, while Daniel will have to pay Klara 1,000 USD. Determine which of the following statements is NOT correct.

(A) Daniel locks the purchasing price of the piece of gold at 1,000 USD after one year.

(B) After one year, if the value of the piece of gold depreciates to 900 USD, then Klara experiences a loss of 100 USD.

(C) The underlying asset is the piece of gold.

(D) Their agreement is called the forward contract.

(E) After one year, if the value of the piece of gold appreciates to 1,100 USD, then Daniel experiences a gain of 100 USD.

**Solution:**
1.2 Forward Contracts

What is a forward contract?

1.2.1 A forward contract on an underlying asset is an agreement between two parties to buy or sell the underlying asset at an expiration date and at a forward price; the party who longs the forward contract will have to buy the underlying asset, while the party who shorts the forward contract will have to sell the underlying asset. Both parties bear their obligations once the contract is signed.

1.2.2 The four key terms describe the obligation on a forward contract for a party.

- Underlying asset: The asset that the forward is written on.
- Expiration date: The time when the forward will be settled with the forward price.
- Forward price: The amount which will be paid/received to settle the forward at the expiration date.
- Long/Short: The position of the party who will buy/sell the underlying asset.

1.2.3 Suppose that the time when the forward contract is signed is $t = 0$. Denote the price of the underlying asset by $S_t$, for $t \geq 0$, with $S_0$ known as the spot price; denote the expiration date by $T \geq 0$; denote the forward price by $F_{0,T}$.

1.2.4 Recall the example in 1.1.2: Alfred and John sign a forward contract that, after one week, Alfred will pay John 70 USD for buying a barrel of crude oil. The underlying asset is the barrel of crude oil; the expiration date $T = 1$ week; the forward price is $F_{0,T} = 70$ USD; Alfred longs the forward, while John shorts it.

Payoff and profit.

1.2.5 For the long party, his payoff of the forward contract at the expiration date is given by:

\[ \text{Payoff of long forward} = S_T - F_{0,T}. \]

For the short party, his payoff of the forward contract at the expiration date is given by:

\[ \text{Payoff of short forward} = F_{0,T} - S_T. \]

Signing the forward contract is a zero-sum game for these two parties, since the sum of payoffs of long and short positions is zero.

1.2.6 The profit of a derivative contract at the expiration date is given by:

\[ \text{Profit of contract} = \text{Payoff of contract} - FV_{0,T} (\text{Cost of contract}). \]
The payoffs (resp. profits) can be pictured by payoff (resp. profit) diagrams as functions of \( S_T \). In particular, the payoff diagram of the forward contract is always a linear function, with slope 1 for long or slope \(-1\) for short, and passing through \( F_{0,T} \) on the \( S_T \)-axis:

**Example 2** [SOA Introductory Derivatives Sample Question 68]: For a non-dividend-paying stock index, the current price is 1,100 and the 6-month forward price is 1,150. Assume the price of the stock index in 6 months will be 1,210. Which of the following is true regarding forward positions in the stock index?

(A) Long position gains 50  
(B) Long position gains 60  
(C) Long position gains 110  
(D) Short position gains 60  
(E) Short position gains 110

*Solution:*

**Exercise 2** [Modified from SOA Introductory Derivatives Sample Question 68]: For a non-dividend-paying stock index, the current price is 500 and the 1-year forward price is 600. Assume the price of the stock index in 1 year will be 450. Which of the following is true regarding forward positions in the stock index?

(A) Long position losses 50  
(B) Long position losses 100  
(C) Long position losses 150  
(D) Short position losses 150  
(E) Short position losses 100

*Solution:*
Is there any initial payment for a forward contract?

1.2.8 Given the underlying asset, the expiration date, the long/short position of the forward contract being specified, as long as the forward price is fair, the forward contract requires no initial payment or premium. A similar forward contract, known as the prepaid forward contract, which requires an initial payment, known as the prepaid forward price, will be discussed in the next section.

1.2.9 Since the forward contract requires no initial payment or premium, its profit, which is given by its payoff minus the future value of its cost, equals to its payoff:

- Profit of long forward = Payoff of long forward = $S_T - F_{0,T}$.
- Profit of short forward = Payoff of short forward = $F_{0,T} - S_T$.

No-arbitrage principle.

1.2.10 The followings provide a general pricing theory of financial instruments, which is applicable to derivative contracts, in particular both forward/futures and option contracts.

1.2.11 With an investment strategy $\pi$, denote the value process of the investment portfolio, or simply the portfolio value, by $V^\pi_t$, for $t \geq 0$. The investment strategy $\pi$ is an arbitrage strategy if

\[ V^\pi_0 = 0, \quad \mathbb{P}(V^\pi_T \geq 0) = 1, \quad \text{and} \quad \mathbb{P}(V^\pi_T > 0) > 0. \]

Essentially, the arbitrage strategy $\pi$ generates a non-negative, and possibly positive, cash flow from the portfolio value at the expiration date $T$, with no-cost at time 0. The arbitrage portfolio $\pi$ is thus basically a deterministic money making machine. If such an arbitrage strategy exists in the financial market, then the market has an arbitrage opportunity, or informally speaking, a free lunch.

1.2.12 Throughout this course, if not in the entire mathematical finance community, the no-arbitrage principle is adopted:

\[ \text{NO-ARBITRAGE PRINCIPLE:} \]

The financial market does not have any arbitrage opportunities.

The no-arbitrage pricing principle states that prices of financial instruments are determined in a way that the financial market is arbitrage-free.

1.2.13 Necessarily, if two investment strategies generate the same portfolio values at the expiration date $T$, then their portfolio values at time 0 must also be the same:

\[ \text{Law of one-price:} \quad V^\pi_1 = V^{\pi_2}_T \Rightarrow V^\pi_0 = V^{\pi_2}_0. \]
Indeed, assume, in the contrary, that $V_{0}^{\pi_1} \neq V_{0}^{\pi_2}$, and without loss of generality, assume that $V_{0}^{\pi_1} > V_{0}^{\pi_2}$. Then construct an investment strategy $\pi$ which consists of (1) short 1 investment strategy $\pi^1$, (2) long 1 investment strategy $\pi^2$, and (3) lend $(V_{0}^{\pi_1} - V_{0}^{\pi_2})$, or equivalently long 1 zero-coupon bond matured at $T$ with a face-value $(V_{0}^{\pi_1} - V_{0}^{\pi_2}) \times e^{rT}$. The portfolio value with this investment strategy $\pi$ at time 0 is

$$V_{0}^{\pi} = -V_{0}^{\pi_1} + V_{0}^{\pi_2} + (V_{0}^{\pi_1} - V_{0}^{\pi_2}) = 0,$$

while the value at the expiration time $T$ is deterministically

$$V_{T}^{\pi} = -V_{T}^{\pi_1} + V_{T}^{\pi_2} + (V_{0}^{\pi_1} - V_{0}^{\pi_2}) \times e^{rT} = (V_{0}^{\pi_1} - V_{0}^{\pi_2}) \times e^{rT} > 0,$$

where $r$ is the continuously compounded risk-free interest rate. This shows that the investment strategy $\pi$ is an arbitrage strategy, which contradicts the no-arbitrage principle, and thus $V_{0}^{\pi_1} = V_{0}^{\pi_2}$. Such the investment strategy $\pi$ is constructed based on the buy-low-sell-high argument.

**Example 3**: Alfred and John sign a forward contract at time 0 that, after one week, Alfred will pay John 70 USD for buying a barrel of crude oil. Assume further that the price for a barrel of crude oil is 65 USD at time 0 and the continuously compounded risk-free interest rate is 5%. Construct an arbitrage strategy for Alfred or John, if any.

**Solution**: 


Exercise 3: Alfred and John sign a forward contract at time 0 that, after one week, Alfred will pay John 65 USD for buying a barrel of crude oil. Assume further that the price for a barrel of crude oil is 65 USD at time 0 and the continuously compounded risk-free interest rate is 5%. Construct an arbitrage strategy for Alfred or John, if any.

Solution:

1.2.14 Let $\pi^1$ and $\pi^2$ be two investment strategies. If the investment portfolio $\pi^1$ consists of only 1 derivative contract while $V^\pi_T = V_T^{\pi^2}$, then the investment portfolio $\pi^2$ is called the replicating portfolio which provides a perfect hedge for the derivative contract. By the law of one-price:

The price of the derivative contract at time 0 equals to the time-0 cost for the replicating portfolio.

What is a fair forward price?

1.2.15 For a forward contract written on a non-dividend-paying stock, the fair forward price is given by:

Forward price on non-dividend-paying stock: $F_{0,T} = S_0 \times e^{rT}$.
Indeed, consider two investment strategies $\pi^1$ and $\pi^2$, where $\pi^1$ consists of long 1 forward contract, while $\pi^2$ consists of (1) long 1 non-dividend-paying stock and (2) short 1 zero-coupon bond matured at $T$ with a face value $F_{0,T}$. Then, the portfolio values generated by these two investment strategies at the expiration date are the same:

$$V_T^{\pi^1} = V_T^{\pi^2} = S_T - F_{0,T}.$$ 

By the law of one-price, their portfolio values at time 0 must also be the same:

$$0 = V_0^{\pi^1} = V_0^{\pi^2} = S_0 - F_{0,T} \times e^{-rT},$$

which deduces that $F_{0,T} = S_0 \times e^{rT}$. The replicating portfolio $\pi^2$, long stock and short bond, is known as the synthetic long forward.

1.2.16 The forward price on non-dividend-paying stock can also be derived by considering an investment strategy $\pi^1$ with shorting 1 forward contract. The replicating portfolio $\pi^2$, which consists of (1) long 1 zero-coupon bond matured at $T$ with a face value $F_{0,T}$ and (2) short 1 non-dividend-paying stock, is known as the synthetic short forward.

Example 4 [Modified from SOA Introductory Derivatives Sample Question 73]: The current price of a non-dividend-paying stock is 100. The annual effective risk-free interest rate is 4%, and there are no transaction costs. The stock’s two-year forward price is mis-priced at 108. Construct an arbitrage strategy.

Solution:

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Exercise 4 [SOA Introductory Derivatives Sample Question 56]: Determine which of the following positions has the same cash flows as a short stock position.

(A) Long forward and long zero-coupon bond
(B) Long forward and short forward
(C) Long forward and short zero-coupon bond
(D) Long zero-coupon bond and short forward
(E) Short forward and short zero-coupon bond

Solution:

1.2.17 For a forward contract written on a stock paying discrete dividends \(D_{t_1}, D_{t_2}, \ldots, D_{t_n}\), for \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T\), the fair forward price is given by:

\[
F_{0,T} = S_0 \times e^{rT} - FV_{0,T}(\text{Div}) = S_0 \times e^{rT} - \sum_{i=1}^{n} D_{t_i} \times e^{r(T-t_i)}.
\]

Indeed, consider two investment strategies \(\pi^1\) and \(\pi^2\), where \(\pi^1\) consists of long 1 forward contract, while \(\pi^2\) consists of (1) long 1 stock paying the discrete dividends and (2) short \(n+1\) zero-coupon bonds, matured at \(t_1, t_2, \ldots, t_n, T\), with face values \(D_{t_1}, D_{t_2}, \ldots, D_{t_n}, F_{0,T}\). Then, the portfolio values generated by these two investment strategies at the expiration date are the same:

\[
V_{\pi^1}^T = V_{\pi^2}^T = S_T - F_{0,T}.
\]

By the law of one-price, their portfolio values at time 0 must also be the same:

\[
0 = V_{\pi^1}^0 = V_{\pi^2}^0 = S_0 - \sum_{i=1}^{n} D_{t_i} \times e^{-r t_i} - F_{0,T} \times e^{-r T},
\]

which deduces that \(F_{0,T} = S_0 \times e^{rT} - \sum_{i=1}^{n} D_{t_i} \times e^{r(T-t_i)}\).

Example 5 [SOA Introductory Derivatives Sample Question 37]: A one-year forward contract on a stock has a price of $75. The stock is expected to pay a dividend of $1.50 at two future times, six months from now and one year from now, and the annual effective risk-free interest rate is 6\%. Calculate the current stock price.

Solution:
Exercise 5 [SOA Introductory Derivatives Sample Question 20]: The current price of a stock is 200, and the continuously compounded risk-free interest rate is 4%. A dividend will be paid every quarter for the next 3 years, with the first dividend occurring 3 months from now. The amount of the first dividend is 1.50, but each subsequent dividend will be 1% higher than the one previously paid. Calculate the fair price of a 3-year forward contract on this stock.

Solution:

1.2.18 For a forward contract written on a stock paying continuous dividend with the dividend yield $\delta$, the fair forward price is given by:

Forward price on stock paying continuous dividend: $F_{0,T} = S_0 \times e^{(r-\delta)T}$.

Indeed, consider two investment strategies $\pi^1$ and $\pi^2$, where $\pi^1$ consists of long 1 forward contract, while $\pi^2$ consists of (1) long $e^{-\delta T}$ stock paying the continuous dividend and (2) short 1 zero-coupon bond matured at $T$ with a face value $F_{0,T}$. Then, the portfolio values generated by these two investment strategies at the expiration date are the same:

$V_{T}^{\pi^1} = V_{T}^{\pi^2} = S_T - F_{0,T}$.

By the law of one-price, their portfolio values at time 0 must also be the same:

$0 = V_{0}^{\pi^1} = V_{0}^{\pi^2} = S_0 \times e^{-\delta T} - F_{0,T} \times e^{-rT}$,

which deduces that $F_{0,T} = S_0 \times e^{(r-\delta)T}$. 

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Example 6 [SOA Introductory Derivatives Sample Question 21]: A market maker in stock index forward contracts observes a 6-month forward price of 112 on the index. The index spot price is 110 and the continuously compounded dividend yield on the index is 2%. The continuously compounded risk-free interest rate is 5%. Describe actions the market maker could take to exploit an arbitrage opportunity and calculate the resulting profit (per index unit).

(A) Buy observed forward, sell synthetic forward, Profit = 0.34
(B) Buy observed forward, sell synthetic forward, Profit = 0.78
(C) Buy observed forward, sell synthetic forward, Profit = 1.35
(D) Sell observed forward, buy synthetic forward, Profit = 0.78
(E) Sell observed forward, buy synthetic forward, Profit = 0.34

Solution:
Exercise 6: The spot price of a stock index paying continuous dividend is 88, while its 4-month forward price is 89. Assume that the continuously compounded risk-free rate is 4%. Determine the continuously compounded dividend yield of the index.

Solution:

**Relationship between the forward price and the expected future stock price.**

1.2.19 From their expressions, the forward prices are intuitively related to the expected future stock price. However, instead of using the expected rate of return on the stock, the expressions adopt the risk-free rate of return.

1.2.20 Let $\alpha$ be the continuously compounded **expected rate of return** on the stock, in the sense that $E[S_T] = S_0 \times e^{\alpha T}$.

1.2.21 Due to the riskiness of stock price, the expected rate of return on the stock $\alpha$ should be naturally greater than the risk-free rate of return $r$. The difference $\alpha - r > 0$ is known as the **risk premium**.

1.2.22 The expected stock price at the expiration date is always greater than the forward prices:

\[
E[S_T] = S_0 \times e^{\alpha T} > S_0 \times e^{r T} \geq S_0 \times e^{r T} - FV_{0,T}(\text{Div}) = S_0 \times e^{(r-\delta)T}.
\]

**Example 7** [SOA Introductory Derivatives Sample Question 70]: Investors in a certain stock demand to be compensated for risk. The current stock price is 100. The stock pays dividends at a rate proportional to its price. The dividend yield is 2%. The continuously compounded risk-free interest rate is 5%. Assume there are no transaction costs. Let $X$ represent the expected value of the stock price 2 years from today. Assume it is known that $X$ is a whole number. Determine which of the following statements is true about $X$.

(A) The only possible value of $X$ is 105.
(B) The largest possible value of $X$ is 106.
(C) The smallest possible value of $X$ is 107.
(D) The largest possible value of $X$ is 110.
(E) The smallest possible value of $X$ is 111.

**Solution:**
Exercise 7 [SOA Introductory Derivatives Sample Question 6]: The following relates to one share of XYZ stock:

- The current price is 100.
- The forward price for delivery in one year is 105.
- $P$ is the expected price in one year.

Determine which of the following statements about $P$ is TRUE.

(A) $P < 100$
(B) $P = 100$
(C) $100 < P < 105$
(D) $P = 105$
(E) $P > 105$

Solution:
1.3 Variations on Forwards

Four alternative ways to own an asset.

1.3.1 The simplest way to acquire an underlying asset by a time $T$ is the outright purchase: the payment is made at time 0, while the asset is delivered at time 0, and the investor holds the asset till time $T$. In other words, this strategy longs 1 asset at time 0 and holds until time $T$. The payment at time 0 is obviously the spot price $S_0$.

1.3.2 The forward contract is another extreme of acquiring the underlying asset by the time $T$: the payment is made at time $T$, while the asset is delivered at time $T$. The payment at time $T$ is known as the forward price $F_{0,T}$, which is set at time 0 a priori.

1.3.3 There are two more alternative ways to acquire the underlying asset by the time $T$, namely, the fully leveraged purchase and the prepaid forward contract.

1.3.4 For the fully leveraged purchase, the payment is made at time $T$, while the asset is delivered at time 0, and the investor holds the asset till the time $T$. Since the investor funds the required investment for the outright purchase of $S_0$ at time 0 by making a full loan to be repaid at time $T$, the payment for the fully leveraged purchase at time $T$ is thus $FV_{0,T}(S_0) = S_0 \times e^{rT}$.

1.3.5 For the prepaid forward contract, the payment is made at time 0, while the asset is delivered at time $T$. The payment at time 0 is known as the prepaid forward price $F_{0,T}^P$.

1.3.6 The relationship between the forward price and the prepaid forward price is given by:

$$F_{0,T} = FV_{0,T}(F_{0,T}^P) \text{ or } F_{0,T}^P = PV_{0,T}(F_{0,T}).$$

Indeed, in parallel with the argument of deriving the fully leveraged purchase price $FV_{0,T}(S_0)$ from the outright purchase price $S_0$, since the investor funds the required investment for the prepaid forward contract of $F_{0,T}^P$ at time 0 by making a full loan to be repaid at time $T$, the payment for the forward contract at time $T$ is thus $FV_{0,T}(F_{0,T}^P)$, which equals to $F_{0,T}$. Therefore,

Prepaid forward price on non-dividend-paying stock : $F_{0,T}^P = S_0$.

Prepaid forward price on stock paying discrete dividends :

$$F_{0,T}^P = S_0 - PV_{0,T}(\text{Div}) = S_0 - \sum_{i=1}^{n} D_{t_i} \times e^{-r t_i}.$$ 

Prepaid forward price on stock paying continuous dividend :

$$F_{0,T}^P = S_0 \times e^{-\delta T}.$$ 

1.3.7 The following table summarizes the four alternative ways to own the underlying asset by time $T$.  

<table>
<thead>
<tr>
<th>Prepaid forward price on non-dividend-paying stock</th>
<th>$F_{0,T}^P = S_0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prepaid forward price on stock paying discrete dividends</td>
<td>$F_{0,T}^P = S_0 - PV_{0,T}(\text{Div}) = S_0 - \sum_{i=1}^{n} D_{t_i} \times e^{-r t_i}$.</td>
</tr>
<tr>
<td>Prepaid forward price on stock paying continuous dividend</td>
<td>$F_{0,T}^P = S_0 \times e^{-\delta T}$.</td>
</tr>
</tbody>
</table>
Example 8 [SOA Introductory Derivatives Sample Question 71]: A certain stock costs 40 today and will pay an annual dividend of 6 for the next 4 years. An investor wishes to purchase a 4-year prepaid forward contract for this stock. This first dividend will be paid one year from today and the last dividend will be paid just prior to delivery of the stock. Assume an annual effective interest rate of 5%. Calculate the price of the prepaid forward contract.

Solution:

Exercise 8 [SOA Introductory Derivatives Sample Question 29]: The dividend yield on a stock and the interest rate used to discount the stock’s cash flows are both continuously compounded. The dividend yield is less than the interest rate, but both are positive. The following table shows four methods to buy the stock and the total payment needed for each method. The payment amounts are as of the time of payment and have not been discounted to the present date.

<table>
<thead>
<tr>
<th>METHOD</th>
<th>TOTAL PAYMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outright purchase</td>
<td>A</td>
</tr>
<tr>
<td>Fully leveraged purchase</td>
<td>B</td>
</tr>
<tr>
<td>Prepaid forward contract</td>
<td>C</td>
</tr>
<tr>
<td>Forward contract</td>
<td>D</td>
</tr>
</tbody>
</table>

Determine which of the following is the correct ranking, from smallest to largest, for the amount of payment needed to acquire the stock.

(A) $C < A < D < B$
(B) $A < C < D < B$
(C) $D < C < A < B$
(D) $C < A < B < D$
(E) $A < C < B < D$

Solution:
What is a futures contract?

1.3.8 A futures contract is essentially an exchange-based forward contract, which also represents an obligation to long or short an underlying asset at some future time.

1.3.9 However, the futures contract is different from the otherwise identical forward contract in the following aspects.

- **Mark-to-market**: While the forward contract is settled at the expiration date, the futures contract is settled daily. Frequent marking-to-market and settlement of the futures contract can lead to pricing differences between the futures and forward contracts.

- **Liquidity**: Due to daily settlement and marking-to-market, the futures contract is liquid, and thus it is possible to offset an obligation on a given date by entering into the opposite position.

- **Standardization**: While the OTC forward contract can be customized to suit the buyer or seller, the exchange-based futures contract is standardized on the expiration date, size, underlying asset, etc.

- **Credit risk minimized**: Due to daily settlement and marking-to-market, the futures contract minimizes the effects of credit risk.

- **Price limit**: A price limit, which is imposed in the futures contract, is a move in the futures price that triggers a temporary halt in trading.

<table>
<thead>
<tr>
<th>Example 9 [SOA Introductory Derivatives Sample Question 30]: Determine which of the following is NOT a distinguishing characteristic of futures contracts, relative to forward contracts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) Contracts are settled daily, and marked-to-market.</td>
</tr>
<tr>
<td>(B) Contracts are more liquid, as one can offset an obligation by taking the opposite position.</td>
</tr>
<tr>
<td>(C) Contracts are more customized to suit the buyer’s needs.</td>
</tr>
<tr>
<td>(D) Contracts are structured to minimize the effects of credit risk.</td>
</tr>
<tr>
<td>(E) Contracts have price limits, beyond which trading may be temporarily halted.</td>
</tr>
</tbody>
</table>

How does the frequent settlement work?

1.3.10 To guarantee the daily mark-to-market payments for the futures contract, each party has to set up a margin account.

1.3.11 Denote the time-$t$ price of the futures contract with the expiration date $T$ by $F_{t,T}$, for $0 \leq t \leq T$. Let $B_t$ be the margin balance at time $t$, for $0 \leq t \leq T$.

1.3.12 The initial margin balance $B_0$ is a percentage of the notional value of the contract,
which is the product of multiplier $N$ and futures price at time 0:

$$B_0 = \text{initial margin percentage} \times N \times F_{0,T}.$$ 

1.3.13 On a daily basis, the margin balance earns interest and is debited or credited for the mark-to-market settlement:

- **Long futures**: $B_t = B_{t-\frac{1}{365}} \times e^{\frac{r}{365}} + N \times \left(F_{t,T} - F_{t-\frac{1}{365},T}\right),$ for $t = \frac{1}{365}, \frac{2}{365}, \ldots, T.$

- **Short futures**: $B_t = B_{t-\frac{1}{365}} \times e^{\frac{r}{365}} + N \times \left(F_{t-\frac{1}{365},T} - F_{t,T}\right),$ for $t = \frac{1}{365}, \frac{2}{365}, \ldots, T.$

1.3.14 There is usually a maintenance margin balance $m$, which is a minimum level of the margin account balance and is a high percentage of the initial margin balance:

$$m = \text{high percentage} \times B_0,$$

1.3.15 Once the margin account balance $B_t$ falls below the maintenance margin balance $m$, a margin call is issued to the investor. The investor can then either (1) make an additional deposit, by at least $m - B_t$, to bring the margin account balance back to or above the initial margin level $B_0$, or (2) refuse to make the deposit, but his position is closed and the remaining margin account balance $B_t$ is returned to the investor.

**Exercise 9** [SOA Introductory Derivatives Sample Question 69]: Determine which of the following statements about futures and forward contracts is false.

- (A) Frequent marking-to-market and settlement of a futures contract can lead to pricing differences between a futures contract and an otherwise identical forward contract.
- (B) Over-the-counter forward contracts can be customized to suit the buyer or seller, whereas futures contracts are standardized.
- (C) Users of forward contracts are more able to minimize credit risk than are users of futures contracts.
- (D) Forward contracts can be used to synthetically switch a portfolio invested in stocks into bonds.
- (E) The holder of a long futures contract must place a fraction of the cost with an intermediary and provide assurances on the remaining purchase price.
Example 10 [SOA Introductory Derivatives Sample Question 32]: Judy decides to take a short position in 20 contracts of S&P 500 futures. Each contract is for the delivery of 250 units of the index at a price of 1,500 per unit, exactly one month from now. The initial margin is 5% of the notional value, and the maintenance margin is 90% of the initial margin. Judy earns a continuously compounded risk-free interest rate of 4% on her margin balance. The position is marked-to-market on a daily basis. On the day of the first marking-to-market, the value of the index drops to 1,498. On the day of the second marking-to-market, the value of the index is $X$ and Judy is not required to add anything to the margin account. Calculate the largest possible value of $X$.

Solution:
Exercise 10 [SOA Introductory Derivatives Sample Question 45]: An investor enters a long position in a futures contract on an index \( F \) with a notional value of \( 200 \times F \), expiring in one year. The index pays a continuously compounded dividend yield of 4%, and the continuously compounded risk-free interest rate is 2%. At the time of purchase, the index price is 1,100. Three months later, the investor has sustained a loss of 100. Assume the margin account earns an interest rate of 0%. Let \( S \) be the price of the index at the end of month three. Calculate \( S \).

Solution: