PATCHING THE NORM RESIDUE ISOMORPHISM THEOREM

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Abstract. We provide a patch to complete the proof of the Voevodsky-Rost Theorem, that the norm residue map is an isomorphism. (This settles the motivic Bloch-Kato conjecture). This patch is designed to be read by experts, to check that the theorem has indeed been proven.

Introduction

The purpose of this paper is to patch up the sketched proof in [MC/l] of the “Voevodsky-Rost Theorem” that the norm residue map $K^M_n(k)/\ell \longrightarrow H^n_\ell(k, M^{\otimes n})$ is an isomorphism, for any prime $\ell > 2$ and for any field $k$ such that $1/\ell \in k$. This result is sometimes called the “Bloch-Kato conjecture.”

It is based upon Voevodsky’s 2003 preprint [MC/l]. That preprint gave a proof modulo three missing results (Lemmas 2.2, 2.3 and Theorem 6.3). The third ingredient, the existence of a Rost variety for $a$ (defined in Definition 1.1 below), is no longer a problem. A Rost variety $X_a$ was constructed for any $a$ by Markus Rost in his 1998 “Chain Lemma” preprint [7]; the proof that $X_a$ satisfies these properties was published in [10].

The main innovation is that we consider integral to modular cohomology operations from $H^{2n,n}(X, Z)$ to $H^{p,q}(X, Z/\ell)$ in order to provide substitutes for the two missing lemmas 2.2 and 2.3 (see 5.1 and 3.10 below). Our substitute for 2.2 is inspired by the theorem of Henri Cartan [1] that in ordinary topology all such operations $\phi(x)$ are polynomials in the standard Steenrod operations $P^I(x)$, where (in the notation of [9]) $I = (\epsilon_0, s_1, \ldots, s_k)$ is admissible (so $s_k \geq 1$ and $\epsilon_k = 0$).

We will be interested in the Lefschetz motives $L^\bullet_R = R_{tr}(A^\bullet/k^n - 0)$, and in the underlying pointed spaces $K_n = uL^\bullet_R$ which represent integral motivic cohomology in the sense that $H^{2n,n}(X, Z) = \text{Hom}_{DM}(Z_{tr}X, L^n) \cong [X_+, K_n]$. To explain the notation, recall that there is a functor $R_{tr}$ from the Morel-Voevodsky $A^1$-homotopy category of pointed spaces (of [5]) to the triangulated category $\text{DM}^{tr}(R)$, which is left adjoint to the underlying (forgetful) functor $u$. (See [3], [16] or [6] for example.) This is the viewpoint presented in [RPO, 2.1].

It follows that the set of (unstable) cohomology operations $\phi$ from $H^{2n,n}(-, Z)$ to $H^{p,q}(-, R)$ is in 1–1 correspondence with the elements of the group $\tilde{H}^{p,q}(K_n, R)$. (All cohomology operations $\phi$ satisfy $\phi(0) = 0$, from naturality in $X_+ \rightarrow *$.)

Example 0.1. When $a = 0$, $K_0$ is the wedge of copies of $S^0$ indexed by $Z_0$. Hence cohomology operations $\phi : H^{0,0}(X, Z) \rightarrow H^{p,q}(-, R)$ correspond to elements $\{\phi_j\}$ of $H^{p,q}(\bigvee S^0, R) = \prod_{j \neq 0} H^{p,q}(k, R)$. If $X$ is connected, then $\phi : Z \rightarrow H^{p,q} \rightarrow H^{p,q}(X, R)$ is $\phi(j) = \phi_j$. 

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More precisely, \((R_{tr}, u)\) is an adjunction between (simplicial) pointed radditive functors and (positive chain complexes of) presheaves with transfers. It is a Quillen adjunction with respect to both the local and \(A^1\)-local closed model structures. In particular, \(R_{tr}\) preserves cofibrations and \(u\) preserves fibrations. Applying this to the simplicial suspension sequence \(V \to \text{cone}(V) \to \Sigma V\), we see that \(R_{tr}(\Sigma V) \simeq R_{tr}(V)[1]\), and that \(u(M[1]) \simeq B(uM)\). In particular, \(u(\mathbb{L}^a[1]) \simeq B\mathbb{A}\).

**Example 0.2.** The Eilenberg-MacLane space (simplicial set) \(K(A, n)\) represents \(H^{n,0}(-, A)\). Thus cohomology operations from \(H^{n,0}(-, A)\) to \(H^{p,q}(-, R)\) correspond to elements of \(H^{p,q}(K(A, n), R)\). In particular, motivic cohomology operations \(H^{n,0}(-, A) \to H^{p,0}(-, R)\) are the same as the usual (unstable) topological cohomology operations described for example in [1]. A useful case is \(n = 1\) and \(A = \mathbb{Z}\); since \(K(\mathbb{Z}, 1) = S^1\) there are no cohomology operations \(H^{1,0}(-, \mathbb{Z}) \to H^{p,0}(-, R)\) for \(p \neq 0\).

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**Notation** The integer \(n\) and the prime \(\ell > 2\) will be fixed. We will work over a fixed field \(k\) in which \(\ell\) is invertible. The integer \(d\) will always be \(\ell^{n-1} - 1\) and \(b\) will always be \(d/(\ell - 1) = 1 + \cdots + \ell^{n-2}\).

We fix the sequence of units \(\underline{a} = (a_1, \ldots, a_n)\), and \(X_{\underline{a}}\) will always denote a \(d\)-dimensional Rost variety relative to \(\underline{a}\), satisfying Axioms 1.3.

We will work in the triangulated category of motives \(\text{DM}^\text{eff}\) described in [4], usually inside the full subcategory generated by simplicial presheaves with transfers. The Lefschetz motive is \(\mathbb{L} = \mathbb{Z}(1)[2]\). Unless explicitly stated otherwise, motivic cohomology will always be taken with coefficients \(\mathbb{Z}(\ell)\). The notation \(H^{p,q}_{\text{et}}(-)\) refers to the étale motivic cohomology \(H^{p}_{\text{et}}(-, \mathbb{Z}(\ell)(q))\) defined in [4, 10.1].
1. The outer shell of the proof

We begin by setting the scene for the later sections. The material in this section is largely taken from [17]. We will inductively assume that the norm residue maps $K^M_i(L)/\ell \to H^i_{et}(L, \mu_{\ell}^{\otimes i})$ are isomorphisms for $i \leq n$ and all fields $L$ containing $1/\ell$.

From now on, $k$ will denote a field of characteristic zero, and $a = \{a_1, \ldots, a_n\}$ will be a nonzero element of $K^M_n(k)/\ell$.

Recall [10, 1.20] that a $\nu_{n-1}$-variety over a field $k$ is a smooth projective variety $X$ of dimension $d = \ell^{n-1} - 1$, with $\deg s_d(X) \neq 0 \pmod{\ell^2}$. Here $s_d(X)$ is the characteristic class of the tangent bundle $T_X$ corresponding to the symmetric polynomial $\sum t^a_j$ in the Chern roots $t_j$ of $T_X$; see [RPO, 14.3].

**Definition 1.1.** A Rost variety for a sequence $a = (a_1, \ldots, a_n)$ of units in $k$ is a $\nu_{n-1}$-variety $X = X_a$ such that: $\{a_1, \ldots, a_n\}$ vanishes in $K^M_n(k(X))/\ell$; for each $i < n$ there is a $\nu_i$-variety mapping to $X$; and the motivic homology sequence
\begin{equation}
\label{eq:1.2}
H_{-1,-1}(X^2) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1,-1}(X) \to H_{-1,-1}(k) \quad (= k^\times),
\end{equation}
is exact. As mentioned above, Rost varieties exist for every $H_{n-1}$-variety, which we shall dub the “symmetric Rost motive” (see 1.13). As observed in [MC/l, 6.6], it follows from [MC/2, 6.9(1)] that:
\begin{equation}
\label{eq:1.5}
H^{p,q}(\Sigma X; \mathbb{Z}/\ell) = 0 \text{ when } (p, q) \text{ is in the region } q < n, p \leq 1 + q.
\end{equation}

Let $X$ denote the simplicial Čech scheme $\tilde{C}(X_a) : p \mapsto X_{a^p+1}$; see [MC/2, 9.1]. By abuse of notation, we will regard $X$ as a chain complex, and hence as an element of $DM_{et}(k, \mathbb{Z}(\ell))$. We set $b = d/(\ell - 1) = 1 + \ell + \cdots + \ell^{n-2}$.

**Definition 1.3.** A Rost motive for $a$ is a motive $M$ satisfying the following axioms:

(a) $M$ is a direct summand of a Rost variety $X_a$, defined over $\mathbb{Z}(\ell)$.

(b) The evident duality map $M^* \otimes \mathbb{L}^d \to M$ is an isomorphism.

(c) There is a motive $D$, related to the structure map $y : M \to X$ and its dual $Dy$ by two distinguished triangles:
\begin{align}
\label{eq:1.3.1}
& D \otimes \mathbb{L}^b \to M \xrightarrow{y} X \to, \\
\label{eq:1.3.2}
& X \otimes \mathbb{L}^d \xrightarrow{Dy} M \to D \to .
\end{align}

Given (1.3.1), triangle (1.3.2) is equivalent to the duality $D^* \otimes \mathbb{L}^{d-b} \cong D$.

When $\ell = 2$, triangle (1.3.1) was constructed in [MC/2, 4.4] using $D = X$, and (1.3.2) is its dual; axioms 1.3(a–b) hold by a result of Rost (cited as [MC/2, 4.3]).

The following theorem, which summarizes the contents of section 6 of [MC/l], was the main theorem in my 2006 preprint [17].

**Theorem 1.4.** Let $n$ and $\ell$ be such that the norm residues maps are isomorphisms for all $i < n$. Suppose that a Rost motive exists for every $a$ in $K^M_n(k)/\ell$.

Then the norm residue map $K^M_n(k)/\ell \to H^*_{et}(k, \mu_{\ell}^{\otimes n})$ is an isomorphism.

In the rest of this section we define a candidate for a Rost motive, following Voevodsky [MC/l], which we shall dub the “symmetric Rost motive” (see 1.13). As observed in [MC/l, 6.6], it follows from [MC/2, 6.9(1)] that:
\begin{equation}
\label{eq:1.15}
H^{p,q}(\Sigma X; \mathbb{Z}/\ell) = 0 \text{ when } (p, q) \text{ is in the region } q < n, p \leq 1 + q.
\end{equation}

**Lemma 1.6.** If $X$ is the simplicial Čech scheme of a $\nu_{n-1}$-variety, then for all $i < n$ the Margulis homology sequence is exact:
\[
\cdots \xrightarrow{Q_i} H^{p,q}(\Sigma X; \mathbb{Z}/\ell) \xrightarrow{Q_i} \cdots .
\]

**Proof.** This is proven in [MC/2, 3.2]. The proof is repeated in [MC/l, 4.3].
Lemma 1.7. The motivic cohomology groups $H^\ast \ast(X,\mathbb{Z})$ have exponent $\ell$.

Lemma 1.7 is implicit in [MC/2, 9.3], and is proven in [17, 2.3], using [10]. It implies that the integral cohomology $H^p_q(X,\mathbb{Z})$ may be identified with the kernel of the Bockstein $\beta: H^p_q(X,\mathbb{Z}/\ell) \to H^{p+1}_q(X,\mathbb{Z}/\ell)$.

We now consider the cohomology operation $Q_i$, which has bidegree $(2\ell^i-1,\ell^i-1)$. Since $Q_i$ anticommutes with $\beta$, it follows that $Q_i$ sends the subgroup $H^\ast \ast(X,\mathbb{Z})$ of $H^\ast \ast(X,\mathbb{Z}/\ell)$ to itself. (This was observed in [MC/2, 7.2].)

**Lemma 1.8.** The operations $Q = Q_{n-2} \cdots Q_0 : H^{n,n-1}(X;\mathbb{Z}/\ell) \to H^{2b+1,b}(X;\mathbb{Z}/\ell)$ and $Q_{n-1} : H^{2b+1,b}(X;\mathbb{Z}/\ell) \to H^{2b+2,b+1}(X;\mathbb{Z}/\ell)$ are injections.

*Proof.* (Voevodsky) Using Lemma 1.6 and (1.5), it is routine to check that $Q_0 = \beta$ is injective on $H^{n+1,n-1}(X,\mathbb{Z})$, and $Q_i$ is injective on the group $H^{p,q}(X,\mathbb{Z})$ containing $Q_{i-1} \cdots Q_0 H^{n+1,n-1}(X,\mathbb{Z})$ when $i < n$, because the preceding term in 1.6 is zero by (1.5). Since $H^{p,q}(X) \cong H^{p+1,q}(\Sigma X)$ for $p > q$, we are done. \(\square\)

If $q \neq 0$ in $K^M_0(k)/\ell$, Voevodsky shows in [MC/1, 6.5] that its norm residue symbol in $H^\ast \ast(_{\ell}k,\mu_{\ell} \otimes n)$ lifts to a nonzero element $\delta \in H^{n,n-1}(X,\mathbb{Z}/\ell)$, which follows from 1.8 and 1.7 that $\beta(\delta) \in H^{n+1,n-1}(X,\mathbb{Z})$ and $\mu = Q(\delta) \in H^{2b+1,b}(X,\mathbb{Z})$ are nonzero. Regarding $\mu$ as a morphism in $\text{DM}$ from $Rtr(\Sigma X)$ to $Rtr(\Sigma X) \otimes \mathbb{L}^b[1]$, we define the motive $A$ associated to $\mu$ by the distinguished triangle

$$(1.9)\quad X \otimes \mathbb{L}^b \xrightarrow{x} A \xrightarrow{y} Rtr(\Sigma X) \xrightarrow{\mu} Rtr(\Sigma X) \otimes \mathbb{L}^b[1].$$

We now suppose that $R$ is either $\mathbb{Z}/\ell$ or $\mathbb{Z}/\ell$, so that $(\ell - 1)!$ is invertible in $R$.

**Definition 1.10.** If $m < \ell$, the symmetrizing idempotent $e = \sum \sigma / \sigma_\ell$ acts on $M^{\otimes m}$ for each object $M$ of $\text{DM}$, and we set $S^m(M) = e \cdot M^{\otimes m}$.

For each $(p,q)$, Voevodsky constructs a motivic cohomology operation $\phi \psi$ from $H^{2p+1,q}(-,R)$ to $H^{2p+2,q}(-,R)$ in [MC/1, 3.1-3.2] (we use $m = \ell - 1$). In particular, the element $\phi \psi(\mu)$ is computed from (1.9) as follows. Consider the transfer map associated to $\Sigma_{m-1} \subset \Sigma_m$ acting on $A^{\otimes m}$ (see [15, 6.7.16]):

$$(1.11)\quad S^m(A) \to S^{m-1}(A) \otimes A, \quad (a_1 \otimes \cdots) \mapsto \sum (\cdots \otimes \bar{a}_j \otimes \cdots) \otimes a_j,$$\

Composing with $1 \otimes y$ gives a map $u : S^m(A) \to S^{m-1}(A)$. Composing $1 \otimes x$ with the corestriction map $S^{m-1}(A) \otimes A \to S^m(A)$ gives a map $v : S^{m-1}(A) \otimes \mathbb{L}^b \to S^m(A)$. Voevodsky uses the slice filtration to prove the following result in [MC/1, 3.1].

**Lemma 1.12.** For $m < \ell$, there exist unique morphisms $r$ and $s$ fitting into distinguished triangles:

$$S^m(A) \otimes \mathbb{L}^d \xrightarrow{\text{corestr}} S^{m-1}(A) \xrightarrow{r} Rtr(\Sigma X) \otimes \mathbb{L}^d[1]$$

By 1.12, $S^{\ell-1}(A)$ satisfies axiom 1.3(c), with $D = S^{\ell-2}(A)$. The point of [MC/1], and of this paper, is that $S^{\ell-1}(A)$ is a Rost motive in the sense of 1.3, so that Theorem 1.4 applies to prove the Voevodsky-Rost Theorem. This motivates:

**Definition 1.13.** The symmetric Rost motive associated to $\mu$ is defined to be the symmetric product $S^{\ell-1}(A)$ of $A$. 


2. Symmetric Products

If $X$ is a normal quasiprojective variety over $k$, the symmetric products $S^m(X) = X^m/S_m$ are also normal. Thus $S^m$ is a functor from the category $\text{Norm}$ of normal quasiprojective varieties to itself. (If $X \neq \emptyset$, then $S^0(X) = \text{Spec}(k).$) We have:

\[(2.1) \quad S^m(X \amalg Y) = \amalg S^i(X) \times S^j(Y).\]

Since $R_{tr}(X)$ makes sense for any scheme $X$ of finite type ([4, 2.11]), it makes sense to write $S^m_{tr}R_{tr}(X)$ for $R_{tr}(S^mX)$. Thus $S^m_{tr}R_{tr}$ is a functor from $\text{Norm}$ to presheaves with transfers. If $X$ is a simplicial object in $\text{Norm}$, we define $S^m(X)$ and $S^m_{tr}R_{tr}(X)$ degewise.

It is easy to see that $S^m_{tr}R_{tr}$ extends to finite correspondences between objects. It also respects $\mathbb{A}^1$-local equivalences. If $(l-1)!$ is invertible in $R$ and $i < \ell$, a transfer argument for $X^i \to S^i(X)$ shows that $S^m_{tr}R_{tr}(X)$ is the symmetric power $S^i(R_{tr}(X)) = e : R_{tr}(X)^{\otimes i}$ of 1.10.

**Lemma 2.2.** If $M = R_{tr}(X)$ and $N = R_{tr}(Y)$ for normal $X$ and $Y$, we have

\[S^m_{tr}(M \oplus N) \cong \oplus_{i+j=m} S^i_{tr}(M) \otimes S^j_{tr}(N).\]

**Proof.** This is immediate from (2.1). \qed

If $V$ is based, there is a canonical map $S^{m-1}(V) \to S^m(V)$, and we write $S^\infty(V)$ for the colimit of the $S^m(V)$. When the basepoints are disjoint, the filtration by the $S^m(V)$ splits, and the filtration of $S^m_{tr}R_{tr}(V)$ splits naturally, as we now show.

We write $X_+$ for the disjoint union of $X$ and a basepoint $* = \text{Spec}(k)$, considered as a based space. Then the decomposition $S^m(X_+) = \underset{i=0}{\overset{m}{\amalg}} S^i(X)$ of (2.1) yields a split sequence of pointed spaces

\[(2.3) \quad S^{m-1}(X_+) \to S^m(X_+) \to \overset{\cdot}{S}^m(X_+),\]

where $\overset{\cdot}{S}^m$ is the functor $\overset{\cdot}{S}^m(X_+) = (S^mX)_+$ on $\text{Norm}_+$. By naturality, (2.3) extends to a degewise split sequence of simplicial objects in $\text{Norm}_+$.

As $X_+$ is a based space, $R_{tr}(X_+) = R_{tr}(X \amalg *)/R = R_{tr}(X)$ is well defined and we have $S^m_{tr}R_{tr}(X_+) = S^m_{tr}R_{tr}(X) = R_{tr}(S^m(X)) = R_{tr}(S^m(X_+))$.

**Lemma 2.4.** Applying $R_{tr}$ to (2.3) yields a naturally split exact sequence for every $V = X_+$ in $\text{Norm}_+$:

\[0 \to R_{tr}S^{m-1}(V) \to R_{tr}S^m(V) \to S^m_{tr}R_{tr}(V) \to 0.\]

**Proof.** By (2.3), the sequence is split exact as a functor of $X$. To split it functorially in $X_+$, we have to consider maps like $(X \amalg Y)_+ \to X_+$. Recall that, as in (1.11), for each $X$ and $i < m$ the transfer maps for $S^i(X) \times X^{m-i} \to S^m(X)$ and the structure map $\pi_X : X \to *$ induce maps $R_{tr}S^m(X) \to R_{tr}S^i(X)$.

The alternating sum (over $i$) of the transfer maps defines a map

\[R_{tr}\overset{\cdot}{S}^m(X_+) = R_{tr}S^m(X) \to \oplus_{i=1}^\infty S^i_{tr}R_{tr}(X) = S^m_{tr}R_{tr}(X_+).\]

We claim that this map is natural in $X_+$. To see this, it suffices to consider $(X \amalg Y)_+ \to X_+$ arising from $\pi_Y$, and show that for $j > 0$ and $a < i$ the composition from the summand $S^a_{tr}R_{tr}(X) \otimes S^i_{tr}R_{tr}(Y)$ of $R_{tr}S^m(X \amalg Y)$ to $S^a_{tr}R_{tr}(X)$ is zero. This composition factors through $S^a_{tr}R_{tr}(X) \otimes S^b_{tr}R_{tr}(Y)$ for $b = 0, \ldots, j$, and the result follows from $\sum_{b=0}^j (-1)^b \binom{j}{b} = 0$. \qed
Since $R_{tr}S^0(V) = 0$, it follows from 2.4 that for any simplicial object $V$ of $\text{Norm}_+$ we have a natural isomorphism

\begin{equation}
R_{tr}(S^\infty V) \cong \bigoplus_{i=1}^{\infty} R_{tr}(\hat{S}^i V) = \bigoplus_{i=1}^{\infty} S_{tr}^i R_{tr}(V).
\end{equation}

**Theorem 2.6.** Let $V$ be a simplicial object of $\text{Norm}_+$ such that the simplicial sets $\text{Hom}(X, V)$ are connected for all $X$. Then the morphism $S^\infty(V) \to u\mathbb{Z}_{tr}(V)$ is a global weak equivalence.

Example 0.1 shows that the connected hypothesis is crucial in Theorem 2.6.

**Proof.** This follows easily from [11, 6.8], which says that $u\mathbb{Z}_{tr}(V)^{eff} \cong S^\infty(V)$ is an isomorphism of (simplicial) presheaves, together with the fact that $S^\infty(V)$, being a connected simplicial $H$-space, has a homotopy inverse. \qed

**Corollary 2.7.** For $n \geq 1$, $R_{tr}(K_n) \cong \bigoplus_{i=1}^{\infty} S_{tr}^i(L^n)$, and $R_{tr}(BK_n) \cong \bigoplus_{i=1}^{\infty} S_{tr}^i(L^n[1])$.

**Proof.** For $V = \mathbb{A}^n/(\mathbb{A}^n - 0)$ we have $L^n = \mathbb{Z}_{tr}(V)$ and $K_n = uL^n = u\mathbb{Z}_{tr}(V)$, so $R_{tr}(K_n) \cong R_{tr}S^\infty(V)$ by 2.6. Similarly, $BK_n = u(L^n[1]) = u\mathbb{Z}_{tr}(\Sigma V)$ so $R_{tr}(BK_n) \cong R_{tr}S^\infty(\Sigma V)$. Now use (2.5). \qed

**Example 2.8.** The $i$-th power operation $x \mapsto x^i$ from $H^{2n,n}(X, \mathbb{Z})$ to $H^{2n,n,i}(X, \mathbb{Z})$ (and hence to $H^{2n,n,i}(X, R)$) has a natural interpretation in terms of this structure. In particular, if $i < \ell$ then $S_{tr}^i(L^n) \cong L^{ni}$ by 1.10 and [4, 15.8], and the power map corresponds to the generator of $\text{Hom}(S_{tr}^i(L^n), \mathbb{L}^{ni}) = \mathbb{Z}$.

Consider the diagonal $V \to V^\times \to S^i(V)$ for $V = \mathbb{A}^n/(\mathbb{A}^n - 0)$; applying $\mathbb{Z}_{tr}$ yields the motivic power map $L^n \to L^{ni}$ (see [4, 3.11]), and the symmetrizing map $L^{ni} \to S_{tr}^i(L^n)$. Applying $u$ to $L^n \to L^{ni}$ yields the map $K_n \to K_n$, representing the power map, and the adjoint of this may be identified with the map $S_{tr}^\infty(L^n) \to S_{tr}^i(L^n) \to L^{ni}$ by 2.7.

**Corollary 2.9.** Cohomology operations $\phi : H^{2n,n}(-, \mathbb{Z}) \to H^{p,q}(-, R)$ are in 1–1 correspondence with elements of

$$H^{p,q}(K_n, R) \cong \text{Hom}_{DM}(\bigoplus_{i=1}^{\infty} S_{tr}^i(L^n), R(q)[p]) = \prod_{i=1}^{\infty} \text{Hom}_{DM}(S_{tr}^i(L^n), R(q)[p]).$$

**Example 2.9.1.** We will see in Theorem 3.4 and 3.7.1 that $(\mathbb{Z}/\ell)_{tr}(K_n)$ is a sum of Tate motives, and that the only summands of weight $< n + \ell$ are $\mathbb{L}^{n+\ell-1}$, $\mathbb{L}^{n+\ell-1}[1]$ and the $S_{tr}^i(L^n) = L^{ni}$ for $i < 1 + \ell/n$. It is easy to see from 2.8 and [RPO] that these correspond to the cohomology operations sending $x$ to $P^1(\bar{x})$, $\beta P^1(\bar{x})$ and $\bar{x}^i$.

**Corollary 2.10.** Cohomology operations $\phi : H^{2,\ell}(-, \mathbb{Z}) \to H^{p,q}(-, R)$ are in 1–1 correspondence with homogeneous polynomials $f(t) = \sum_{i>0} a_i t^i$ of bidegree $(p, q)$ in $H^{*,*}(k, R)[t]$, where $t$ has bidegree $(2, 1)$. The operation corresponding to $f$ is $\phi(x) = \sum a_i x^i$.

**Proof.** It is well known that $R_{tr}(\mathbb{P}^1) \cong R \oplus L^1$ and $S^i\mathbb{P}^1 \cong \mathbb{P}^i$. Using Lemma 2.2, we obtain $S_{tr}^i(L^1) \cong L^i$. By 2.9, the classifying space $K_1 = uL^1$ has cohomology:

$$H^{p,q}(K_1, R) = \prod_{i=1}^{\infty} \text{Hom}_{DM}(L^i, R(q)[p]) = \prod_{i=1}^{\infty} H^{p-2i, q-i}(k, R).$$

In fact, $K_1 \cong \mathbb{P}^\infty$ by [5, 3.8]. The need for $a_0 = 0$ follows from $\phi(0) = 0$. \qed
3. Contents of “Operations II”

This section is devoted to an exposé of those results in [V-06] that we shall need in order to establish the Künneth formula 3.10 for $H^{*,*}(K_n \wedge \cdots \wedge K_n, R)$. These are: a description (in 3.1) of $S^{m}_{tr}$ in terms of $S^{\ell}_{tr}$; a description (in 3.4) of $S^{m}_{tr}(\mathbb{L}^n)$, and the structure theorem 3.7 that each $S^{n}_{tr}(\mathbb{L}^n)$ is a sum of proper Tate motives.

Let $G$ be any subgroup of the symmetric group $\Sigma_m$, and $H$ any subgroup of $\Sigma_n$. We set $S^{G}(X) = X^m/G$ and $S^{G}_{tr}(X_+) = R_{tr}(S^{G}(X_+))$. The usual embedding $\Sigma_m \times \Sigma_n \subset \Sigma_{m+n}$ induces

$$S^{G \times H}(X_+) = S^{G}(X_+) \times S^{H}(X_+) \quad \text{and} \quad S^{G \times H}_{tr}(M) = S^{G}_{tr}(M) \otimes S^{H}_{tr}(M),$$

where $M = R_{tr}(X_+)$. Similarly, the wreath product $G \wr H = G^n \times H \subset \Sigma_{mn}$ acts on $\{1, \ldots, mn\}$ by decomposing the set into $n$ blocks of $m$ elements, with $H$ permuting the blocks and $G$ acting inside the blocks. It is easy to see that

$$S^{H}(S^{G}(X_+)) = S^{G \wr H}(X_+).$$

Using the $\ell$-adic expansion $m = m_0 + m_1 \ell + \cdots + m_r \ell^r$ with $0 \leq m_i < \ell$, it is well known and easy to verify that the group

$$(3.0a) \quad G = \Sigma_{m_0} \times (\Sigma_{m_1} \wr \Sigma_{\ell}) \times (\Sigma_{m_2} \wr \Sigma_{m_0} \wr \Sigma_{\ell}) \times \cdots \times (\Sigma_{m_n} \wr \Sigma_{m_0})$$

contains a Sylow $\ell$-subgroup of $\Sigma_m$. By the above remarks, if $M = R_{tr}(V)$ then $S^{G}_{tr}(M)$ equals:

$$(3.0b) \quad S^{m_0}_{tr}(S^{\ell}_{tr}(M)) \otimes S^{m_1}_{tr}(S^{\ell}_{tr}(M)) \otimes \cdots \otimes S^{m_n}_{tr}(S^{\ell}_{tr}(M)).$$

**Proposition 3.1.** If $R$ is $\mathbb{Z}/\ell$ or $\mathbb{Z}/\ell$, and $M = R_{tr}(V)$ for a simplicial based $V$, then $S^{m}_{tr}(M)$ is a direct summand of the $S^{G}_{tr}(M)$ displayed in (3.0b).

**Proof.** (Voevodsky) The display is just $S^{G}_{tr}(M)$, and $S^{G}(V) = V^m/G$ is a ramified covering of $S^m(V) = V^m/\Sigma_m$ of degree $[\Sigma_m : G]$ prime to $\ell$. Its inverse is a finite correspondence, and the composition is multiplication by $[\Sigma_m : G]$ on $S^{m}_{tr}(V)$.

3.2. Let $C$ be the cyclic group $\mathbb{Z}/\ell$, set $A = \text{Aut}(C) = (\mathbb{Z}/\ell) \times$ and identify $G = C \times A$ with the affine subgroup of $\Sigma_\ell$. Since $[\Sigma_\ell : G] = (\ell - 2)!$, standard transfer arguments show that $S^{m}_{tr}(\mathbb{L}^n)$ is a summand of $S^{G}_{tr}(\mathbb{L}^n)$, which in turn is $(S^{\ell}_{tr}(\mathbb{L}^n))^A$.

We briefly recall the computation of $H^{*,*}(B_{gm\mu_\ell})$ and $H^{*,*}(B_{gm\mu_\ell})$ in [RPO, \S 6]. Suppose that $k$ has $\ell$-th roots of unity, so that we may identify $C$ with the algebraic group $\mu_\ell$. In [RPO, 6.10], Voevodsky showed that (for odd $\ell$):

$$H^{*,*}(B_{gm\mu_\ell}, R) \cong H^{*,*}([u,v]/(u^2), \quad v = \beta(u),$$

where $H^{*,*}$ denotes $H^{*,*}(k, R)$ and $u$ and $v$ have bidegrees $(1, 1)$ and $(2, 1)$, respectively. The group $A$ acts by algebra maps, and $a \in A$ satisfies: $a \cdot u = au, a \cdot v = av$; see [RPO, 6.11]. Thus for $c = u^\ell - 2$ and $d = -v^{\ell - 1}$ we have:

$$H^{*,*}(B_{gmG}, R) \cong H^{*,*}((B_{gm\mu_\ell}, R)^A \cong H^{*,*}([c,d]/(c^2), \quad d = \beta(c).$$

By [RPO, 13.14], $c$ and $d$ lift to $H^{*,*}(B_{gm}, R)$. This implies that the canonical map $H^{*,*}(B_{gm}, R) \to H^{*,*}(B_{gmG}, R) = H^{*,*}(B_{gmC}, R)^A$ is an isomorphism.

The following two theorems should be compared to the formulas announced in the 1996 preprint [13, 3.4.4–3.4.6].
Theorem 3.3. When $R = \mathbb{Z}/\ell$ and $n > 0$, $S^n_{tr}(L^n)$ is $\mathbb{A}^1$-equivalent to
\[ L^n \oplus \bigoplus_{i=1}^{n-1} \left( L^{n+i(\ell-1)} \oplus L^{n+i(\ell-1)}[1] \right). \]

Proof. This is proven by Voevodsky in [V-06, 4.5]. Voevodsky first proves in [V-06, 7.1] that $S^n_{tr}(L^n) \cong L^n \otimes R_{tr}(V - 0)/C[1]$, where $V$ is the direct sum of $n$ copies of the reduced regular representation of $C$.

The map $(V - 0)/C \to B_{gm}\mu_\ell$ of [5, §4.2] induces a map from $H^{\ast,\ast}(B_{gm}\mu_\ell, R)$ to $H^{\ast,\ast}((V - 0)/C, R)$; by [RPO, 6.1], it is an isomorphism in cohomology up to $n(\ell - 1)$. Voevodsky proves in [V-06, 7.5] that $R_{tr}(V - 0/C)$ is a sum of Tate motives with $u$ and $v$ corresponding to $L$ and $L[-1]$, so that:
\[ H^{\ast,\ast}((V - 0)/C, R) = H^{\ast,\ast}[u, v]/(u^2, v^{n(\ell - 1)}). \]

It follows that $H^{\ast,\ast}((V - 0)/G, R) = H^{\ast,\ast}[c, d]/(c^2, d^n)$. Translating this into motivic language proves that $R_{tr}(V - 0/G)$ is the sum of the $L^{(\ell - 1)}[-1]$ and $L^{\ell-1}$, and the theorem follows.

By 2.10, $S^n_{tr}(L^1) \cong L^{ni}$. This is the case $n = 1$ of the following calculation.

**Theorem 3.4.** When $R = \mathbb{Z}/\ell$ and $n > 0$, the natural map $S^n_{tr}(L^n) \to S^n_{tr}(L^n)$ is an isomorphism. Thus $S^n_{tr}(L^n)$ is given by the display in Theorem 3.3.

Proof. By the usual transfer argument, $S^n_{tr}(L^n)$ is a direct summand of $S^n_{tr}(L^n)$. Since the category of proper Tate motives is idempotent complete, it suffices to show that there is an isomorphism in motivic cohomology.

The quotient map $(V - 0)/\Sigma_{\ell} \to B_{gm}\Sigma_{\ell}$ induces a map from $H^{\ast,\ast}(B_{gm}\Sigma_{\ell})$ to the cohomology of $(V - 0)/\Sigma_{\ell}$, fitting into the commutative diagram:
\[ H^{\ast,\ast}(B_{gm}\Sigma_{\ell}, R) \to H^{\ast,\ast}(V - 0)/\Sigma_{\ell}, R) \to \text{Hom}(S^n_{tr}(L^n), R(\ast)[\ast] \otimes L^n[-1]) \]
\[ \cong \text{onto} \quad \cong \quad \text{into} \]
\[ H^{\ast,\ast}(B_{gm}G, R) \to H^{\ast,\ast}(V - 0)/G, R) \to \text{Hom}(S^n_{tr}(L^n), R(\ast)[\ast] \otimes L^n[-1]). \]

The left vertical map is an isomorphism by 3.2. We saw that the bottom map is a surjection in the proof of Theorem 3.3. Since the right vertical maps are injections by a transfer argument, they are isomorphisms by a diagram chase.

**Remark 3.4.1.** A different proof of 3.4 is given in the 2007 versions of [V-06]; it suffices to check that the composite $S^n_{tr}(L^n) \to S^n_{tr}(L^n) \to S^n_{tr}(L^n)$ is an isomorphism.

The following result is proven by Voevodsky in [V-06, 3.13–3.17], using the filtration of $S^n_{tr}$ for the cone of a map. Part (a) is immediate from $S^i(R_{tr}(X)) = e \cdot R_{tr}(X^i)$.

**Proposition 3.5.** Let $T$ be a motive for which the switch involution $\tau$ on $T \otimes T$ is $\pm 1$. Then (a) for $i < \ell$ the projection $\pi : T^{\otimes i} \to S^i_{tr}(T)$ of 1.10 is an isomorphism for $\tau = +1$ and zero for $\tau = -1$;

(b) for $\ell > 2$ there is an distinguished triangle:
\[ T^{\otimes \ell}[1] \to S^0_{tr}(T)[1] \to S^i_{tr}(T)[1] \to T^{\otimes \ell}[2], \quad \tau = +1, \]
\[ T^{\otimes (\ell - 1)} \to S^i_{tr}(T)[1] \to S^i_{tr}(T)[1] \to T^{\otimes (\ell - 1)}, \quad \tau = -1. \]
Now there is a topological realization functor, sending $\mathbb{A}^n/\mathbb{A}^n - 0$ to the based sphere $S^{2n}$, $L^{a}[b]$ to $\hat{C}_*(S^{a+b}, R)$ and $S^m_{tr}(L^{a}[b])$ to $\hat{C}_*(S^m(S^{a+b}), R)$. Thus we can refer to topology to calculate the maps in 3.5.

**Example 3.5.1.** When $T = R = \mathbb{Z}/\ell$, the map $\delta : R \to R$ is an isomorphism and we get $S^i_{tr}(R[1]) = 0$, and hence $S^i_{tr}(R[2]) \cong R[2\ell]$. These reflect the cohomology of the topological spaces $S^1 = K(\mathbb{Z}, 1)$ and $BS^1 = K(\mathbb{Z}, 2)$. Comparison with the cohomology of the Eilenberg Mac Lane spaces $K(\mathbb{Z}, n)$, described in [1], shows that for higher $b$ the maps $\delta$ are zero. This yields the inductive formula for $S^i_{tr}(R[b])$:

$$S^i_{tr}(R[b]) = \left\{ \begin{array}{ll}
\bigoplus_{i=1}^{b} R[b + 2i(\ell - 1)] \otimes (R \oplus R[1]), & \text{b odd,} \\
R[b\ell] \oplus \bigoplus_{i=1}^{b/2-1} R[b + 2i(\ell - 1)] \otimes (R \oplus R[1]), & \text{b even.}
\end{array} \right.$$

Thus the canonical map $S^i_{tr}(R[b])[1] \to S^i_{tr}(R[b + 1])$ is a split injection for all $b \neq 0$ (and the zero map for $b = 0$). Note that the singular cohomology $H^*(\hat{S}^i(b), R)$ can be read off from this formula using $\text{Hom}(S^i_{tr}(R[b]), R[*])$.

It is instructive to associate the terms in $S^i_{tr}(R[b])$ with the topological Steenrod operations $P^i$ and $\beta P^i (i < b/2)$ of [1] and [9].

**Corollary 3.6.** When $R = \mathbb{Z}/\ell$ and $a > 0$, $S^i_{tr}(L^{a}[b])[1] \to S^i_{tr}(L^{a}[b + 1])$ is a split injection for all $b$, and we have:

$$S^i_{tr}(L^{a}[1]) = \bigoplus_{i=1}^{a} \{L^{a+i(\ell-1)+1} \oplus L^{a+i(\ell-1)+2}\};$$

$$S^i_{tr}(L^{a}[b]) = S^i_{tr}(L^{a}[1])[b - 1] \oplus \bigoplus_{i=1}^{b} \{L^{a}[2i\ell + 1] \oplus L^{a}[2i\ell + 2]\}, \quad b = 2k + 1;$$

$$S^i_{tr}(L^{a}[b]) = S^i_{tr}(L^{a}[b - 1])[1] \oplus L^{a}[b\ell], \quad b \geq 2 \text{ even.}$$

**Proof.** Set $T = L^{a}[b]$ for $a > 0$. We will assume the result is true for $T[1]$ using the triangles in 3.5(b). When $b$ is odd, any map $T^{\otimes \ell} \delta[1] \to S^i_{tr}(T)[1]$ is zero for weight reasons, and hence the second triangle in 3.5 splits. This reduces us to the case of even $b \geq 0$.

When $b$ is even, we claim that the map $\delta$ is zero in the first sequence of 3.5(b), so the sequence splits. The initial case $S^i_{tr}(L^{a}[1])$ as well as the inductive case $S^i_{tr}(L^{a}[b + 1])$ will then follow from Theorem 3.3. Using the topological realization functor, this claim follows from the description of $H^*(\hat{S}^i(S^n), \mathbb{Z}/\ell)$ in 3.5.1. \qed

Here is the main theorem of [V-06], which was originally stated in [13, 3.4.1]. By a **proper Tate motive** we mean a direct sum of motives of the form $L^{a}[b]$ with $a \geq 0$ and $b \geq 0$. When $R$ is a field, the full subcategory of proper Tate motives is idempotent complete.

**Theorem 3.7.** When $R = \mathbb{Z}/\ell$, $S^\infty_{tr}(L^{a}) = R_{tr}(K_n)$ is a proper Tate motive. There are only finitely many summands $L^{a}[b]$ of any fixed weight $a$.

**Proof.** Combine 2.9, 3.1 and 3.6 with idempotent completion. \qed

**Remark 3.7.1.** It also follows from 3.6 that the terms of smallest weight in $S^\infty_{tr}(L^{a})$ are the $L^{a}[b]$, $L^{a+\ell - 1}[b]$ and $L^{a+\ell - 1}[1]$. 
Theorem 3.8. (Pure Küneth formula) Suppose that $X$ and $Y$ are pointed simplicial schemes such that $R_{tr}(Y)$ is a direct sum of motives $R(q_\alpha)[p_\alpha]$, and that for each $q$ there are only finitely many $\alpha$ with $q_\alpha = q$. Then the Küneth homomorphism is an isomorphism:

$$H^{*,*}(X, R) \otimes_{H^{*,*}(k, R)} H^{*,*}(Y, R) \to H^{*,*}(X \times Y, R).$$

It induces $\tilde{H}^{*,*}(X, R) \otimes \tilde{H}^{*,*}(Y, R) \cong \tilde{H}^{*,*}(X \wedge Y, R)$.

Proof. By assumption, we have $R_{tr}(X \times Y) \cong \oplus R_{tr}(X)(q_\alpha)[p_\alpha]$ and hence

$$H^{n,i}(X \times Y, R) \cong \prod \operatorname{Hom}_{DM}(R_{tr}(X)(q_\alpha)[p_\alpha], R(i)[n]).$$

The terms with $q_\alpha > i$ are summands of $H^{*,0}(X \times k^{q_\alpha-i}/(k^{q_\alpha-i} - 0), R)$ by the Cancellation Theorem [4, 16.25], and they vanish by [RPO, 3.5]. This leaves the finitely many terms with $q_\alpha \geq i$ which, by Cancellation, are:

$$H^{n-p_\alpha,i-q_\alpha}(X, R).$$

The case $X = \text{Spec}(k)$ shows that $H^{*,*}(Y, R)$ is a free graded $H^{*,*}(k, R)$-module with generators in bidegrees $(p_\alpha, q_\alpha)$, and the result follows.

Corollary 3.9. If $1/m! \in R$, then $H^{*,*}(S^m(Y), R) \cong \text{Sym}^m H^{*,*}(Y, R)$ for every $Y$ such that $R_{tr}(Y)$ is a sum of Tate motives.

Proof. By 3.8, the Küneth map $H^{*,*}(Y, R) \otimes \cdots \otimes H^{*,*}(Y, R) \xrightarrow{\sim} H^{*,*}(Y^m, R)$ is an isomorphism of free $H^{*,*}(k, R)$-modules. The symmetric group acts on both sides, and the Küneth map is equivariant, so the symmetric parts are equal. The symmetric part of $H^m$ is $\text{Sym}^m(H)$, whence the result.

The following result replaces the unproven “Lemma 2.3” in [MC/l].

Proposition 3.10. The Küneth homomorphisms for $R = \mathbb{Z}/\ell$ and $p > 0$,

$$H^{*,*}(K_n, \mathbb{Z}/\ell) \otimes_{H^{*,*}(k)} \cdots \otimes_{H^{*,*}(k)} H^{*,*}(K_n, \mathbb{Z}/\ell) \to H^{*,*}(K_n^{\otimes p}, \mathbb{Z}/\ell),$$

are isomorphisms. Hence $\tilde{H}^{*,*}(K_n, \mathbb{Z}/\ell) \otimes \cdots \otimes \tilde{H}^{*,*}(K_n, \mathbb{Z}/\ell) \cong \tilde{H}^{*,*}(K_n^{\otimes p}, \mathbb{Z}/\ell)$.

Proof. By Theorem 3.7, $R_{tr}(K_n)$ is a proper Tate motive, so 3.8 applies.

Remark 3.10.1. This was originally announced (in 1996) for $K_n/\ell$ in [13, 3.15].

4. Scalar weight

The goal of this section is to prove Theorem 4.2, which is our replacement for Lemma 2.2 of [MC/l]. It is based upon the notion of the scalar weight of an integral-to-modular cohomology operation.

The monoid $(\mathbb{Z}, \times)$ acts (via multiplication by scalars) on any abelian group and more generally on any presheaf $F$ of simplicial abelian groups (such as $\mathbb{Z}(q)[p]$) and hence on its underlying presheaf of simplicial sets $uF$. In particular, this monoid acts on $K_n = u\mathbb{L}^n$ and $BK_n = u\mathbb{L}^n[1]$. The induced action on motivic cohomology groups such as $H^{2n,n}(X, \mathbb{Z}) = [X, K_n]$ is just addition. We will be interested in the action on the cohomology operations represented by the groups $H^{*,*}(K_n, R)$ and $H^{p,q}(BK_n, R)$. Note that if $\alpha \in H^{2n,n}(X, \mathbb{Z})$ then $(\phi \cdot m)(\alpha) = \phi(m\alpha)$. 
Definition 4.1. An element \( \phi \in H^{\ast,\ast}(K_n, \mathbb{Z}/\ell) \) has scalar weight \( s \) (0 \leq s < \ell - 1) if \( \phi \cdot m = m^s \phi \) for all integers \( m \). Such a \( \phi \) has \( \phi \cdot \ell = 0 \), and \( \phi \cdot m = \phi \cdot m' \) if \( m \equiv m' \pmod{\ell} \). Hence \( \mathbb{Z}/\ell^\infty \) acts on the elements which have scalar weight \( s \).

Example 4.1.1. A cohomology operation \( \phi : H^{0,0}(-, \mathbb{Z}) \to H^{p,q}(-, R) \) is represented by a sequence \((\phi_j)\) in \( R \); see Example 0.1. The cohomology operation \( \phi \cdot m \) is represented by the sequence \((\phi_{mj})\). Most of these have no scalar weight.

Example 4.1.2. Any cohomology operation \( H^{2,1}(-, \mathbb{Z}) \to H^{p,q}(-, R) \) is represented by a homogeneous polynomial \( \sum a_m t^m \) of bidegree \((p, q)\) by Example 2.10. The operation \( \phi(x) = ax^m \), corresponding to the monomial \( at^m \), has scalar weight \( m \) modulo \((\ell - 1)\).

Example 4.1.3. Any stable cohomology operation, such as \( P^l \) or \( \beta \), is additive by [RPO, 2.9], so it has scalar weight one. If \( \phi_i \) has scalar weight \( s_i \), then the monomial \( x \mapsto \phi_1(x) \cdots \phi_m(x) \) has scalar weight \( \sum s_i \pmod{\ell - 1} \).

Example 4.1.4. The cohomology operation \( \phi_\ast : H^{2a+1,b}(-, R) \to H^{2a+1,b}(-, R) \) constructed in [MC/1, §3] has scalar weight one, because it satisfies \( \phi \cdot m = m^s \phi \) by [MC/1, 3.5]. (See Corollary 6.2 below.)

Recall from 2.9 that cohomology operations on \( H^{2n,n}(-, \mathbb{Z}) \) are sums of operations corresponding to elements of \( \text{Hom}(S^n_0(L^n), \mathbb{Z}/\ell \epsilon(*)[*]) \), arising from the decomposition \( R_{tr}(K_n) \iso \bigoplus S^n_0(L^n) \) in (2.5).

Theorem 4.2. The cohomology operations \( H^{2n,n}(-, \mathbb{Z}) \to H^{\ast,\ast}(-, \mathbb{Z}/\ell) \) corresponding to elements of \( \text{Hom}(S^n_0(L^n), \mathbb{Z}/\ell \epsilon(*)[*]) \) have scalar weight \( m \) \( \pmod{\ell - 1} \).

The proof of Theorem 4.2 will occupy the rest of this section. We first dispose of the cases \( m \leq \ell \).

Example 4.2.1. The cohomology operation corresponding to \( a \in H^{p-2n,q,n}(k, R) = \text{Hom}(S^n_0(L^n), R(q)[p]) \) is \( x \mapsto ax \). This is additive, and so has scalar weight one. Similarly, if \( 1 < m < \ell \) then it follows from Example 2.8 and 3.5(a) that the cohomology operation corresponding to \( a \in H^{\ast,\ast}(k, R) = \text{Hom}(S^n_0(L^n), R(*)[*]) \) is \( x \mapsto ax^m \); these have scalar weight \( m \).

Proposition 4.3. The cohomology operations \( \phi : H^{2n,n}(-, \mathbb{Z}) \to H^{p,q}(-, \mathbb{Z}/\ell) \) corresponding to elements of \( \text{Hom}(S^n_0(L^n), \mathbb{Z}/\ell \epsilon(*)[*]) \) are additive. Hence they have scalar weight one.

Proof. Set \( T = \mathbb{Z}/\ell \epsilon(q)[p] \). By 3.6, the map \( S^n_0(L^n)[1] \to S^n_0(L^n)[1] \) is a split injection, so \( \text{Hom}(S^n_0(L^n)[1], T[1]) \to \text{Hom}(S^n_0(L^n), T) \) is onto. Hence \( \phi \) lifts to a cohomology operation \( \phi_1 : H^{2n+1,n}(-, \mathbb{Z}) \to H^{p+1,q}(-, \mathbb{Z}/\ell) \), in the sense that the suspension \( \Sigma \phi(x) = \phi_1(\Sigma x) \). If \( x, y \in H^{2n,n}(X, Z) \) then by [RPO, 2.9], the cohomology operation \( \phi_1 \) is additive on \( H^{2n+1,n}(\Sigma X, Z) \), so:

\[
\Sigma \phi(x + y) = \phi_1(\Sigma (x + y) + \phi_1(\Sigma x + \Sigma y)) = \phi_1(\Sigma x) + \phi_1(\Sigma y) = \Sigma \phi(x) + \Sigma \phi(y).
\]

Since the suspension \( \Sigma \) is an isomorphism of groups, we are done.

Continuing the proof of Theorem 4.2, we next show that operations coming from \( S^n_0(L^n) \) may be factored using the \( \ell \)-adic expansion of \( m \). By Example 4.1.3, this reduces the proof of Theorem 4.2 to \( m = \ell^\nu \).

Lemma 4.4. Write \( m = m_0 + m_1 \ell + \cdots + m_r \ell^r \) with \( 0 \leq m_i < \ell \). Every cohomology operation \( \phi \) corresponding to an element of \( \text{Hom}(S^n_0(L^n), \mathbb{Z}/\ell \epsilon(*)[*]) \) is a sum of operations \( x \mapsto \phi_0(x) \phi_1(x) \cdots \phi_r(x) \), where the \( \phi_i \) are of \( S^n_0(L^n), \mathbb{Z}/\ell \epsilon(*)[*]) \).
Proof. By 3.1, $S^m_{tr}(\mathbb{L}^n)$ is a summand of $S^G_{tr}(\mathbb{L}^n)$, where $G$ is given in (3.0a). Hence $\phi$ is induced from a map $S^G_{tr}(\mathbb{L}^n) \to R(*)[s]$. By 3.8 and (3.0b),

$$\text{Hom}(S^G_{tr}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[s]) \cong \otimes_{i=0}^r \text{Hom}(S^{m_i}_{tr}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[s]),$$

so $\phi$ is induced by a sum of terms $\phi_0 \otimes \phi_1 \otimes \cdots \otimes \phi_r$. □

We now establish the case $m = \ell^r$ by induction on $\nu$, the case $\nu = 1$ being 4.3.

**Proposition 4.5.** Any cohomology operation coming from $\text{Hom}(S^e_{tr}(\mathbb{L}^n), R(*)[s])$ has scalar weight one.

Proof. Recall from 3.1 that $S^e_{tr}(\mathbb{L}^n)$ is a direct summand of $S^G_{tr}(S^{e-1}_{tr}(\mathbb{L}^n))$. Thus it suffices to treat cohomology operations of the form $S^e_{tr}(\mathbb{L}^n) \to S^G_{tr}(S^{e-1}_{tr}(\mathbb{L}^n)) \to T$. Write $S^{e-1}_{tr}(\mathbb{L}^n)$ as a sum of $\mathbb{L}^a[b]$. Then $S^e_{tr}S^{e-1}_{tr}(\mathbb{L}^n)$ is a sum of $S^e_{tr}(\mathbb{L}^a[b])$, which are additive by Lemma 4.6 below, and terms of the form

$$S^e_{tr}(\mathbb{L}^{a_1}[b_1]) \otimes \cdots \otimes S^e_{tr}(\mathbb{L}^{a_k}[b_k]), \quad \sum r_i = \ell.$$

These latter terms correspond to cohomology operations which are sums of monomials $\phi_1(x) \cdots \phi_k(x)$ which have scalar weight $\sum r_i = \ell \equiv 1$ (mod $\ell - 1$) by our inductive hypothesis. □

**Lemma 4.6.** Let $\mathbb{L}^a[b]$ be a summand of $S^{e-1}_{tr}(\mathbb{L}^n)$. Then the motivic cohomology operations on $H^{2n,a}(-, \mathbb{Z})$ corresponding to elements of $\text{Hom}(S^e_{tr}(\mathbb{L}^a[b]), \mathbb{Z}/\ell(*)[s])$ are additive. Hence they have scalar weight one.

Proof. By 3.5, the map from $S^e_{tr}(S^{e-1}_{tr}(\mathbb{L}^n)[1])$ to $S^e_{tr}(S^{e-1}_{tr}(\mathbb{L}^a)[1])$ restricts to a map $S^e_{tr}(\mathbb{L}^a[b])[1] \to S^e_{tr}(\mathbb{L}[b+1])$. We now argue as in the proof of Proposition 4.3. By 3.6, the map $\text{Hom}(S^e_{tr}(\mathbb{L}^a[b+1]), T[1]) \to \text{Hom}(S^e_{tr}(\mathbb{L}^a[b]), T)$ is onto. Hence $\phi$ lifts to a cohomology operation $\phi_1 : H^{2n+1,a}(-, \mathbb{Z}) \to H^{p+1,q}(-, \mathbb{Z}/\ell)$. But then $\phi$ is additive by [RPO, 2.9]. □

The proof of Theorem 4.2 is completed by the next result.

**Corollary 4.7.** If $a < \ell$, then every operation coming from $\text{Hom}(S^a_{tr}(\mathbb{L}^n), R(*)[s])$ has scalar weight $a$.

Proof. By 3.1, every such operation has the form $S^a_{tr}(\mathbb{L}^n) \to S^e_{tr}(S^{e-1}_{tr}(\mathbb{L}^n)) \to T$. By 3.9, every element of $\text{Hom}(S^a_{tr}(S^e_{tr}(\mathbb{L}^n)), T)$ is a sum of monomials $\phi_1 \cdots \phi_a$ where the $\phi_i$ belong to $\text{Hom}(S^e_{tr}(\mathbb{L}^n), T)$. By 4.5 and Example 4.1.3, these monomials have scalar weight $a$. □
5. Uniqueness of $\beta P^n$

The goal of this section is to prove Theorem 5.6, which is our replacement for Theorem 2.1 of [MC/l]. Our exposition follows §2 of [MC/l], except that Lemma 2.2 is replaced by the contents of the previous section, and the equations (2.6), (2.7) and (2.8) of [MC/l] are strengthened to inequalities when $m \geq \ell$ in the following result.

**Theorem 5.1.** If $R(q)[p]$ is a Tate summand of $S_{tr}^n(L^n)$ and $m \equiv s \mod (\ell - 1)$ for $m \geq 1$ and $0 \leq s \leq \ell - 2$, then:

1. $q \geq ns$, with equality iff $m < \ell$;
2. $q \geq n(\ell - 1)$ if $s = 0$, with equality iff $m = \ell - 1$;
3. $p \geq 2q \geq 2n$.

**Proof.** Recall from Theorem 3.7 that $R(q)[p] = \mathbb{L}^q[b]$ for $b \geq 0$, so $p = 2q + b$. As $m \geq 1$, (1) and (2) imply (3). If $m < \ell$ then $S_{tr}^n(L^n) = \mathbb{L}^{mn}$ and $q = mn$ by 3.5(a). This yields the ‘if’ part of (1) and (2). To prove the ‘only if’ parts of (1) and (2), suppose that $m > \ell$ and write $m = \sum m_i\ell$, noting that $\sum m_i > m_0$, $\sum m_i \equiv m \mod (\ell - 1)$. We also have $q \geq (\sum m_i)n + (\ell - 1)$ by 3.1 and 3.4. Since $\sum m_i \geq s$, we have $q \geq ns$. If $s = 0$ then $\sum m_i \geq \ell - 1$ and we have $q \geq (n + 1)(\ell - 1)$.

We now turn to the cohomology of $K_n \wedge \cdots \wedge K_n$ in scalar weight one. The following presentation is entirely due to Voevodsky and is taken from [MC/l, §2].

**Lemma 5.2.** The scalar weight one part of $\bar{H}^{p,q}(K_n^\wedge r, \mathbb{Z}/\ell)$ vanishes if $q < n\ell$ and $r > 1$, and also if $q = n\ell$ and $p < 2n\ell$.

**Proof.** ([MC/l, 2.7 and 2.8]) By 3.10 and 3.7 it suffices to consider the monomials $x_1 \otimes \cdots \otimes x_r$, where the $x_i$ are in $\text{Hom}(S^{m_i}(L^n), R(q_i)[p_i])$, with $\sum p_i = p$, $\sum q_i = q$ and $\sum m_i \equiv 1 \mod (\ell - 1)$. In the case $q < n\ell$ and $r \geq 2$ we have $m_i \neq 0$ by 5.1(2) and we must have $\sum m_i \geq \ell$, which is excluded by 5.1(1) as $q \geq n\sum m_i$. When $q = n\ell$ and $p < 2n\ell$, the vanishing comes from 5.1(3).

We now analyze the motivic cohomology $\bar{H}^{2n\ell + 2, n\ell}(BK_n, \mathbb{Z}/\ell)$, where $BK_n$ is the simplicial classifying space $[r] \mapsto K_n^r$. The standard spectral sequence for the cohomology of a simplicial space with coefficients $\mathbb{Z}/(n\ell)$ is

$$E_1^{r,s} = \bar{H}^{s,n\ell}(K_n^\wedge r, \mathbb{Z}/\ell) \Rightarrow \bar{H}^{r+s,n\ell}(BK_n, \mathbb{Z}/\ell).$$

The spectral sequence is bounded and converges for $n > 0$ by [MC/l, 2.6], because $E_1^{r,s} = 0$ for $r > \ell$. (This is because $n \geq 1$, and $K_n^\wedge r$ is $nr$-fold $T$-connected by 3.8.) Using Lemma 5.2, the relevant part of the spectral sequence looks like this:

$$
\begin{array}{c|c|c}
0 & \bar{H}^{2n\ell + 1, n\ell}(K) & \rightarrow \\
s = 2n\ell & 0 & \bar{H}^{2n\ell, n\ell}(K) \rightarrow \bar{H}^{2n\ell, n\ell}(K \wedge K) \rightarrow \bar{H}^{2n\ell, n\ell}(K \wedge K \wedge K) \\
s < 2n\ell & 0 & 0 \\
\end{array}
$$

(nothing in scalar weight one)

The $E_1$ page of the spectral sequence converging to $\bar{H}^{*, n\ell}(BK_n, \mathbb{Z}/\ell)$
Recall from [RPO, 3.7] that $H^{2n,n}(K_n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ on the fundamental class $\alpha$.

**Lemma 5.4.** For $r \geq 2$, the scalar weight one subgroup $\Gamma_r$ of $\tilde{H}^{2n,n}(K_n^r, \mathbb{Z}/\ell)$ is the free $H^*+(k, \mathbb{Z}/\ell)$-module generated by the monomials of the form $\alpha^1 \wedge \cdots \wedge \alpha^r$, where $\sum i_r = \ell$ and each $i_j > 0$.

**Proof.** This is [MC/l, 2.9].

**Example 5.5.** $\gamma = \alpha^{\ell-1} \wedge \alpha + \cdots + \alpha \wedge \alpha^{\ell-1}$ is an element of $\Gamma_2 \subset \tilde{H}^{2n,n}(K_n \wedge K_n, \mathbb{Z}/\ell)$. A calculation shows that $\gamma$ is a cycle in $E_1$; formally this follows from $\partial(\alpha^\ell) = (\alpha \otimes 1 + 1 \otimes \alpha)^\ell$.

**Theorem 5.6.** Let $\phi : H^{2n,n+1}(-, \mathbb{Z}) \to H^{2n,n+2}(-, \mathbb{Z}/\ell)$ be a cohomology operation such that for all $x \in H^{2n,n+1}(X, \mathbb{Z})$:

1. $\phi(mx) = m\phi(x)$ for all $m \in \mathbb{Z}$;
2. If $x = \sum y$ for $y \in H^{2n,n}(X, \mathbb{Z})$ then $\phi(x) = 0$.

Then $\phi$ is a multiple of $\beta P^n$.

**Proof.** (Voevodsky) We regard $\phi$ as an element of $\tilde{H}^{2n+2,n}(BK_n, \mathbb{Z}/\ell)$. Condition (1) says that $\phi$ has scalar weight one. Condition (2) says that $\phi$, like $\beta P^n$, is in the kernel of the map

$$\tilde{H}^{2n+2,n}(BK_n, \mathbb{Z}/\ell) \to \tilde{H}^{2n+2,n}(\Sigma K_n, \mathbb{Z}/\ell) = \tilde{H}^{2n+1,n}(K_n^\sticks, \mathbb{Z}/\ell)$$

defined by the inclusion of $\Sigma K_n$ in $BK_n$ as the part of simplicial degree one. That is, $\phi$ and $\beta P^n$ both lie in the kernel of the edge map in the spectral sequence, and belong to the subgroup $E^2_{2,2n}$ of $\tilde{H}^{2n+2,n}(BK_n, \mathbb{Z}/\ell)$.

Voevodsky makes the following observation (at the end of §2 in [MC/l]). By a formal calculation due to Lazard in [2, 12.1], the kernel of $E^2_{1,2n} \to E^3_{1,2n}$ is $\mathbb{Z}/\ell$ in scalar weight one, generated by the cycle $\gamma$ displayed in 5.5. Since $\beta P^n$ is nonzero by [1] and [RPO], it follows that every element in the kernel of the edge map must be a multiple of $\beta P^n$.

**Remark 5.6.1.** In topology, $\beta P^n$ is the mod-$\ell$ reduction of an integral cohomology operation $H^{2n+1}(-, \mathbb{Z}) \to H^{2n+2}(-, \mathbb{Z})$; see [1, Thm 5]. We will see in 6.4 that Voevodsky’s operation $\phi_V$ provides such a lift for $\beta P^n$ on $H^{2n+1,n}(-, \mathbb{Z})$. 


6. The Rost motive

We now consider the motivic operation \( \phi_V : H^{2p+1,q}(-, R) \to H^{2p+2,q}(-, R) \), constructed by Voevodsky in [MC/l, 3.1-3.2] for any coefficient ring \( R \) containing \( 1/(\ell - 1) \) (we use \( i = \ell - 1 \)).

**Proposition 6.1.** If \( \gamma \in H^{2r,s}(X, R) \) and \( \sigma \in H^{2p+1,q}(X, R) \) then

\[
\phi_V(\gamma \sigma) = \gamma^f \phi_V(\sigma).
\]

**Proof.** This is just Lemma 3.4 of [MC/l], where \( \gamma \) and \( \sigma \) are interpreted as maps \( R \to R[s][2r] \) and \( R \to R[q][2p + 1] \) in the triangulated category \( \text{DM}(X, R) \) for a smooth simplicial \( X \). \( \square \)

**Corollary 6.2.** (a) For any \( x \in H^{2p+1,q}(X, R) \) and \( m \in R \), \( \phi_V(mx) = m^f \phi_V(x) \).

(b) If \( x = \Sigma y \) for \( y \in H^{2p,q}(X, R) \) then \( \phi_V(x) = 0 \).

**Proof.** ([MC/l, 3.5 and 3.6]) The first assertion is just the case \( \gamma = m \), and the second is just the case \( \sigma \in H^{1,0}(S^1, R) \subset H^{1,0}(X \times S^1, R) \) with the observation that \( \phi_V(\sigma) = 0 \) because \( H^{2,0}(S^1, R) = 0 \). (See Example 6.2.) \( \square \)

**Example 6.2.1.** Suppose that \( n = 0 \). When \( R = \mathbb{Z}/(\ell) \), then \( \phi_V = 0 \) because \( S^1 = BK_0 \) and \( H^{2,0}(S^1, R) = 0 \). When \( R = \mathbb{Z}/(\ell) \), \( \phi_V \) is the Bockstein \( \beta : H^{1,0}(X, \mathbb{Z}/(\ell)) \to H^{2,0}(X, \mathbb{Z}/(\ell)) \). (This was shown by Voevodsky in [MC/l, 3.7].) It is well known that if \( L = K(\mathbb{Z}/(\ell), 1) \) is the Lens space then \( \beta(\tau) \neq 0 \), where \( \tau \in H^{1,0}(L, \mathbb{Z}/(\ell)) \cong \mathbb{Z}/(\ell).

**Example 6.2.2.** The following argument, implicit in [MC/l, 3.7], is taken from [9]. If \( X = (\mathbb{P}^N)^n \times L \) and \( x \in H^{2,1}(\mathbb{P}^N, \mathbb{Z}/(\ell)) \) is the generator, then \( x_1 \otimes \cdots \otimes x_n \otimes \tau \in H^{2n+1,n}(X, \mathbb{Z}/(\ell)) \) satisfies

\[
\phi_V(x_1 \otimes \cdots \otimes x_n \otimes \tau) = x_1^\ell \cdots x_n^\ell \beta(\tau)
\]

by 6.1, and this is nonzero if \( N \geq \ell \) by 6.2.1.

Let \( \tilde{\phi}_V \) denote the mod-\( \ell \) reduction of \( \phi_V \), considered as an operation from \( H^{2p+1,q}(-, \mathbb{Z}) \) to \( H^{2p+2,q}(-, \mathbb{Z}/(\ell)) \). Example 6.2.1 shows that \( \tilde{\phi}_V = 0 \) when \( n = 0 \). Thus the argument of [MC/l, 3.7], using Example 6.2.2, does not apply to show that \( \tilde{\phi}_V \neq 0 \). We substitute the following argument.

**Proposition 6.3.** For any \( n \geq 1 \), \( \tilde{\phi}_V \) is nonzero on \( H^{2n+1,n}(-, \mathbb{Z}) \). That is, there exists an \( X \) and an \( x \in H^{2n+1,n}(X, \mathbb{Z}) \) so that \( \tilde{\phi}_V(x) \neq 0 \) in \( H^{2n,nt}(X, \mathbb{Z}/(\ell)) \).

**Proof.** It suffices to consider the case \( n = 1 \), by the trick of Example 6.2.2. Let \( L \) be the Lens space, and \( \tau \in H^{1,0}(L, \mathbb{Z}/(\ell)) \) as in Example 6.2.1. We saw in 3.2 that \( u \in H^{1,1}(B_{gm}(\mathbb{Z}/(\ell)), \mathbb{Z}/(\ell)) \) has \( v = \beta(u) \). Set \( X = L \times B_{gm}(\mathbb{Z}/(\ell)), \) and consider the element \( x = \beta(u) \tau \) of \( H^{1,1}(X, \mathbb{Z}) \); the mod-\( \ell \) reduction of \( x \) is \( \beta(u) \tau = v \tau - u \beta(\tau) \). Invoking 6.1, we have

\[
\tilde{\phi}_V(x) = \phi_V(v \tau) - \phi_V(u \beta(\tau)) = v^\ell \beta(\tau) - v \cdot \beta(\tau)^\ell,
\]

and this is nonzero by the Künneth formula of [RPO]; see 3.2 or 3.8. \( \square \)

**Corollary 6.4.** The cohomology operations \( \tilde{\phi}_V \) and \( \beta P^n \) are non-zero multiples of each other.

**Proof.** By 6.3, \( \tilde{\phi}_V \) is a nonzero element of the vector space \( H^{2n+1,2nt}(K_n, \mathbb{Z}/(\ell)) \). By 6.2 and Theorem 5.6, \( \tilde{\phi}_V \) is a multiple of \( \beta P^n \). \( \square \)
Corollary 6.4 is our replacement for Theorem 3.8 in [MC/1]. We can now follow the presentation given by Voevodsky in [MC/1, §5].

**Lemma 6.5.** There are maps $\lambda : \Ztr(X) \to S^{t-1}(A)$ such that the inclusion $X \hookrightarrow \mathfrak{X}$ factors in $\text{DM}$ as

$$\Ztr(X) \xrightarrow{\lambda} S^{t-1}(A) \xrightarrow{S^{t-1}y} \mathfrak{X}.$$  

**Proof.** (Voevodsky [MC/1, 5.11]) Applying $\text{Hom}(\Ztr(X), -)$ to the triangle (1.9) yields the exact sequence

$$\text{Hom}(\Ztr(X), A) \xrightarrow{\nu} \text{Hom}(\Ztr(X), \Ztr(\mathfrak{X})) \xrightarrow{\tau} \text{Hom}(\Ztr(X), \mathfrak{X} \otimes L^b[1]) = 0.$$  

(The group on the right vanishes since it equals $H^{2b+1}(X, \mathbb{Z}) = 0$.) Hence $\tau$ factors as $y\lambda_1$ for some $\lambda_1 : \Ztr(X) \to A$. Inductively, we use the second triangle of 1.12 to get

$$\text{Hom}(\Ztr(X), S' A) \xrightarrow{\nu} \text{Hom}(\Ztr(X), S^{t-1}(A)) \xrightarrow{\tau} \text{Hom}(\Ztr(X), \mathfrak{X} \otimes L^b[1]) = 0.$$  

Again, the group on the right is $H^{2b+1,i}(X, \mathbb{Z}) = 0$, so we see that there are maps $\lambda_i : \Ztr(X) \to S'(A)$ for $i < \ell$ such that $\lambda_{i-1} = u\lambda_i$. By the construction of $u$, $yu^i = S'y : S'(A) \to \mathfrak{X}$. \qed

Recall that $\text{Hom}(L^d, \Ztr(X)) \cong H^0(X, \mathbb{Z})$ by Duality, so there is a fundamental class $\tau : L^d \to \Ztr(X)$. Since $\Ztr(X) \otimes \Ztr(\mathfrak{X}) \cong \Ztr(X)$, we may view it as a map from $\Ztr(\mathfrak{X}) \otimes L^d$ to $\Ztr(X)$.

**Proposition 6.6.** The composition $\Ztr(\mathfrak{X}) \otimes L^d \xrightarrow{\tau} \Ztr(X) \xrightarrow{\lambda} S^{t-1}(A)$ is not divisible by $\ell$.

**Proof.** (Voevodsky, [MC/1, 5.12]) By Lemma 1.8, $\alpha = Q_{n-1}(\mu)$ is a nonzero element of $H^{bd+2, bd}(\mathfrak{X}, \mathbb{Z}/\ell)$. Since $\alpha^2 = 0$, $Q_{n-1}(\alpha) = 0$. By the definition of $\phi_V$ in terms of the map $s$ of 1.12, the restriction of $\phi_V$ to $S^{t-1}(A)$ is zero. By 6.4, $\beta P^b$ also vanishes there. Since the $Q_i$ anticommute we have $Q_i(\mu) = 0$ for $i \leq n - 2$. By the definition of $Q_{n-1}$ we have

$$\alpha = Q_{n-1}(\mu) = Q_{n-2}(P^{\alpha-2} \mu) = \cdots = \beta P^b(\mu),$$

and we have seen that $(S^{t-1}y)^*(\alpha) = 0$ By [MC/1, 4.4], applied to $\alpha \neq 0$ and 6.5, the mod-$\ell$ reduction of the map $\Ztr(\mathfrak{X}) \otimes L^d \to S^{t-1}(A)$ is nonzero. \qed

Because $\mu : \mathfrak{X} \to \mathfrak{X} \otimes L^b[1]$ is a map between Tate objects, it is self-dual $(\mu = \mu^* \otimes L^b)$. It follows that $A \cong A^* \otimes L^b$. Since $S'(M)^* \cong (S'M)^*$ for every $M$ we also have $S^i(A) \cong S^i(A)^* \otimes L^b$. Cf. [MC/1, 5.7]. We write $D\lambda$ for the dual map

$$D\lambda : S^{t-1}(A) \cong S^{t-1}(A)^* \otimes L^d \xrightarrow{\lambda^*} \Ztr(X)^* \otimes L^d \cong \Ztr(X).$$

**Theorem 7.** The composition $\lambda \circ D\lambda$ is an isomorphism on the symmetric Rost motive $M = S^{t-1}(A)$ (with coefficients $\Ztr(\ell)$ or $\mathbb{Z}/\ell$), and there is an integer $c \neq 0$ (mod $\ell$) so that the following diagram commutes:

$$\begin{array}{ccc}
M & \xrightarrow{\lambda D \circ \lambda} & M \\
\downarrow S^{t-1}y & & \downarrow S^{t-1}y \\
R_{tr}(\mathfrak{X}) & \xrightarrow{c} & R_{tr}(\mathfrak{X}).
\end{array}$$

In particular, $M$ is a direct summand of $R_{tr}(X)$. 
Proof. This is proven by Voevodsky in [MC/l, 5.15].

Corollary 6.8. When \( R = \mathbb{Z}(\ell) \), the maps \( \lambda \) and \( D \lambda \) make \( M = S_{tr}(X) \) into a direct summand of \( R_{tr}(X) \), and the following composition is an isomorphism.

\[
M \cong M^* \otimes L^d \xrightarrow{\lambda^*} R_{tr}(X)^* \otimes L^d \cong R_{tr}(X) \xrightarrow{\lambda} M.
\]

Indeed, this is just a restatement of Theorem 6.7 in the form of axioms 1.3(a,b). Since axiom 1.3(c) holds by 1.12, \( M \) is a Rost motive for \( \underline{a} \). By Theorem 1.4, it follows that the norm residue homomorphisms \( K^n_{\alpha}(k)/\ell \rightarrow H^i_{et}(k, \mu_{\ell^n}) \) are isomorphisms, verifying the Bloch-Kato conjecture.

References


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