Weight structures, weight filtrations, weight spectral sequences, and weight complexes (for motives and spectra)

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Abstract

The goal of the current paper is to generalize certain results of the previous paper [9] to a large class of triangulated categories and functors. Our basic notion is the new definition of a weight structure for a triangulated $\mathcal{C}$. We prove that a weight structure defines Postnikov towers of objects of $\mathcal{C}$; these towers are canonical and functorial "up to cohomology zero". For $\mathcal{H}$ being the heart of the weight structure we define a canonical conservative weakly exact functor $t$ from $\mathcal{C}$ to a certain weak category of complexes $K_w(\mathcal{H})$. For any (co)homological functor $H : \mathcal{C} \to A$ for an abelian $A$ we construct a weight spectral sequence $T : H(X^i[j]) \Rightarrow H(X[i + j])$ where $(X^i) = t(X)$; it is canonical and functorial starting from $E_2$. This spectral sequences specializes to the usual weight spectral sequences for "classical" realizations of (Voevodsky’s) motives. We prove that $K_0(\mathcal{C}) \cong K_0(\mathcal{H})$ in the bounded case if $\mathcal{H}$ is idempotent complete. Under certain restrictions, we prove a similar equality for $K_0(\text{End} \mathcal{C})$.

These result give us a better understanding of Voevodsky’s motives (that were studied in [9]) and also of the stable homotopy category $SH$. In particular, we calculate very explicitly the groups $K_0(SH_{fin})$ and $K_0(\text{End} SH_{fin})$ (and also certain $K_0(\text{End}^n SH_{fin})$ for $n \in \mathbb{N}$). In this case we also have $K_w(\mathcal{H}) = K(\mathcal{H}) \cong K(\text{Ab}_{fin, fr})$. Besides we obtain a certain "weight filtration" on homotopy groups of spectra (and the corresponding "weight" spectral sequence).

The definition of a weight structure for a triangulated category $\mathcal{C}$ is almost dual to those of a $t$-structure; yet some properties of these

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definitions are surprisingly distinct. Under certain conditions for a weight structure \( w \) one can construct a certain \( t \)-structure which is \emph{adjacent} to \( w \). Vice versa, for a \( t \)-structure one can often construct adjacent weight structures (such that either \( C_{w \leq 0} = C_{t \leq 0} \) or \( C_{w \geq 0} = C_{t \geq 0} \)). In particular, this is the case for the Voevodsky’s category \( DM^{eff} \) (one obtains certain \emph{Chow} weight and \( t \)-structures) and for the stable homotopy category. The hearts of \emph{adjacent} structures are dual in a very interesting sense.

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Introduction

In [9] for a triangulated category $C$ with a (negative) differential graded enhancement a conservative exact weight complex functor $t_0 : C \to K(Hw)$ for a certain additive $Hw$ was constructed. For any 'enhancement' realization functor $G : C \to K(A)$ for an abelian $A$ a spectral sequence starting from the cohomology of $t_0(X)$ and converging to the cohomology of $G(X)$ was constructed. It was proved that $K_0(C) \cong K_0(Hw)$ if $Hw$ is idempotent complete.
The goal of the current paper is to generalize these results to categories and functors that do not (necessarily) have a differential graded enhancement. Our basic notion is the new definition of a weight structure. We obtain the generalizations wanted (up to certain modifications) as well as several new results. The main idea is that a weight structure defines Postnikov towers of objects; these towers are canonical and functorial "up to cohomology zero". These results give us a better understanding of Voevodsky's motives (that were studied in [9]) and of the stable homotopy category \( SH \) (that doesn't possess any enhancements of this sort!). In particular, we calculate very explicitly the groups \( K_0(SH_{fin}) \) and \( K_0(\text{End} \ SH_{fin}) \) (and also certain \( K_0(\text{End}^n \ SH_{fin}) \) for \( n \in \mathbb{N} \)). Besides we obtain a certain "weight filtration" on homotopy groups of spectra (and the corresponding "weight" spectral sequence). The author doesn't think that all of these results are new; yet they illustrate our methods very well.

The definition of a weight structure for a triangulated category \( \mathcal{C} \) is almost dual to those of a \( t \)-structure; yet some properties of these definitions are surprisingly distinct. The heart \( Hw \) of a weight structure is defined in the same way as the heart of a \( t \)-structure; yet if \( A \to B \to C \to A[1] \) is a distinguished triangle in \( \mathcal{C} \) whose terms belong to \( Hw \) then it necessarily splits. Besides, any weight structure defines a canonical conservative weakly exact functor \( t \) (the weight complex functor) from \( \mathcal{C} \) to a certain weak category of complexes \( K_w(Hw) \) (which is a factor of \( K(Hw) \)). For \( \mathcal{C} = SH \) we have \( K_w(Hw) = K(Hw) \cong K(\text{Ab}_f) \) (the homotopy category of complexes of free abelian groups); so \( t \) is exact.

For any (co)homological functor \( H \) and \( X \in \text{Obj} \mathcal{C} \) one has a spectral sequence \( T: H(X^i[j]) \to H(X[i + j]) \) where \( (X^i) = t(X) \). This spectral sequence is canonical and functorial starting from \( E_2 \); it specializes to the usual weight spectral sequences for "classical" realizations of motives.

Under certain conditions (see §4) for a weight structure \( w \) one can construct a certain \( t \)-structure which is adjacent to \( w \). Vice versa, for a \( t \)-structure one can often construct adjacent weight structures (such that either \( C^{w \leq 0} = C^{t \leq 0} \) or \( C^{w \geq 0} = C^{t \geq 0} \)). In particular, this is the case for the Voevodsky's category \( DM_{eff} \) and for the stable homotopy category. The hearts of adjacent structures are dual in a very interesting sense (see Theorem 4.3.2).

A predecessor of the definition of a weight structure were the classical notions of "filtration bete" (see §3.1.7 of [4]) and of connective spectra (see §7 of [19]). Yet our axiomatics and all basic results are completely new. In [9] a weight structure was (essentially) constructed for Voevodsky's motives. A similar construction for Hanamura's motives was described in §1 of [14] (see property (6) in the end of the §1 loc. cit.); note that in §4 of [9] it was proved that Hanamura's category of motives is anti-equivalent to (the rational hull
of) those of Voevodsky.

A general example of a category which has a natural weight structure is the category of twisted complexes over a negative differential graded category; all these notions are defined in section 6. In [9] these concepts were studied in detail; it was shown that the Voevodsky's category of motives $DM^* \subset DM^{eff}$ is an example of our situation (without mentioning weight structures explicitly); we also recall those results here. The relevant definitions and constructions are described in subsection 6.4 independently from [9]; yet an interested reader should certainly compare the differential graded versions of proofs (presented in [9]; see also [5]) with those here.

Another important example of a category with adjacent weight and $t$-structures is the stable homotopy category $SH$. The weight complex functor in this case computes singular homology and cohomology of spectra, see §4.5. Note that $SH$ certainly cannot have a differential graded description! The spherical weight structure is "generated" by the sphere spectrum; it is left adjacent to the (usual) Postnikov $t$-structure on $SH$. Note also that the spherical weight structure is also defined on the category of finite spectra, whence the Postnikov $t$-structure is not. The reason for this is that the sphere spectrum is finite while Eilenberg-Maclane spectra aren't.

Our results in [9] easily yield a Chow weight structure for $DM_{gm}^{eff}$ (and hence also for $DM_{gm}$) such that $\omega^{w=0} = Chow^{eff}$; we describe it in §6. We also prove in §7.1 that there is a Chow $t$-structure on $DM_{gm}^{eff}$ that is right adjacent to the Chow weight structure. Moreover, an easy application of the weight spectral sequence yields that if the "mixed motivic" cohomology of motives exists, then its images have certain canonical weight filtration, which behaves well under regulators.

Now we list the contents of the paper.

In section 1 we give the definition of a weight structure $w$ in a triangulated category $\mathcal{C}$. We describe some other basic definitions and prove their (relatively) simple properties. Our central objects of study are weight decompositions of objects and morphisms. We also describe certain Postnikov towers for object of $\mathcal{C}$ that come from weight structures.

In section 2 we describe the weight spectral sequence $T(H, X)$ (for $X \in \text{Obj}\mathcal{C}$ and a (co)homological functor $H : \mathcal{C} \to A$) that comes from the Postnikov towers described. It is canonical and functorial starting from $E_2$. It specializes to the "usual" weight spectral sequences for "classical realizations" of varieties (or motives; at least with rational coefficients). Moreover, in this case the spectral sequence degenerates at $E_2$ and its $E_2$-terms are exactly the graded pieces of the weight filtration.

In section 3 we define the weight complex functor. Its target is a certain
"weak category of complexes" $K_w(Hw)$. $K_w(Hw)$ is a factor of $K(Hw)$ which is no longer triangulated; yet the kernel of the projection $K(Hw) \to K_w(Hw)$ is an ideal whose square is zero so our ("weak") weight complex functor is not much worse than the "strong" one of [9]. In particular, it is conservative, weakly exact and preserves the filtration given by the weight structure. Still we conjecture that the "strong" weight complex functor exists also; see Remark 3.3.4 and §8.2. Besides, in some cases (for example, for all subcategories of $SH$ mentioned in this paper) we have $K_w(Hw) = K(Hw)$. Our main tool of study is the weight decomposition functor $WD : C \to K_w^{[0,1]}(C)$; see Theorem 3.2.2.

In section 4 we prove that weight structures are closely related to $t$-structures. In particular, in several cases a triangulated category possesses simultaneously a $t$-structure and a weight structure which are "dual" in a very interesting sense. Besides, a weight structure could often be described in terms of some "negative" additive subcategory of $C$.

In §4.5 we apply our results to the study of the stable homotopy category. It turns out that the weight complex for it calculates the singular (co)homology of spectra. Besides our results immediately yield a certain "weight filtration" on homotopy groups of spectra (and the corresponding "weight" spectral sequence).

In section 5 we prove that a bounded $C$ is idempotent complete iff $Hw$ is; the idempotent completion of a general bounded $C$ has a weight structure whose heart is the idempotent completion of $Hw$. If $C$ is bounded and idempotent complete then $K_0(C) \cong K_0(Hw)$. In §5.4 we study a certain Grothendieck group of endomorphisms in $C$. Though it is not always isomorphic to $K_0(End Hw)$, it is if $Hw$ is regular in a certain sense. Besides, we can still say something about $K_0(End C)$ in the general case. In particular, this allows us to generalize Theorem 3.3 of [7] (on independence of $l$ for traces of "open correspondences"); see also §8.4 of [9]. As an application of our results, we also calculate explicitly the groups $K_0(SH_{fin})$ and $K_0(End SH_{fin})$ (along with their ring structure). We also extend these results to the calculation of certain $K_0(End^n SH_{fin})$ for $n \in \mathbb{N}$.

In section 6 we translate the results of [9] into the language of weight structures. In particular, we show that Voevodsky's $DM_{gm}^{eff}$ ($\subset DM_{gm}$) admits the Chow weight structure whose heart it $Chow^{eff}$ (resp. Chow). This allows us to prove that the weight spectral sequence for realizations (described in §7 of [9]) exists for any realizations (not necessarily admitting a differential graded enhancement) and does not depend on the choice of enhancements.

In section 7 we show that the Chow weight structure of $DM_{gm}^{eff}$ extends
to $DM_{eff}^c$ and admits a right adjacent Chow $t$-structure (whose heart is the category $\text{Chow}_{eff}^c = \text{AddFun}(\text{Chow}_{eff}^c, Ab) \supset \text{Chow}_{eff}^c$). We prove that any possible (conjectural) "mixed motivic" $t$-structure induces a canonical "weight filtration" on the values of the corresponding homological functor $DM_{eff}^c \to MM$. Lastly, we prove that the weight complex functor could be defined on $DM_{eff}^c$ without using the resolution of singularities (so one could define it for motives over any perfect field).

In section 8 we show that a weight structure $w$ on $\underline{C}$ which induces a weight structure on a triangulated $\underline{D} \subset \underline{C}$ yields also a weight structure on the localization $\underline{C}/\underline{D}$. Next we prove (by an argument due to A. Beilinson) that any $f$-category enhancement of $\underline{C}$ yields a "strong" weight complex functor $\underline{C} \to K(Hw)$.

We also describe our ideas on the (possible) "higher truncation functors" in our setting (related to those described in [9]) and on other possible sources of conservative "weight complex-like" functors.

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**Notation.** For a category $C$, $A, B \in \text{Obj}C$, we denote by $C(A,B)$ the set of $A$-morphisms from $A$ into $B$.

For categories $C, D$ we write $C \subset D$ if $C$ is a full strict subcategory of $D$. Recall that $D$ is called *strict* if it contains all objects in $\text{Obj}C$ isomorphic to those from $\text{Obj}D$.

For a category $C$, $X, Y \in \text{Obj}C$ we say that $X$ is a retract of $Y$ if $id_X$ could be factorized through $Y$. Note that if $C$ is triangulated or abelian then $X$ is a retract of $Y$ iff $X$ is its direct summand. For an additive $D \subset C$ the subcategory $D$ is called *Karoubi-closed* in $C$ if it contains all retracts of its objects in $C$.

$X \in \text{Obj}C$ will be called compact if $X^\ast = \underline{C}(X, -)$ commutes with arbitrary direct sums.

For a category $\underline{C}$ we denote by $\underline{C}^{\text{op}}$ the opposite category.

$\underline{C}$ will usually denote a triangulated category; usually it will be endowed with a weight structure $w$ (see Definition 1.1.1 below). We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a for a functor that preserves the structures of triangulated categories). We will call a covariant additive functor $\underline{C} \to A$ for an abelian $A$, homological if it converts distinguished triangles into long exact sequences; homological functors $\underline{C}^{\text{op}} \to A$ will be called cohomological when considered as contravariant functors $\underline{C} \to A$. 

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For $f \in C(X,Y)$, $X,Y \in \text{Obj} \mathcal{C}$ we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of $f$. Note that different choices of cones are connected by non-unique isomorphisms, cf. IV.1.7 of [12]. Besides, in $C(A)$ we have canonical cones of morphisms, see section §III.3 of [12].

We will often specify a distinguished triangle by two of its morphisms. The author apologizes for this as well as for absence of certain diagrams (mostly 'octahedron diagrams') that could help to understand this text.

When dealing with triangulated categories we (mostly) use conventions and auxiliary statements of [12]. For a set of objects $C_i \in \text{Obj} \mathcal{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full subcategory containing all $C_i$; for $D \subset \mathcal{C}$ we will write $\langle D \rangle$ instead of $\langle \text{Obj} D \rangle$.

We will say that $C_i$ generate $\mathcal{C}$ if $\mathcal{C}$ equals $\langle C_i \rangle$. We will say that $C_i$ weakly generate $\mathcal{C}$ if for $X \in \text{Obj} \mathcal{C}$ we have $\mathcal{C}(C_i[j], X) = 0$ $\forall i \in I$, $j \in \mathbb{Z}$ $\implies X = 0$.

In this paper all complexes will be cohomological i.e. the degree of all differentials is $+1$; respectively, we will use cohomological notation for their terms.

For an additive category $A$ we denote by $C(A)$ the unbounded category of complexes over $A$; $K(A)$ is the homotopy category of $C(A)$ i.e. the morphisms of complexes are considered up to homotopy equivalence; $C^{-}(A)$ denotes the category of complexes over $A$ bounded above; $C^{b}(A) \subset C^{-}(A)$ is the subcategory of bounded complexes; $K^{b}$ denotes the homotopy category of bounded complexes. We will denote by $C(A)^{\leq i}$ (resp. $C(A)^{\geq i}$) the unbounded category of complexes concentrated in degrees $\leq i$ (resp. $\geq i$).

For an abelian $A$ we will denote by $D(A)$, $D^{-}(A)$, $D^{b}(A)$ the corresponding versions of the derived category of $A$.

$A b$ is the category of abelian groups; $A b_{fg}$ is the category of free abelian groups; $A b_{fg,fr}$ is the category of finitely generated free abelian groups.

For additive $C,D$ we denote by $\text{AddFun}(C,D)$ the category of additive functors from $C$ to $D$ (we will be always able to assume that $C,D$ are small). For an additive $A$ we will denote by $A^{*}$ the category $\text{AddFun}(A, A b)$ and by $A_{*}$ the category $\text{AddFun}(A, A b^{op})$. Note that both of these are abelian. Moreover, Ioneda’s lemma gives full embeddings of $A$ into $A_{*}$ and of $A^{op}$ into $A^{*}$ (these send $X \in \text{Obj} A$ to $X_{*} = A(\cdot, X)$ and to $X^{*} = A(X, \cdot)$, respectively). $A'_{*}$ will denote the full abelian subcategory of $A_{*}$ generated by $A$.

It is easily seen that any object of $A$ becomes projective in $A_{*}$. Besides, any object of $A_{*}$ has a resolution by (infinite) direct sums of objects of $A$. These fact are rather easy; the proofs can be found in the beginning of §8 of [22].
$\text{Corr}_{\text{rat}}$ will denote the (homological) category of rational correspondences. Its objects are smooth projective varieties; the morphisms are morphisms in $\text{SmCor}$ up to homotopy equivalence. The category $\text{Chow}^{\text{eff}}$ is the idempotent completion of $\text{Corr}_{\text{rat}}$; it was shown in Proposition 2.1.4 of [25] that $\text{Chow}^{\text{eff}}$ is naturally isomorphic to the usual category of effective homological Chow motives. Finally, $\text{Chow}$ denotes the full category of Chow motives, i.e. we invert the Tate twist in $\text{Chow}^{\text{eff}}$.

The definition of a cocompact object is dual to those of a compact one: $X \in \text{Obj} \mathcal{C}$ is cocompact if $\mathcal{C}(\prod_{i \in I} Y_i, X) = \bigoplus \mathcal{C}(Y_i, X)$ for any set $I$ and any $Y_i \in \text{Obj} \mathcal{C}$ such that the product exists.

We list the main definitions of this paper. Weight structures, $\mathcal{C}^{w \geq 0}, \mathcal{C}^{w \leq 0}$, and weight decompositions of objects are defined in Definition 1.1.1; $Hw$ (the heart of $w$), $\mathcal{C}^{w=0}, \mathcal{C}^{w \geq 1}, \mathcal{C}^{w \leq 1}, \mathcal{C}^{[j,i]}$, non-degenerate, and bounded (above, below or both) weight structures are defined in Definition 1.2.1; $X^{w \leq i}$ and $X^{w \geq i+1}$ are defined in Remark 1.2.2; $\mathcal{C}^-, \mathcal{C}^+$, and $\mathcal{C}^b$ are defined in Definition 1.3.3; several notation and definitions for weight decomposition of morphisms, (infinite) weight decomposition of objects, and Postnikov towers for objects are introduced in §1.5; the weight filtration of functors is introduced in Definition 2.1.1; the weight complex of objects is defined in Definition 2.2.1; weight spectral sequences are introduced in §2.3 and §2.4; the weak category of complexes $K_w(A)$, distinguished triangles in it, and weakly exact functors are defined in Definition 3.1.4; the weight decomposition functor $WD$ and the weight complex functor $t$ are described in Theorem 3.2.2; negative subcategories, Karoubi-closures, and small envelopes are introduced in Definition 4.1.1; the categories $SH$ and $SH_{\text{fin}}$ of spectra are mentioned in Corollary 4.1.3; $t$-structures are recalled in §4.2; adjacent (weight and $t$-structures) are defined in Definition 4.3.1; negatively well-generating sets of objects are defined in Definition 4.4.1; more categories of spectra, singular cohomology, and singular homology $H^{\text{sing}}$ of spectra are considered in §4.5; we discuss idempotent completions in §5.1; $K_\sigma$-groups of $Hw, \mathcal{C}, \text{End} Hw, \text{End} \mathcal{C}, \text{End}^a Hw$, and $\text{End}^a \mathcal{C}$ are defined in §5.3 and §5.4; regular additive categories are defined in Definition 5.4.2; differential graded categories and twisted complexes over them are defined in §6.1; truncation functors $t_N$ are constructed in §6.3; the spectral sequence $S(H, X)$ is considered in §6.4; we recall $\text{SmCor}, J, \mathfrak{S}, DM^{\text{eff}}, DM^*, DM_{\text{gm}}^{\text{eff}}, DM_{\text{gm}},$ and $\mathfrak{S}$ in §6.5.
1 Weight structures in triangulated categories: basic definitions and results; auxiliary statements

In this section we give the definition of a weight structure \( w \) in a triangulated category \( \mathcal{C} \) (in §1.1) (this includes the notion of a weight decomposition of an object). We give other basic definitions and prove their certain simple properties in §1.2 and §1.3. We recall certain auxiliary statements that will help us to prove that the weight decomposition is functorial (in a certain sense) in §1.4. We study weight decompositions of morphisms, infinite weight decompositions and Postnikov towers for objects in §1.5.

1.1 Weight structures: definition and simple examples

Definition 1.1.1 (Definition of a weight structure). A pair of subclasses \( \mathcal{C}^{w \geq 0}, \mathcal{C}^{w \leq 0} \subset \text{Obj} \mathcal{C} \) for a triangulated category \( \mathcal{C} \) will be said to define a weight structure \( w \) if \( \mathcal{C}^{w \geq 0}, \mathcal{C}^{w \leq 0} \) satisfy the following conditions:

(i) \( \mathcal{C}^{w \geq 0}, \mathcal{C}^{w \leq 0} \) are Karoubi-closed (i.e. contain all retracts of their objects that belong to \( \text{Obj} \mathcal{C} \)).

(ii) 'Semi-invariance' with respect to translations. \( \mathcal{C}^{w \geq 0} \subset \mathcal{C}^{w \geq 0}[1], \mathcal{C}^{w \leq 0}[1] \subset \mathcal{C}^{w \leq 0}. \)

(iii) Orthogonality. For any \( X \in \mathcal{C}^{w \geq 0}, Y \in \mathcal{C}^{w \leq 0}[1] \) we have \( \mathcal{C}(X, Y) = 0. \)

(iv) Weight decomposition. For any \( X \in \text{Obj} \mathcal{C} \) there exists a distinguished triangle

\[
B[-1] \to X \to A \xrightarrow{j} B
\]

such that \( A \in \mathcal{C}^{w \leq 0}, B \in \mathcal{C}^{w \geq 0}. \)

The triangle (1) will be called a weight decomposition of \( X. \)

The basic example of a weight structure is given by the stupid filtration on the homotopy category of complexes over an arbitrary additive category \( A. \) We will omit \( w \) in this case and denote by \( K(A)^{\leq 0} \) (resp. \( K(A)^{\geq 0} \)) the set of complexes that are (up to homotopy) retracts of complexes whose representatives are concentrated in degrees \( \leq 0 \) (resp. \( \geq 0 \)). Its heart (see Definition 1.2.1 below) lies in the idempotent completion of \( A \) (it is its small envelope; see part 3 of Definition 4.1.1 below). Moreover, we will see below (cf. Theorems 3.2.2, 3.3.1, and Remark 3.3.4) that this example is "almost universal" if one fixes the heart.
Note that in the case when \( A \) is an abelian category with enough projectives and injectives then often the appropriate version of \( D(A) \) is equivalent to \( K(I) \) and \( K(P) \) where \( P \) and \( I \) denote the categories of projective and injective objects of \( A \). Hence we see that some triangulated categories can support more then one weight structure; note that their hearts are usually not isomorphic.

1.2 Other definitions

The following definitions are also very important.

**Definition 1.2.1.** [Other basic definitions]

1. A category \( Hw \) whose objects are \( C^{w=0} = C^{w \geq 0} \cap C^{w \leq 0} \), \( Hw(X, Y) = C(X, Y) \) for \( X, Y \in C^{w=0} \), will be called the heart of the weight structure \( w \). We will see below that \( Hw \) is additive.

2. \( C_{w>1} \) (resp. \( C_{w \leq 1} \)) will denote \( C^{w \geq 0}[-1] \) (resp. \( C^{w \leq 0}[-1] \)).

3. For all \( i, j \in \mathbb{Z}, i \geq j \) we define \( C^{[j,i]} = C_{w \geq j} \cap C_{w \leq i} \). B abuse of notation, we will sometimes identify \( C^{[j,i]} \) with the corresponding full additive subcategory of \( C \).

4. \( w \) will be called non-degenerate if

\[
\cap_i C^{w \geq i} = \cap_i C^{w \leq i} = \{0\}.
\]

5. \( w \) will be called bounded above (resp. bounded below) if \( \cup_i C^{w \leq i} = \text{Obj}_C \) (resp. \( \cup_i C^{w \geq i} = \text{Obj}_C \)).

6. \( w \) will be called bounded if it is bounded both above and below.

There is an important difference between ‘decompositions of objects’ with respect to \( t \)-structures and weight structures.

**Remark 1.2.2.** Note that (in contrast to the \( t \)-structure situation) the presentation of \( X \) in the form (1) is (almost) never unique. The only exception is the following totally degenerate situation; for any \( X \in \text{Obj}_C \) there exist \( Y \in \cap_i C^{w \leq i} \) and \( Z \in \cap_i C^{w \geq 1} \) such that for some \( f \in C(Y, Z) \) we have \( X \approx \text{Cone}(f) \). Indeed, otherwise we can replace \( (A, B, f) \) by \( (A \oplus D, B \oplus D, f \oplus id_D) \) for any \( D \in C^{w=0} \). It could be easily seen that \( C^{w=0} \) is zero only in the totally degenerate case (we don’t give the proof since degenerate cases are not very interesting).

Yet we will need to choose some \( (A, B, f) \) several times. We will write that \( A = X^{w \leq 0}, B = X^{w \geq 1} \) if there exists a distinguished triangle (1). In Theorem 3.2.2 below we will verify that \( X \rightarrow (A, B, f) \) is a functor ‘up to zero cohomology’.

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We will also often denote \( X[-i]_{w \leq 0} \) by \( X^w_{\leq i} \) and \( X[i]_{w \geq 1} \) by \( X^w_{\geq i+1} \) for all \( i \in \mathbb{Z} \). Note that we have \( X^w_{\leq i} \in C^w_{\leq 0} \) and \( X^w_{\geq i} \in C^w_{\leq 0} \).

Below we will introduce a similar convention for the weight complex of \( X \).

### 1.3 Simple basic properties of weight structures

For any \( C, w \) also the following fundamental properties are fulfilled. These properties (except part 7) are parallel to those of \( t \)-structures; part 7 shows the distinction between these notions.

**Proposition 1.3.1.** 1. If \( C(Y, X) = 0 \) for some \( X \in \text{Obj}_C \) and all \( Y \in \bigcup_{w \geq 1} \) then \( X \in \bigcup_{w \leq 0} \).

2. Vice versa, if \( C(X, Y) = 0 \) for some \( X \in \text{Obj}_C \) and any \( Y \in \bigcup_{w \leq -1} \) then \( X \in \bigcup_{w \geq 0} \).

3. If \( A \to B \to C \to A[1] \) is a distinguished triangle and \( A, C \in \bigcup_{w \geq 0} \) (resp. \( A, C \in \bigcup_{w \leq 0} \), resp. \( A, C \in \bigcup_{w = 0} \)) then \( B \in \bigcup_{w \geq 0} \) (resp. \( B \in \bigcup_{w \leq 0} \), resp. \( B \in \bigcup_{w = 0} \)).

4. All \( \bigcup_{w \leq i} \) are closed with respect to arbitrary (small) direct products (those, which exist in \( C \)).

5. All \( \bigcup_{w \geq i} \) are closed with respect to arbitrary (small) direct sums (those, which exist in \( C \)).

6. For any weight decomposition of \( X \in \bigcup_{w \geq 0} \) (see (1)) we have \( A \in \bigcup_{w = 0} \), \( B \in \bigcup_{w \geq 0} \).

7. If \( A \to B \to C \to A[1] \) is a distinguished triangle and \( A, C \in \bigcup_{w = 0} \) then \( B = A \oplus C \).

**Proof.** 1. Let \( B[-1] \to X \to A \to B \) be a weight decomposition of \( X \). Since \( C(B[-1], X) = 0 \) we obtain that \( X \) is a retract of \( A \); hence \( X \in \bigcup_{w \leq 0} \).

2. The proof is similar to those of part 1 and could be obtained by dualization. If \( B[-1] \to X[-1] \to A \to B \) is a weight decomposition of \( X[-1] \) then \( C(X[-1], A) = 0 \). So \( X \) is a retract of \( B \).

3. Let \( A, C \in \bigcup_{w \geq 0} \). For any \( Y \in \text{Obj}_C \) we have a (long) exact sequence \( \cdots \to C(C(Y), Y) \to C(B, Y) \to C(A, Y) \to \cdots \); hence by part (ii) of Definition 1.1.1 we obtain that \( C(B, Y) = 0 \) for any \( Y \in \bigcup_{w \leq -1} \). Now assertion 2 implies that \( B \in \bigcup_{w \geq 0} \).

The proof for the case \( A, C \in \bigcup_{w \geq 0} \) could be obtained by dualization.

The statement for the case \( A, C \in \bigcup_{w = 0} \) now follows immediately from the definition of \( \bigcup_{w = 0} \).

4. Obviously, assertion 1 implies that \( \bigcup_{w \leq i} = \{ Y \in \text{Obj}_C : C(X, Y) = 0 \forall X \in \bigcup_{w \geq i+1} \} \). This yields the result immediately.

5. Similarly, by assertion 2 we have \( \bigcup_{w \geq i} = \{ X \in \text{Obj}_C : C(X, Y) = 0 \forall Y \in \bigcup_{w \leq i-1} \} \); this yields the result.
6. $A \in C_{w \leq 0}$ by definition. Since we have a distinguished triangle $X \to A \to B \to X[1]$, assertion 3 implies that $A \in C_{w \geq 0}$.

7. Since $C \in C_{w \geq 0}$ and $A[1] \in C_{w \leq -1}$ from the distinguished triangle we obtain the claim. \hfill \Box

**Remark 1.3.2.** 1. We try to answer the questions when a morphism $b[-1] \in C(B[-1], X)$ for $B \in C_{w \geq 0}$ extends to a weight decomposition of $X$ and $a \in C(X, A)$ for $A \in C_{w \leq 0}$ extends to a weight decomposition of $X$ (i.e. Cone($f$) $\in C_{w \geq 0}$) using parts 1 and 2 of Proposition 1.3.1.

We apply the long exact sequence corresponding to the functor $C^*$ for $C \in C_{w \geq 0}$ (resp. to $C_*$ for $C \in C_{w \leq 0}$). In the first case we obtain that $b[-1]$ extends to a weight decomposition iff the map $C(C[i], B[-1]) \to C(C[i], X)$ induced by $b$ is bijective for $i = -2$ and is surjective for $i = -1$ for all $C \in C_{w \geq 0}$. Dually, $a$ extends to a weight decomposition iff for any $C \in C_{w \leq 0}$ the map $C(A, C) \to C(X, C)$ induced by $a$ is bijective for $i = 1$ and is injective for $i = 0$.

Moreover, in many important cases (cf. section 4 below) it suffices to check the conditions of part 1 (resp. part 2) of Proposition 1.3.1 only for $Y = C[i]$ for $C \in C_{w = 0}$, $i < 0$ (resp. for $i > 0$). Then these conditions are equivalent to the bijectivity of all maps $C(C[i], B[-1]) \to C(C[i], X)$ induced by $b$ for $i < -1$ and their surjectivity for $i = -1$ for all $C \in C_{w = 0}$ (resp. to the bijectivity of all maps $C(A, C) \to C(X, C)$ induced by $a$ for $i > 0$ and their injectivity for $i = 0$).

We will use these statements below.

2. In particular, parts 4 and 5 imply that all $C_{w \geq i}, C_{w \leq i}, C_{w = i}$ are additive (i.e. closed with respect to direct sums of two objects) for any $i \in \mathbb{Z}$.

3. Since all (co)representable functors are additive, for any class of $C \subset ObjC$ the classes of $X \in ObjC$ satisfying $C(X, Y) = 0$ for all $Y \in C$ and $C(Y, X) = 0$ for all $Y \in C$ are Karoubi-closed in $C$. We will use this fact below.

**Definition 1.3.3.** We consider $C^- = \bigcup C_{w \leq i}$ and $C^+ = \bigcup C_{w \geq i}$

We call $C^b = C^+ \cap C^-$ the set of bounded objects of $C$.

**Proposition 1.3.4.** 1. $C^-, C^+, C^b$ give full triangulated subcategories of $C$.

2. $w$ induces weight structures for $C^-, C^+, C^b$ whose hearts equal $Hw$.

3. $w$ is non-degenerate when restricted to $C^b$.

**Proof.** 1. By parts 3,4 of Proposition 1.3.1 $C^-, C^+, C^b$ give full triangulated subcategories of $C$. 

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2. It suffices to note that for any object $X$ of $C^-, C^+, C^b$, respectively, components of all its weight decompositions belong to the corresponding category.

Let a distinguished triangle $B[-1] \to X \to A \to B \to X[1]$ be a weight decomposition of $X$, i.e $A \in \underline{C}^{w \leq 0}$, $B \in \underline{C}^{w \geq 0}$.

If $X$ in $\underline{C}^{w \leq i}$ for some $i > 0$ then part 3 of Proposition 1.3.1 implies $B \in \underline{C}^{w \leq i-1}$. Similarly, if $X$ in $\underline{C}^{w \geq i}$ for some $i \leq 0$ then $A \in \underline{C}^{w \geq i}$. We obtain the claim.

3. Let $X \in \text{Obj} \underline{C}^b \cap (\cap \underline{C}^{w \leq i})$; in particular, $X \in \underline{C}^{w \leq j}$ for some $j \in \mathbb{Z}$. Then, by the orthogonality property for $w$, we have $\underline{C}(X, X) = 0$, hence $X = 0$.

A similar argument proves that $\text{Obj} \underline{C}^b \cap (\cap \underline{C}^{w \leq i}) = \{0\}$.

$\underline{C}^b$ is especially important; note that it equals $\underline{C}$ if $(\underline{C}, w)$ is bounded.

Lastly we prove a simple statement on comparison of weight structures.

**Lemma 1.3.5.** Suppose that $v, w$ are weight structures for $\underline{C}$; let $\underline{C}^{w \leq 0} \subset \underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0} \subset \underline{C}^{w \geq 0}$. Then $v = w$ (i.e. the inclusions are equalities).

**Proof.** Let $X \in \text{Obj} \underline{C}^{w \leq 0}$; let $B[-1] \xrightarrow{h} X \to A \to B$ be a weight decomposition of $X$ with respect to $v$. Since $B[-1] \in \underline{C}^{w \geq 1}$, the orthogonality property for $w$ implies $h = 0$. Hence $X$ is a retract of $A$. Since $\underline{C}^{w \leq 0}$ is Karoubi-closed, we have $X \in \underline{C}^{w \leq 0}$.

We obtain that $\underline{C}^{w \leq 0} = \underline{C}^{w \leq 0}$. The equality $\underline{C}^{w \geq 0} = \underline{C}^{w \geq 0}$ is proved similarly.

\[\square\]

1.4 Some auxiliary statements: 'almost functoriality' of distinguished triangles

We will prove below that the weight decomposition is functorial in a certain sense ("up to zero cohomology"). We will need some (general) statements on 'almost functoriality' of distinguished triangles for this. This means that a morphism of between (two) vertices of two distinguished triangles can often be completed to a large commutative diagram.

**Lemma 1.4.1.** Let $T : X \to A \to B \to X[1]$ and $T' : X' \to A' \to B' \to X[1]'$ be distinguished triangles.

1. Let $\underline{C}(B, A'[1]) = 0$. Then for any morphism $g : X \to X'$ there exist $h : A \to A'$ and $i : B \to B'$ completing $g$ to a morphism of triangles $T \to T'$.

2. Let moreover $\underline{C}(B, A') = 0$. Then $g$ and $h$ are unique.
Proof. This fact could be easily deduced from Proposition 1.1.9 of [4] (or Corollary IV.1.4 of [12]); we use the same argument here.

1. Since the sequence $C(B, A') \to C(B, B') \to C(B, X'[1]) \to C(B, A'[1])$ is exact, there exists $i : B \to B'$ such that $g'[0] = g[1] \circ b_0$. By axiom Tr3 (see §IV.1 of [12]) there also exist a morphism $h : A \to A'$ that completes $(g, i)$ to a morphism of triangles.

2. Now we also have $C(B, A') = 0$. Hence the exact sequence mentioned in the proof of part I now also yields the uniqueness of $i$.

The condition on $h$ is that $h \circ a_0 = a' \circ f$. We have an exact sequence $C(B, A') \to C(A, A') \to C(X, A')$. Since $C(B, A') = 0$, we obtain that $h$ is unique also. \qed

**Proposition 1.4.2.** [3 × 3-Lemma]

Any commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{a} & A \\
\downarrow{g} & & \downarrow{h} \\
X' & \xrightarrow{a'} & A'
\end{array}
$$

could be completed to the following $3 \times 3$ diagram (we will mainly need its upper left $3 \times 3$ part)

$$
\begin{array}{cccc}
X & \xrightarrow{a} & A & \xrightarrow{f} & B & \longrightarrow & X[1] \\
\downarrow{g} & & \downarrow{h} & & \downarrow{i} & & \downarrow{g[1]} \\
X' & \xrightarrow{a'} & A' & \xrightarrow{f'} & B' & \longrightarrow & X'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & (2) \\
X'' & \xrightarrow{a''} & A'' & \xrightarrow{f''} & B'' & \longrightarrow & X''[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\end{array}
$$

such that all rows and columns are distinguished triangles and all squares are commutative, except the right lowest square which anticommutes.

Proof. The proof is mostly a repetitve use of the octahedron axiom. Yet it requires certain unpleasant diagrams. It is written in [4], Proposition 1.1.11. \qed
We will also apply the octahedron axiom (see §IV.1.1 of [12]) directly. We recall that it states that any diagram \( X \xrightarrow{f} Y \xrightarrow{g} Z \) can be completed to an octahedron diagram. In particular, there exists a distinguished triangle \( \text{Cone}(g \circ f) \to \text{Cone}(g) \to \text{Cone}(f)[1] \), whence the map \( \text{Cone}(g) \to \text{Cone}(f)[1] \) is obtained by composing of two of the maps in the distinguished triangles that define \( \text{Cone}(f) \) and \( \text{Cone}(g) \) (see §IV.1.8 of [12]).

We will also need the following easy statements on the existence of countable (homotopy) limits.

**Lemma 1.4.3.** Suppose that we have a sequence of objects \( Y_i \) and maps \( \phi_i : Y_i \to Y_{i+1} \). Suppose that there exists a direct sum \( A = \oplus_{i \geq 0} Y_i \). Then there exists a direct limit of \( Y_i \) i.e. any \( Y \in \text{Obj} \mathbb{C} \) such that for any \( C \in \text{Obj} \mathbb{C} \) we have \( C(Y, C) = \lim C(Y_i, C) \).

**Proof.** The construction is fairly standard, see Definition 1.6.4 of [20]. We take \( D = \oplus_{i \geq 0} Y_i \). For any \( C \) we have \( C(D, C) = \prod C(Y_i, C) \). We take a map \( d : \oplus \text{id}_{Y_i} + \oplus (-\phi_i) : D \to D \) (note that its \( i \)-th component is could be easily factorized as a composition \( Y_i \to Y_i \oplus Y_{i+1} \to D \)). We denote the cone of \( a \) as \( Y \). It is easily seen that for any \( C \in \text{Obj} \mathbb{C} \) we have
\[
C(Y, C) = \{(s_i) : s_i \in C(Y_i, C), s_{i+1} = s_i \circ \phi_i \} = \lim C(Y_i, C).
\]

\( \Box \)

Lastly, we prove that the isomorphism class of a cone of a morphism \( f \) (in \( \mathbb{C} \)) is completely determined by the equivalence class of \( f \) in \( K(\mathbb{C}) \).

### 1.5 Weight decomposition of morphisms; multiple weight decomposition of objects

Starting from this moment the triangle
\[
T_{k}[k] : X[k] \xrightarrow{a_k} X^{w \leq k} \xrightarrow{b_k} X^{w \geq k+1} \xrightarrow{b_k} X[k+1]
\]
will be a weight decomposition of \( X[k] \) for some \( X \in \text{Obj} \mathbb{C} \), \( k \in \mathbb{Z} \); \( T'_{k}[k] : X'[k] \xrightarrow{a'_k} X'^{w \leq k} \xrightarrow{b'_k} X'^{w \geq k+1} \xrightarrow{b'_k} X'[k+1] \) will be a weight decomposition of \( X'[k] \). Sometimes we will drop the index \( k \) in the case \( k = 0 \).

**Lemma 1.5.1.** 1. Let \( l \leq m \). Then for any morphism \( g : X \to X' \) there exist \( h : X^{w \leq m}[-m] \to X'^{w \leq l}[-l] \) and \( i : X^{w \geq m+1}[-m] \to X'^{w \geq l+1}[-l] \) completing \( g \) to a morphism of triangles \( T_m \to T'_l \).

2. Let \( l < m \). Then \( h \) and \( i \) are unique.

3. For \( l = m \) any two choices \((h, i)\) and \((h', i')\) we have \( h - h' = (s \circ f_m)[-m] \) and \( i - i' = (f'_m \circ s')[-m] \) for some \( s, s' \in \text{C}(X^{w \geq m+1} \to X'^{w \geq m}) \).
Proof. 1.2: Immediate from Proposition 1.4.1.

3. If suffices to consider the case $g, h, i = 0$. Since $a_k[-k] \circ h = 0$, $T_k$ is a distinguished triangle, we obtain that $h'$ can be presented as $(s \circ f_m)[-m]$. Dually, $i'$ can be presented as $(f'_m \circ s')[-m]$.

\[\Box\]

Remark 1.5.2. 1. For $l < m$ we will denote $i, h$ constructed by $g_{X^w \leq m, X^{u} \leq l}$ and $g_{X^w \geq m+1, X^{u} \geq l+1}$, respectively.

For $l = m = 0$ we will call any pair $(h, i)$ a weight decomposition of $g$.

2. The statement of part 3 is the best possible in a certain sense. It is not possible (in general) to choose $s = s'$. In particular, one can take

\[X = \begin{array}{c} X' = \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \in \text{Obj}C^{[0,1]}((\mathbb{Z}/8\mathbb{Z}) - \text{mod}) \subset C(A)\end{array}\]

Then for $g = 0$ there exists a pair $(h, i) = (\times 4, 0)$ that is not homotopic to 0. Certainly, this example could be generalized to $X = X' = R/r^3R$ for any commutative ring $R, r \in R$, such that $r^2 \not\mid r^3$. In particular, this problem is not "torsion".

Note that the example of the weight decomposition described is obviously not a "nice" one. In particular, it cannot be extended to a $3 \times 3$ diagram. Yet adding this example to the obvious weight decomposition of $id_X$ one obtains another weight decomposition of $id_X$ that is not homotopy equivalent to the first one; yet it does not seem to be "bad" in any sense.

Still one could check that extending morphisms $X \to X'$ to $(X^{w \leq i}, X^{u \geq i}) \to (X^{w \leq i}, X^{u \geq i})$ using part 1 suffices to prove the functoriality of the cohomology of the weight complex of $X$ as defined in §2.2 below (the cohomology objects belong to $Hw'$, see the Notation and part 2 of Remark 3.1.5 below).

We check that $g_{X^w \leq m, X^{u} \leq l}$ and $g_{X^w \geq m+1, X^{u} \geq l+1}$ are "functorial".

**Lemma 1.5.3.** Let $T''[j] : X''[j] \to X''[j+1]$ be a weight decomposition of $X''[j]$ for $X'' \in \text{Obj}C'$ for some $j \leq i \leq 0$; let $p \in C(X, X')$ and $q \in C(X', X'')$.

If $j < 0$ then for any choice of $(h', h'')$ satisfying

\[h' \circ a_0 = a'_i \circ p \text{ and } h'' \circ a'_i = a''_i \circ q\]

we have $(q \circ p)_{X^w \leq i} = h' \circ h$, while for any choice of $(i', i'')$ satisfying

\[b'_i \circ i' = p[1] \circ b_0 \text{ and } b''_i \circ i'' = q[1] \circ b''_i \text{ we have } (q \circ p)_{X^w \leq j} = i' \circ i'.\]

**Proof.** We apply the uniqueness proved in the previous lemma.

Both sides of the first equality calculate the only map $h$ that satisfies $h \circ a_0 = a'_i \circ (q \circ p)$, while both sides of the second equality calculate the only map $i$ that satisfies $b''_i \circ i = (q \circ p)[1] \circ b_0$.

\[\square\]
To prove the (weak) exactness of the weight complex functor (below), we will need the following lemma.

**Lemma 1.5.4.** Let $DT : C \rightarrow X \xrightarrow{g} X'$ be a distinguished triangle; let $T_i[i]$ and $T_{i-1}[i - 1]$ be weight decompositions of $X[i]$ and $X'[i - 1]$, respectively. Then $DT$ could be completed to a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{a_i[i]} & X^w \leq [i - 1] \\
\downarrow{g} & & \downarrow{g_{X^w \leq [i], X^w \leq [i - 1]}} \\
X' & \xrightarrow{a_{i-1}[1-i]} & X^w \leq [1 - i] \\
\downarrow & & \downarrow \\
C[i] & \longrightarrow & C_i[i - 1] \\
\end{array}
$$

whose rows and columns are distinguished triangles, all squares commute, $C_i, C_i' \in \text{Obj}_C$. Moreover, the last row (shifted by $[i - 1]$) gives a weight decomposition of $C[i]$.

Besides, the choice of the part of (5) consisting of six upper objects and arrows connecting them is unique (even if we don’t demand that this part could be completed to the whole (5)).

**Proof.** By part 2 of Lemma 1.5.1, $g$ could be uniquely completed to a morphism of triangles that are the first two rows of (5). Since the left upper square of (5) is commutative, it could be completed to a $3 \times 3$-diagram (see Proposition 1.4.2). Hence the first two rows of this diagram will be as in (5). It remains to study the third row.

By part 3 of Proposition 1.3.1, the second column gives we obtain $C_i \in C^w_{\leq 0}$, while the third column gives $C_i' \in C^w_{\geq 0}$. Hence $C[i] \rightarrow C_i \rightarrow C_i'$ is a weight decomposition of $C[i]$.

Now we study what happens if one combines more than one weight decompositions $T_k$.

**Proposition 1.5.5.** [Multiple weight decomposition]

1. [Double weight decomposition]

Let $T_k$ be fixed for some $X \in \text{Obj}_C$ for $k$ being equal to some $i, j \in \mathbb{Z}$, $i > j$.

Then there exist unique morphisms $s_{ij} : X^w \leq [j - i] \rightarrow X^w \leq j$, $q_{ij} : X^w \geq j + 1[i - j] \rightarrow X^w \geq j + 1$ making the corresponding triangles commutative. There also exists $X[\bar{i}, j] \in C^{[0, i - j - 1]}$, and distinguished triangles

$$
X^w \leq [j - i] \xrightarrow{s_{ij}} X^w \leq j \xrightarrow{e_{ij}} X^{\bar{[i, j]} \rightarrow} X^w \leq [j - i + 1] \xrightarrow{d_{ij}} X^w \leq [j - i + 1]
$$

(6)
and
\[ X^{[k,j]}[-1] \xrightarrow{z_{ij}} X^{w \geq i+1}[j-i] \xrightarrow{y_{ij}} X^{w \geq j+1} \xrightarrow{y_{ij}} X^{[i,j]} \] (7)
for some $C$-morphisms $c_{ij}, d_{ij}, x_{ij}, y_{ij}$.

2. Infinite weight decomposition

Let $T_k$ be fixed for all $k \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ there exist unique morphisms $s_k : X^{w \leq k}[-1] \to X^{w \leq k-1}$, $q_k : X^{w \geq k+1}[-1] \to X^{w \geq k}$ making the corresponding triangles commutative. There also exists $X^k \in C^{w=0}$, and distinguished triangles
\[ X^{w \leq k}[-1] \xrightarrow{s_k} X^{w \leq k-1} \xrightarrow{c_k} X^k \xrightarrow{d_k} X^{w \leq k} \] (8)
and
\[ X^k[-1] \xrightarrow{x_k} X^{w \geq k+1}[-1] \xrightarrow{q_k} X^{w \geq k} \xrightarrow{y_k} X^k \] (9)
for some $C$-morphisms $c_k, d_k, x_k, y_k$.

Moreover, $c_k$ and $x_k$ can be chosen equal to $y_k \circ f_{k-1}$ and $(f_k \circ d_k)[{-1}]$, respectively.

Proof. 1. Applying Lemma 1.5.1 for $X = X'$ and $g = id_X$ we obtain the existence and uniqueness of $s_{ij}, q_{ij}$. It remains to study cones of these morphisms.

The $3 \times 3$-Lemma (i.e. Proposition 1.4.2) implies that the map of triangles $T_i[i] \to T_j[j]$ could be completed to a $3 \times 3$ diagram whose rows and columns are distinguished triangles. Hence there exists an distinguished triangle $\text{Cone}(id_X) \to \text{Cone}(s_{ij}) \to \text{Cone}(q_{ij})$; hence $\text{Cone}(s_{ij}) \cong \text{Cone}(q_{ij})$.

Part 3 of Proposition 1.3.1 applied to the distinguished triangle (6) implies $X^{[i,j]} \in C^{w \leq i-j-1}$; the same statement applied to the distinguished triangle (7) implies $X^{[i,j]} \in C^{w=0}$.

2. The first part of the assertion is immediate from part 1 applied for $(i, j) = (k+1, k)$ for all $k \in \mathbb{Z}$.

To prove the second part it suffices to complete the commutative triangle $X[k] \xrightarrow{a_k} X^{w \leq k} \xrightarrow{s_k[i]} X^{w \leq k-1}[1]$ to an octahedron diagram.

\[ \square \]

Corollary 1.5.6. $C^{w=0}$ generates $C^b$.

Proof. Since $C^b$ is a triangulated category that contains $Hw$, it suffices to prove that any object of $C^b$ could be obtained from objects of $Hw$ by a finite number of taking cones of morphisms.

Let $X \in C^{w \geq i} \cap C^{w \leq i}$. Then we can take $X^{w \leq k} = 0$ for $k < j$ and $X^{w \geq k} = 0$ for $k > i$ in (3). Then $X = X^{w \leq i}$ and the formula (8) gives a sequence of distinguished triangles implying that $X \in \langle C^{w=0} \rangle$.

\[ \square \]
We will need the following definition several times.

**Definition 1.5.7.** We will denote by $P_0(X)$ (a Postnikov tower for $X$) the following data; all $a_k$ (in (3)) and all triangles (8).

**Remark 1.5.8.** 1. A Postnikov tower (of $X$) is certainly not unique; yet we will prove below that it is 'unique up to a homotopy' in a certain sense.
   2. $P_0(X)$ could be recovered from $a_k$ and $X^{w\leq k}$ uniquely up to a non-canonical isomorphism.

## 2 The weight spectral sequence

The goal of this section is to describe the *weight spectral sequence* $T(H, X)$ (for $X \in \text{Obj}_{\mathcal{C}}$ and a functor $H : \mathcal{C} \to A$) generalizing (in a certain sense) those of §7 of [9]; cf. Remark 7.4.4 loc. cit. and §6.4 below. It will specialize to the "usual" weight spectral sequence for "classical realizations" of varieties (or motives); see part 2 of Remark 2.4.2. Moreover, in this case the spectral sequence degenerates at $E_2$ (rationally) and its $E_2$ terms are exactly the graded pieces of the weight filtration.

In §2.1 we define the "weight filtration" for any functor from $\mathcal{C}$ with an abelian target. This will allow us to prove that the $D_2$ term of the derived exact couple for $T(H, X)$ is functorial in $X$.

In §2.2 we define the weight complex of $X$ in the terms of $P_0(X)$. Its "cohomology" will yield the $E_2$ terms of the weight spectral sequence.

For simplicity we only construct in detail only the spectral sequence for homological functors (in §2.3); dualization immediately extends the result to the cohomological functor case (see §2.4). We conclude the section by noting that our spectral sequence induces the standard weight filtration for the rational étale and Hodge realizations of varieties (and motives); see Remark 2.4.2.

### 2.1 The weight filtration of functors

Let $A$ be an abelian category. We fix some choice of $T_k$ (in the notation of subsection 1.5).

**Definition 2.1.1.** 1. If $H : \mathcal{C} \to A$, $i \in \mathbb{Z}$ is any covariant functor then we define $W_i(H(X)) = \text{Im}(H(X^{w\geq i}[-i]) \to H(X))$.

2. If $H$ is contravariant then we define $W^i(H(X)) = \text{Im}(H(X^{w\leq i}[-i]) \to H(X))$. 

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Proposition 2.1.2. 1. Let $H$ be covariant. Then the correspondence $X 	o W_i(H(X))$ gives a canonical subfunctor of $H(X)$. This means that $W_i(H(X))$ does not depend on the choice of the weight decomposition of $X[i]$ and for any $f : X \to Y$ we have $f_*(W_i(H(X)) \subset W_i(H(Y))$ (for $X, Y \in \text{Obj}_\mathcal{C}$).

2. The same is true for contravariant $H$ and $W^i(H(X))$.

Proof. 1. Part 1 of Lemma 1.5.1 implies that for any choice of weight decompositions of any $X[i], Y[i]$ we have $f_*(W_i(H(X)) \subset W_i(H(Y))$. In particular, taking $Y = X$, $f = id_X$ we obtain that $W_i(H(X))$ does not depend on the choice of the weight decomposition of $X[i]$

2. The statement is exactly the dual of assertion 1.

Remark 2.1.3. 1. The same is true if we replace images by the corresponding kernels. Certainly, in the case when $H$ is (co)homological, this gives the same filtration.

Besides, Proposition 2.1.2 if true if $A$ is any category with well-defined images of morphisms.

2. A partial case of this method for defining weight filtration for cohomology was (essentially) considered in Proposition 3.5 of [14].

3. [Universal functor; semi-motives]

Recall that we have a natural embedding $\mathcal{C} \to \mathcal{C}_\ast$. Then for any $i \in \mathbb{Z}$ and $X \in \text{Obj}_\mathcal{C}$ the functor $A \to W_i(A)(X) : \mathcal{C} \to \text{Ab}$ gives an object of $\mathcal{C}_\ast$; it equals $W_i(Y \to Y_s)(X)$. We obtain a sequence of functors $W_i : \mathcal{C} \to \mathcal{C}_\ast$. The usual lonedo isomorphism $F(X) \cong \text{Mor}_{\text{Fun}(\mathcal{C}_\ast)}((X \to X_s), F)$ for a functor $F : \mathcal{C} \to A$ can be easily generalized to $\overline{W}_iF(X) \cong \text{Mor}_{\text{Fun}(\mathcal{C}_\ast)}((X \to W_i(X_s)), F)$. Hence the sequence of functors $\ast, (W_i, \ast)$ is universal in the category of (functors from $\mathcal{C}_\ast$ their weight filtrations).

In particular, one could apply this construction to the Chow weight filtration of Voevodsky’s motives (see 6.5 below). For any motif $X$ (an so, for any variety) one obtains a sequence of objects of $DM^{eff}_{gm}$ which could be called semi-motives. These objects contain important cohomological information on $X$. In particular, they have realizations! The author plans to study this example further.

2.2 The definition of the weight complex

Now we describe the weight complex of $X \in \text{Obj}_\mathcal{C}$. We will prove that it is canonical and functorial (in a certain sense) in §3.2 below.

We adopt the notation of subsection 1.5.
**Definition 2.2.1.** We define the morphisms \( h_i : X^i \to X^{i+1} \) as \( c_{i+1} \circ d_i \). We will call \( (X^i, h_i) \) the **weight complex** of \( X \).

Note that all information on \( t(X) \) is contained in \( Po(X) \) (including the ‘connection’ of \( t(X) \) with \( X \)).

**Proposition 2.2.2.** 1. **Weight complex is a complex indeed i.e. an object of \( C(Hw) \).**

2. **If for some \( i \in \mathbb{Z} \) and \( X \in \bigcup_{w \leq i} \) (resp. \( X \in \bigcup_{w \geq i} \)) then there exists a choice of the weight complex of \( X \) belonging to \( C(Hw)^{\leq i} \) (resp. to \( C(Hw)^{\geq i} \)).**

**Proof.** 1. We have

\[
h_{i+1} \circ h_i = c_{i+2} \circ (d_{i+1} \circ c_{i+1}) \circ d_i = c_{i+2} \circ 0 \circ d_i = 0
\]

for all \( i \).

2. Similarly to the proof of Corollary 1.3.4, we can take \( X^{w \geq k} = 0 \) for \( k > i \) (resp. \( X^{w \leq k} = 0 \) for \( k < i \)). Then we would have \( X^{w \leq k} = X \) for \( k \geq i \) (resp. \( X^{w \geq k} = X \) for \( k \leq i \)). Therefore the corresponding choice of the weight complex of \( X \) belongs to \( C(Hw)^{\leq i} \) (resp. to \( C(Hw)^{\geq i} \)) by definition. \( \square \)

**2.3 The weight spectral sequence for homological functors**

Let \( A \) be an abelian category; let \( H : C \to A \) be a homological functor (i.e. a covariant additive functor that transfers distinguished triangles into long exact sequences). The cohomological functor case will be obtained from the homological one by dualization.

Let \( X \in \text{Obj}_C \), \( (X^i, h_i) = t(X) \). We construct a spectral sequence whose \( E_1 \)-terms are \( H(X^i[j]) \) which converges to \( H(X[i + j]) \) in many important cases.

We denote \( H(Y[p]) \) by \( H^p(Y) \) for any \( Y \in \text{Obj}_C \).

First we describe the exact couple. It is obtained by applying \( H \) to the Postnikov tower for \( X \) (see Definition 1.5.7). It is almost the same as the exact couple in §IV2, Exercise 2, of [12].

We take \( E_1^{pq} = H^q(X^p) \), \( D_1^{pq} = H^q(X^{\leq p}) \). Then the distinguished triangles (8) give \( E_1, D_1 \) the structure of an exact couple.

**Theorem 2.3.1.** [The homological weight spectral sequence]

There exists a spectral sequence \( T = T(H, X) \) with \( E_1^{pq} = H^q(X^p) \) which weakly converges to \( H^{p+q}(X) \) such that the map \( E_1^{pq} \to E_1^{p+1q} \) equals \( H^1(h_p) \); the corresponding filtration on \( H(X) \) coincides with those of Definition 2.1.1.
If \( T(H, X) \implies H^{p+q}(X) \) in either of the following cases:

(i) \( X \in \mathcal{C}_b^b \).

(ii) \( H \) vanishes on \( \mathcal{C}^{w\geq q} \) for \( q \) large enough and on \( \mathcal{C}^{w\leq q} \) for \( q \) small enough.

(iii) \( X \in \mathcal{C}^- \) (resp \( \mathcal{C}^+ \)) and \( H \) vanishes on \( \mathcal{C}^{w\leq q} \) for \( q \) small enough (resp. on \( \mathcal{C}^{w\geq q} \) for \( q \) large enough).

III \( T \) is functorial with respect to \( H \) i.e. for any transformation of functors \( H \rightarrow H' \) we have a canonical morphism of spectral sequences \( T(H, X) \rightarrow T(H, X') \); these morphisms respect sums and compositions of transformations.

IV \( T \) is canonical and functorial with respect to \( X \) starting from \( E_2 \).

Proof. I The standard construction of the spectral sequence of an exact couple shows that the boundary maps for \( E_1 \) equal \( H^s(h_p) \) indeed. The induced filtration on \( H(X) \) is the weight filtration (of Definition 2.1.1) by definition. II In case (ii) the exact couple is obviously bounded (this is already true at level 1).

In case (i) this will be also true if we choose the weight complex of \( X \) to be bounded (we can do this by the definition of \( \mathcal{C}_b^b \)). Now, for an arbitrary choice of the weight complex we can connect it with some bounded choice by a quasi-isomorphism; see Remark 3.2.3 below. Applying the statement of part IV we obtain that the spectral sequence will be bounded starting from \( E_2 \) (moreover, the derived exact couple is bounded).

The proof in case (iii) is similar.

III This is obvious since all components of the exact couple are functorial with respect to \( H \).

IV It suffices to check that the correspondence \( X \rightarrow (E_2, D_2) \) (the derived exact couple) defines a functor. \( D_2^q(T) = \text{Im} H^q(X^{\leq p}) \rightarrow H^{p+1}(X^{\leq p-1}) \) is canonical and functorial by Lemma 1.5.3. Indeed, by this Lemma these terms and morphisms between them do not depend on the choices of the corresponding \( h \)-components of weight decompositions of morphisms.

Next, \( E_2 \) factorizes through the weight complex functor \( t \) defined in §3.2 below; see part 3 of Remark 3.1.5. It could be also easily seen that the morphisms that define the derived couple are also functorial in \( X \) (and hence, canonical).
2.4 The weight spectral sequence for cohomological functors

Inverting the arrows in the proof of the previous theorem we obtain the following cohomological analogue.

**Theorem 2.4.1.** [The cohomological weight spectral sequence]

There exists a spectral sequence \( T = T(H, X) \) with \( E_1^{pq} = H^q(X^{-p}) \) which weakly converges to \( H^{p+q}(X) \) such that the map \( E_1^{pq} \to E_1^{p+1q} \) equals \( H^q(h_{-1-p}) \). The corresponding filtration on \( H(X) \) coincides with those of Definition 2.1.1.

- If \( T(H, X) \Rightarrow H^{p+q}(X) \) in either of the following cases:
  - (i) \( X \in \mathcal{C}^b \).
  - (ii) \( H \) vanishes on \( \mathcal{C}^{w \geq q} \) for \( q \) large enough and on \( \mathcal{C}^{w \leq q} \) for \( q \) small enough.
  - (iii) \( X \in \mathcal{C}^- \) (resp \( \mathcal{C}^+ \)) and \( H \) vanishes on \( \mathcal{C}^{w \leq q} \) for \( q \) small enough (resp. on \( \mathcal{C}^{w \geq q} \) for \( q \) large enough).

- If \( T \) is functorial with respect to \( H \) i.e. for any transformation of functors \( H \to H' \) we have a morphism of spectral sequences \( T(H, X) \to T(H, X') \).

- \( IV \) \( T \) is canonical and (contravariantly) functorial with respect to \( X \) starting from \( E_2 \).

**Proof.** It suffices to apply Theorem 2.3.1 to the functor \( H^{op} : \mathcal{C} \to A^{op} \).

**Remark 2.4.2.** 1. Suppose now that there are no maps between different weights i.e. for any \( P, P', Q, Q' \in \mathcal{C}^{w=0}, f \in Hw(P, P'), g \in Hw(Q, Q'), i \neq j \), we have

\[
A(\text{Ker}(H^i(E([P]))) \xrightarrow{f^*} H^i(E([P'])), (\text{Coker}(H^j(E([Q]))) \xrightarrow{g^*} H^j(E([Q']))) = 0.
\]

Then \( T(H, X) \) and all \( T^{a,b}(H, X) \) degenerate at \( E_2 \). Therefore \( H^jF^b_N(X) = W_b(H^j(E(Y)))/W_{b-N-1}(H^j(E(Y))) \).

Note that this condition is fulfilled for the étale and Hodge realizations of \( DM_{gm}^{eff} \) (the category of Voevodsky’s motives; see §6 below) with rational coefficients.

2. We will see in §6 below that \( DM_{gm}^{eff} \) admits a Chow weight structure whose heart is \( Chow^{eff} \). Hence we obtain the weight spectral sequence and weight spectral sequence for any realization of motives. In particular, this is the case for étale and Hodge realizations of motives, and motivic cohomology.

Now, for the rational étale and Hodge realizations there are no non-zero maps between different weights. Hence for any \( H^i(X) \) there cannot exist more than one filtration \( W_j \) on \( H^i(X) \) such that \( W_j/W_{j-1} \) is of weight \( j \).
Therefore for the rational étale and Hodge realizations of motives our weight filtration coincides with the usual one; see §7.4 of [9].

Now we consider the case of motivic cohomology. A simple example of the spectral sequence obtained is the Bloch long exact localization sequence for motivic cohomology; see part 1 of Remark 7.3.1 in [9]. Since the latter is not trivial, the "weight" filtration obtained is non-trivial in this case either; it appears not to be mentioned in the literature. This filtration is compatible with the regulator maps (whose targets are "classical" cohomology theories).

3 The weight complex functor

In §5 of [9] for a triangulated category $\mathcal{C}$ with a (negative) differential graded enhancement a conservative exact weight complex functor $t_0 : \mathcal{C} \to K(H^w)$ (in our notation) was constructed; see §6.3. The goal of this section is to extend this result to the case of arbitrary $(\mathcal{C}, w)$.

The results of §1.5 easily imply that any $g : X \to X'$ where $X, X' \in \text{Obj} \mathcal{C}$ could be extended to a morphism of (any possible) Postnikov towers for $X, X'$. Moreover, the morphisms of Postnikov towers for composition of morphisms could be composed. Yet, as the example of part 2 of Remark 1.5.2 shows, this construction cannot give a canonical morphism of weight complexes in $K(H^w)$. We have to consider a certain factor $K^w(H^w)$ of this category. This factor is no longer triangulated (in the general case; yet cf. Remark 3.3.4); still the kernel of the projection $K(H^w) \to K^w(H^w)$ is an ideal whose square is zero so our ("weak") weight complex functor is not much worse than the "strong" one of [9].

We define and study $K^w(H^w)$ in §3.1. We construct the weight complex functor $t$ in §3.2 and prove its main properties in §3.3. One of our main tools is the weight decomposition functor $WD : \mathcal{C} \to K^w(0, 1)(\mathcal{C})$; see Theorem 3.2.2.

One of the main properties of the functor $t$ is that it calculates the $E_2$-terms of the weight spectral sequence $T$, see part 3 of Remark 3.1.5. In fact, this is why $t$ it called the weight complex; this term was used for the first time in [13] (see §2 and §3.1 of [13]).

3.1 The weak category of complexes

Let $A$ be an additive category.

We will denote by $Z(X, Y)$ for $X, Y \in \text{Obj} K(A)$ the subgroup of $K(A)(X_i, Y_i)$ consisting of morphisms that could be presented as $(s_{i+1} \circ d_X + d_{Y}^{i-1} \circ t_i)$ for some set of $s_i, t_i \in A(X_i, Y_{i-1})$ (here $X = (X_i)$, $Y = (Y_i)$).
Remark 3.1.1. We will often use the fact that \( sd + dt = (s - t)d + (dt + td) \) is homotopy equivalent to \( (s - t)d \); hence we may assume that \( t = 0 \) in the definition of \( Z \).

Now we check that \( Z \) defines a (two-sided) ideal of morphisms in \( K(A) \) whose square is zero. We will abbreviate these properties as \( Z \triangleleft \text{Mor}(K(A)) \) and \( Z^2 = 0 \). A easy argument also shows that all ideals \( Z \triangleleft \text{Mor}_C \) that satisfy \( Z^2 = 0 \) satisfy a collection of nice properties.

**Lemma 3.1.2.** II. \( Z(-, -) \) gives an ideal of morphisms in \( \text{Mor}(K(A)) \).

1. Let \( L, M, N \in \text{Obj}K(A) \), let \( g \in Z(L, M) \subset K(A)(L, M) \), \( h \in Z(M, N) \subset K(A)(M, N) \). Then \( h \circ g = 0 \) (in \( K(A) \)).

II Let \( Z \triangleleft \text{Mor}_C \) for any additive category \( C \); suppose also that \( Z^2 = 0 \); let \( D \) be an additive category. Let \( p : C \rightarrow D \) be an additive functor such that for any \( X, Y \in \text{Obj}_C \) we have \( \text{Ker}(C(X, Y) \rightarrow D(p(X), p(Y))) = Z(X, Y) \).

Then the following statements are valid.

1. The \( p \) is conservative i.e. \( p(g) \) is an isomorphism iff \( g \) is (for any morphism \( g \) in \( C \)).

2. For any \( X \in \text{Obj}_C \) and \( r \in C(X, X) \) if \( p(r) \) is idempotent then it could be lifted to an idempotent \( r' \in C(X, X) \) (i.e. \( p(r') = p(r) \)).

3. If \( C \) is idempotent complete then its image in \( D \) also is. Here we consider a not necessarily full subcategory of \( D \) such that all its objects and morphisms are exactly those that come from \( C \).

**Proof.** II. Obviously, \( Z(-, -) \) is closed with respect to sums and direct sums.

Lastly, let \( d \) denote the differential, let \( f, g = s \circ d \), and \( h \) be composable morphisms; here \( s, t \) are collections of arrows shifting degree by \(-1\). Then we have \( f \circ g = (f \circ s) \circ d \) and \( g \circ h = (s \circ h) \circ d \); note that \( h \) "commutes with the differential".

2. Let \( L = (L_i), M = (M_i), N = N_i \). Suppose that for all \( i \in \mathbb{Z} \) we have \( g_i = s_{i+1} \circ d_i \) for some set of \( s_i \in A(L_i, M_{i-1}) \), whence \( h_i = u_{i+1} \circ d_i M \) for some set of \( u_i \in A(M_i, N_{i-1}) \).

Then \( h_i \circ g_i = u_{i+1} \circ d_i M \circ s_{i+1} \circ d_i L \). Recall now that \( g \) is a morphism of complexes; hence for all \( i \in \mathbb{Z} \) we have \( d_i M \circ s_{i+1} \circ d_i L = d_i M \circ d_i M^{-1} \circ s_i = 0 \). We obtain that \( h \circ g \) is homotopic to \( 0 \).

III. Since \( p \) is a functor, it sends isomorphisms to isomorphisms.

Now we prove the converse statement. Let \( g \in C(X, X') \) for \( X, X' \in \text{Obj}_C \), let \( p(h) \) for some \( h \in C(X', X) \) be the inverse to \( p(g) \). We have \( h \circ g - id_X \in Z(X, X) \) and \( g \circ h - id_{X'} \in Z(X', X') \). It suffices to check that \( h \circ g \) and \( g \circ h \) are invertible in \( C \). The last assertion follows from equalities \((h \circ g - id_X)^2 = 0 \) and \((g \circ h - id_{X'})^2 = 0 \) in \( C \), that yield \((h \circ g)(2id_X - h \circ g) = id_X \) and \((g \circ h)(2id_{X'} - g \circ h) = id_{X'} \).
2. This is just the standard statement that idempotents could be lifted (in rings).

We consider $r' = -2r^3 + 3r^2$. Since $p(r)^2 = p(r)$ in $D$ and $r' = r + (r^2 - r) \circ (id_X - 2r)$, we have $p(r') = p(r)$. Since $r'^2 - r' = (r^2 - r)^2 \circ (4r^2 - 4r - 3id_X)$, we obtain that $r'$ is an idempotent.

3. The assertion follows immediate from II2. Indeed, any idempotent $d$ in the image could be lifted to an idempotent $c$ in $C$. Since $c$ splits in $C$, $p(c) = d$ splits in the image.

\[ \square \]

**Remark 3.1.3.** The assertions of part II remain valid for any nilpotent $Z$. For $l$ that satisfies $l^n = 0$, $n > 0$, the inverse to $id_X - l$ is given by $id_X + l + l^2 + \cdots + l^{n-1}$. If $l^n = 0$, $r^2 - r = l$, then the equality $(x - (x - 1))^{2n-1} = 0$ allows to construct explicitly a polynomial $P(x)$ such that $P \equiv 0 \mod x^nZ[x]$ and $P \equiv 1 \mod (x - 1)^nZ[x]$. Then $P(r)^2 = P(r)$; $P(r) - r$ could be factorized through $l$.

**Definition 3.1.4.** [The definition of $K_w(A)$]

We define the objects of an additive category $K_w(A)$ as the object of $K(A)$ up to a homotopy equivalence; $K_w(A)(X, Y) = K(A)(X, Y)/Z(X, Y)$.

We have the obvious shift functor $[1] : K_w(A) \to K_w(A)$.

A triangle $A \to B \to C \to A[1]$ will be called distinguished if any of its two sides could be lifted to two sides of some distinguished triangle in $K(A)$.

An additive functor $F : C \to K_w(A)$ for a triangulated $C$ will be called _weakly exact_ if it commutes with shifts and sends distinguished triangles to distinguished triangles.

**Remark 3.1.5.** [Why $K_w(A)$ is a category; cohomology]

1. It could be easily seen that $K_w(A)$ is a category indeed. We factorize the class of objects of $K(A)$ with respect to a set of invertible morphisms; whence $Z(-, -)$ is an ideal in $Mor K(A)$.

2. Let $B$ be an abelian category; let $F : A \to B$ be an additive functor. Then any $g \in Z(X, Y)$ gives a zero morphism on cohomology of $F(X)$. It follows that the cohomology of $F_*(X)$ gives well-defined functors $K_w(A) \to B$.

In particular, this is true for the "universal" functor $A \to A'_s$ (recall that $A'_s$ is the full abelian subcategory of $A_s$ generated by $A$). Hence there are well defined cohomology functors $H_i : K_w(A) \to A'_s$.

Obviously, all cohomology functors described translate distinguished triangles in $K_w(A)$ into long exact sequences.

3. Now suppose that for a triangulated $C$ we have a weakly exact functor $u : C \to K_w(A)$. Then the cohomology of $F_*(u(X))$ gives well-defined
functors $C \to B$. Again, distinguished triangles in $C$ are translated into long
exact sequences.

In particular, this statement could be applied to the weight complex func-
tor $t : C \to K_w(Hw)$ described in part II of Theorem 3.2.2 below. This
concludes the proof of (part IV of) Theorem 2.3.1.

The bounded subcategories of $K_w(A)$ are defined in the obvious way.
Lemma 3.1.2 immediately yields the following statement.

**Proposition 3.1.6.** 1. The projection $p : K(A) \to K_w(A)$ is conservative.
2. Let $A$ be idempotent complete. Then $K_w^b(A)$ is idempotent complete
also.

**Proof.** 1. Immediate from part III of Lemma 3.1.2.
2. It is well known that $K_w^b(A)$ is idempotent complete; see, for example,
[3]. Hence part II3 of Lemma 3.1.2 yields the result. \hfill \square

### 3.2 The functoriality of the weight complex

We will use the following simple fact.

**Lemma 3.2.1.** If $X \in C^{w \geq 0}, Y \in C^{w \leq 0}$ then any $f \in C(X,Y)$ could be
factorized through some morphism $X^0 \to Y^0$ (of the zeroth terms of weight
complexes).

**Proof.** Easy from the equality $C(X^{w \geq 1}[-1], Y) = C(X^0, Y^{w \leq -1}[1]) = 0$. \hfill \square

Now we prove that the "single" and the "infinite" weight decompositions
define functors. Let $X, X'$ denote arbitrary objects of $C$.

**Theorem 3.2.2.** 1. The (single) weight decomposition of objects and mor-
phisms gives a functor $WD : C \to K_w^{[0,1]}(C)$ (i.e. the image is concentrated
in degrees 0, 1).

2. Morphisms $g \in C(X, X'), h \in C(X^{w \leq 0}, X'^{w \leq 0})$ and $i : C(X^{w \geq 1}, X'^{w \geq 1})$
give a morphisms of weight decompositions (of $X$ and $X'$) iff $(h, i) = WD(g)$
in $K_w(C)$.

3. The homomorphism $C(X, X') \to K_w^{[0,1]}(C)(WD(X), WD(X'))$ is sur-
jective.

4. For all $X, X' \in \text{Obj}C$ we make the notation

$$T(X, X') = \text{Ker}(C(X, Y) \to K_w(Hw)(WD(X), WD(X'))).$$

Then $T(-, -)$ is a (two-sided) ideal of $\text{Mor}C$; $T^2 = 0$. 

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5. If \( WD(X) \cong WD(X') \) in \( K_w(\mathcal{C}) \) then \( X \cong X' \) in \( \mathcal{C} \).

6. For any \( X \in \text{Obj} \mathcal{C} \), \( p \in \mathcal{C}(X,X) \), if \( WD(p) \) is idempotent then \( WD(p) \) could be lifted to an idempotent \( p' \) in \( \mathcal{C}(X,X) \).

II The infinite weight decomposition of objects and morphisms (cf. the construction described in the proof) gives a functor \( \mathcal{C} \rightarrow K_w(Hw) \).

Proof. 1. By part 1 of Lemma 1.5.1, any morphism \( X \rightarrow X' \) could be extended to a morphism of their (fixed) weight decompositions. This extension is uniquely defined in \( K_w^{0,1}(\mathcal{C}) \) by part 3 loc. cit. One can compose such homomorphisms in \( K_w(\mathcal{C}) \) since one of the possible extensions of the composition of morphisms \( X \rightarrow X' \rightarrow X'' \) (in \( C(\mathcal{C}) \)) is the composition of (arbitrary) extensions.

It remains to check that the image of \( X \) in \( \text{Obj} K_w^{0,1}(\mathcal{C}) \) does not depend on the choice of the weight decomposition. Let \( K, K' \in \text{Obj} K(\mathcal{C}) \) be given by two weight decompositions of \( X \); \( id_X \) induces \( g \in K(\mathcal{C})(K, K') \) and \( h \in K(\mathcal{C})(K', K) \). By part 3 of Lemma 1.5.1, \( h \circ g - id_K \in Z(K, K) \) and \( g \circ h - id_K' \in Z(K', K') \). It suffices to check that \( h \circ g \) and \( g \circ h \) are invertible in \( K(\mathcal{C}); \) this follows from part 1 of Proposition 3.1.6.

2. By definition of \( WD \), the triple \((g, WD(g))\) gives a morphism of weight decompositions.

Now suppose that \((h, i) = WD(g)\) i.e. \((h, i) \in C(\mathcal{C})(WD(X), WD(X')) \) and \((h, i) \equiv WD(g) \mod T(WD(X), WD(X')) \). It follows that \( i \circ f = f' \circ h \) (in the notation of (3)). Besides, there exist \((h', i')\) that give a morphism of weight decompositions; \( h - h' = s \circ f \) and \( i - i' = f' \circ t \) for some \( s, t \in C(X^{w \leq 0}, X^{tw \leq 0}) \). We obtain that \( h \circ a = h' \circ a = a' \circ g \) and \( b' \circ i = b' \circ i' = g[1] \circ b \).

Hence \((g, h, i)\) give a morphism \( T_0 \rightarrow T'_0 \).

3. By definition, any \( h \in K_w^{0,1}(\mathcal{C})(WD(X), WD(X')) \) comes from some commutative square

\[
\begin{array}{ccc}
X^{w \leq 0} & \xrightarrow{f_0} & X^{w \geq 1} \\
\downarrow & & \downarrow \\
X^{tw \leq 0} & \xrightarrow{f'_0} & X'^{tw \geq 1}
\end{array}
\]

Extending this square to a morphisms of triangles \( T_0 \rightarrow T'_0 \) (i.e. of weight decompositions of \( X \) and \( X' \)) immediately yields the result.

4. Since \( WD \) is a functor, \( T \) is an ideal.

We prove that \( T^2 = 0 \) similarly to the proof of I2 of Lemma 3.1.2.

Let \( X, X', X'' \in \text{Obj} \mathcal{C} \), let \( g \in T(X', X'') \subset \mathcal{C}(X, X'), \ h \in T(X', X'') \subset \mathcal{C}(X', X'') \).

We should check that \( h \circ g = 0 \) (in \( \mathcal{C} \)). We can choose any weight decompositions of \( X, X', X' \); denote them by \( T, T, T'' \) (similarly to (3)).
Since $WD(g) = WD(h) = 0$, by assertion I2 we obtain that $(g, 0, 0)$ and $(h, 0, 0)$ give morphisms of weight decompositions. This means that $a' \circ g = a' \circ h = g[1] \circ b = h[1] \circ b' = 0$. Hence $g$ could be presented as $b'[-1] \circ c$ for some $c \in C(X, X^{w \geq 1}[-1])$. Then $h \circ g = (h[1] \circ b')[-1] \circ c = 0$.

5. By assertion I3 any isomorphism $WD(X) \to WD(X')$ is induced by some morphism $X \to X'$. Now by part III of Lemma 3.1.2, $t$ is conservative (we apply assertion I4); this yields the result.

6. Immediate from part II2 of Lemma 3.1.2.

II Exactly the same reasoning as in part I1 will prove the assertion after we verify that morphisms in $C$ give well-defined morphisms of weight complexes (in $K_w(Hw)$).

Applying part I1 to weight decompositions of $X[k], X'[k]$ for $X, X' \in ObjC$ and all $k \in \mathbb{Z}$, we obtain that any $g \in C(X, X')$ gives a (non-unique) family of $g_k: X^{w \leq k} \to X'^{w \leq k}$. Besides, for all $k \in \mathbb{Z}$ we have $g_k \circ s_{k+1} = s'_{k+1} \circ g_{k+1}[-1]$ (see Lemma 1.5.3); here we extend the notation of part 2 of Proposition 1.5.5 to $X'$. These morphisms can be extended to a morphism $Po(X) \to Po(X')$; hence we obtain some morphism $t(g) : t(X) \to t(X')$. It remains to verify that for $g = 0$ we have $t(g) \in Z(t(X), t(X'))$.

We study the possibilities for $g_i : X^i \to X'^i$. Note that $X_i$ depends on the maps $r_k : X^{w \leq k} \to X'^{w \leq k}$ only for $k = i, i - 1$. This dependency is linear. Moreover, any pair of $(r_i, r_{i-1})$ could be presented as $(0, r_{i-1}) + (r_i, 0)$. Hence it suffices to prove that $g_i$ could be presented as $(s_{i+1} \circ h_{iX} + h_{i-1,X} \circ t_i)$ for some $s_{i+1} \in Hw(X^{i+1}, X^n)$, $t_i \in Hw(X^i, X'^{i-1})$ in two cases: either $r_i$ or $r_{i-1}$ equals 0. (Recall that $h$ denotes the boundary of a weight complex).

In the case $r_i = 0$ we can present $g_i[-1]$ as the second component of $WD(0 : X^{w \leq i}[-1] \to X'^{w \leq i}[-1])$. Hence $g_i$ equals $c_{i-1,X'} \circ u_i$ for some $u_i \in C(X^i, X'^{w \leq i-1})$. Note now that $u_i$ could be factorized through $X'^{i-1}$ (see Lemma 3.2.1).

In the case $r_{i-1} = 0$ we can present $g_i$ as the first component of $WD(0 : X^{w \geq i} \to X'^{w \geq i})$. Hence $g_i$ equals $v_{i+1} \circ x_iX[1]$ for some $v_{i+1} \in Hw(X^{w \geq i+1}, X^n)$. It remains to note that $v_i$ could be factorized through $X'^{i-1}$.

Combining the two cases, we obtain our claim.

\[\square\]

Remark 3.2.3. The functoriality of $t$ implies that for any $X \in ObjC$ any two choices for $t(X)$ are connected by a (canonical) isomorphism in $K_w(Hw)$. Then part III of Lemma 3.1.2 (combined with part I2 of the Lemma) implies that they are isomorphic (not necessarily canonically) in $K(Hw)$ i.e. they are homotopy equivalent (in $C(Hw)$).

$WD$ and $t$ "commute" in the following sense.
Proposition 3.2.4. Let $X, X' \in \text{Obj} \mathcal{C}$, $g \in C(X, X')$.

1. Any choice of $(t(i), t(l))$ for $(i, l) = WD(g)$ comes from a truncation of $t(g)$ (here we fix some weight decompositions of $X$ and $X'$ and consider all compatible lifts of $t(g)$ to \text{Mor} \mathcal{C}(Hw))

2. Let some $(r', s') = (t(i'), t(l'))$ for some weight decomposition $(i', l')$ of $g$, let $r + s : t(X) \to t(X')$ be homotopic to $r' + s'$ (here we consider sums of collections of arrows). Then $(r, s) = (t(i), t(l))$ for some (other) weight decomposition $(i, l)$ of $g$.

Proof. 1. By the definition of $t(g)$ (see part II of Theorem 3.2.2) any choice of $(t(i), t(l))$ is a possible truncation of $t(h)$ over $C(Hw)$.

2. It suffices to prove the statement for $g = 0$. Let $(r, s)$ be a truncation $t(0)$. Note that (replacing $r, s$ by equivalent morphisms if needed) we can assume that $r = r_0, s = s_1$ (i.e. they are concentrated in degrees $0, 1$). Hence there exists some $l \in Hw(X^1, X^{01})$ such that $r_0 = l \circ h_0, s_1 = h'_0 \circ l$.

Now it remains to note that the triple $(0, d_0' \circ l \circ c_1, x_0'[1] \circ l \circ y_1)$ gives a weight decomposition of $0 : X \to X'$. This fact follows from the equalities $d_0' \circ l \circ c_1 \circ a_0 = 0 = b'_0 \circ x_0'[1] \circ l \circ y_1$ (see (8) and (9)), whence

\[ f'_0 \circ d_0' \circ l \circ c_1 = x_0'[1] \circ l \circ c_1 = x_0'[1] \circ l \circ y_1 \circ f_0. \]

\[ \square \]

3.3 Main properties of the weight complex

Now we prove the main properties of the weight complex functor.

Theorem 3.3.1. [The weight complex theorem]

I Exactness.

t is a weakly exact functor.

II Nilpotence.

$I(-, -) = \text{Ker} \mathcal{C}(-, -) \to K_w(t(-), t(-))$ defines an ideal in \text{Mor} \mathcal{C}.

For any $i \leq j \in \mathbb{Z}$ the restriction $I^{[i,j]}$ of $I$ to $C^{[i,j]}$ satisfies $I^{[i,j]}j_{j+1} = 0$.

III Idempotents.

If $X \in \mathcal{C}^b$, $g \in C(X, X)$, $t(g) = t(g \circ g)$, then $t(g)$ could be lifted to an idempotent $g' \in C(X, X)$.

IV Filtration.

If $X \in \mathcal{C}^{w \leq i}$ (resp. $\mathcal{C}^{w \geq i}$) for some $i \in \mathbb{Z}$ then $t(X) \in K_w(Hw)^{w \leq i}$ (resp. $K_w(Hw)^{w \geq i}$) i.e. it is homotopy equivalent to a complex concentrated in degrees $\leq i$ (resp. $\geq i$).

If $X$ is bounded from above (resp. from below) then the converse implications are valid also.

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**V Conservativity.**

If $w$ is non-degenerate then the functor $t$ is conservative on $\mathbb{C}^+_{w}$ and $\mathbb{C}^-_{w}$.

VI If $X, Y \in \mathbb{C}^{[0,1]}_{w}$ then $t(X) \cong t(Y) \Rightarrow X \cong Y$.

VII Let $X \in \mathbb{C}_{w}^{\geq a}$ for some $a \in \mathbb{Z}$; consider the homomorphism $t_{*} : C(X, X') \to K_{w}(H_{w})(t(X), t(X'))$. Then the following statements are valid.

1. $t$ is bijective if $X' \in \mathbb{C}^{\leq a}_{w}$.

2. $t$ is bijective if $X' \in \mathbb{C}^{w\leq a+1}_{w}$.

**Proof.** Let $C \xrightarrow{a} X \xrightarrow{f} X' \xrightarrow{b} C[1]$ be a distinguished triangle. We should prove that the triangle $t(C) \xrightarrow{t(a)} t(X) \xrightarrow{t(f)} t(X') \xrightarrow{t(b)} t(C)[1]$ is distinguished. It suffices to construct a triangle of morphisms

$$V : t(X')[-1]) \xrightarrow{m} t(C) \xrightarrow{n} t(X)$$

that splits componentwise (in $C(Hw)$) such that $m$ is some choice for $t(b)[-1]$ and $n$ is some choice for $t(a)$. Indeed, it is a well known fact that any such $V$ gives a distinguished triangle in $K(Hw)$. Hence any two sides of $t(V)$ could be lifted to two sides of a distinguished triangle in $K(Hw)$; so $t(V)$ is distinguished (see Definition 3.1.4).

In order to prove our claim we apply Lemma 1.5.4 for all $i \in \mathbb{Z}$. By the Lemma, the triangles $C[i] \xrightarrow{C[i]} C_{i} \xrightarrow{C'[i]}$ obtained from (5) by shifting the last row are weight decompositions of $C[i]$ for all $i \in \mathbb{Z}$. Hence first two columns could be completed to morphisms $Po(X') \to Po(C)[1] \to Po(X)[1]$.

Now we check that the corresponding map of weight complexes splits componentwise.

We apply Lemma 1.5.4 to the morphism $g_{X^{w\leq i}X'^{w\leq i-1}[i]}$ and the weight decompositions $X^{i} \xrightarrow{d_{i}} X^{w\leq i} \to X^{w\leq i-1}[1]$ and $X^{ti-1}[1]$ $\xrightarrow{d'_{i-1}[1]} X'^{w\leq i-1}[1] \to X'^{w\leq i-2}[2]$ of the corresponding objects. We obtain a diagram

$$
\begin{array}{ccc}
X^{i} & \xrightarrow{d_{i}} & X^{w\leq i} \\
\downarrow & & \downarrow \\
X'^{w\leq i-1}[1] & \xrightarrow{d'_{i-1}[1]} & X'^{w\leq i-1}[1] \\
\downarrow & & \downarrow \\
D_{i}[1] & \xrightarrow{t_{i}[2]} & C_{i-1}[2]
\end{array}
$$

(11)

for some $D_{i} \in \text{Obj}_{\mathbb{C}^+}$ and some $t_{i}$. The first column gives $D_{i} \cong X^{i} \oplus X'^{i-1}$. Hence $C_{i}[-1] \xrightarrow{t_{i}} C_{i-1} \to D_{i}$ is a weight decomposition of $C_{i}[-1]$. Applying the fact that the morphisms $C_{i}[-1] \to C_{i-1}$ that correspond to $id_{C_{i}[-1]}$ is

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unique, we obtain that \( t_i \) equals the corresponding morphism coming from the infinite weight decomposition of \( C \) (see Proposition 1.5.5). Hence we obtain our claim.

II \( I \) is an ideal since \( t \) is an additive functor.

Obviously, it suffices to check that for \( X \in C_{[0,n]} \) the ideal \( J = \{ g \in C(X,X) : t(g) = 0 \} \) of the ring \( C(X,X) \) satisfies \( J^{n+1} = 0 \). We will prove this fact by induction in \( n \). In the case \( n = 0 \) we have \( C_{[0,n]} = Hw \), hence \( J = \{ 0 \} \).

To make the inductive step we consider \( g_0 \circ g_1 \circ \ldots \circ g_n, \ g_i \in J \), let \( r = (g_0 \circ g_1 \circ \ldots \circ g_{n-1})[n-1], \ s = g_n[n-1] \circ r \). By Proposition 3.2.4, we can choose a representative \( (h_i, l_i) \) of \( WD(g_i[n-1]) \) such that \( t(h_i) = 0 \). Then by the inductive assumption we have \( WD(r) = (0, m) \) for some \( m : X^n \to X^n \). Considering the morphism of triangles corresponding to \( WD(r) \) we obtain that \( r = b_{n-1}[-1] \circ q \) for some \( q : X[n-1] \to X^n[-1] \). Next, since \( t(g_n) = 0 \), we can assume that \( t(g_n[n-1]) = (u, 0) \) for some \( u \) (by Proposition 3.2.4).

Hence \( g_n[n-1] = v \circ a_{n-1} \) for some \( v \in C(X^{w \leq n-1}, X[n-1]) \) and we obtain \( s = v \circ (a_{n-1} \circ b_{n-1}[-1]) \circ q = 0 \). The assertion is proved.

III Follows from assertion II by a standard reasoning, see Remark 3.1.3.

IV By part 2 of Proposition 2.2.2 if \( X \in C_{w \leq i} \) (resp. \( C_{w \geq i} \)) then choosing \( X^{w \geq i+1} = 0 \) (resp. \( X^{w \leq i-1} = 0 \)) we obtain that the corresponding choice of \( t(X) \) is concentrated in degrees \( \leq i \) (resp. \( \geq i \)). Now note that all choices of \( t(X) \) are homotopy equivalent by part 1 of Proposition 3.1.6.

Conversely, let \( w \) be non-degenerate, let \( t(X) \in K_w(Hw)^{w \leq i} \). We can assume that \( i = 0 \); let \( X \in C_{w \leq n} \). Then \( t(id_X) \) is homotopy equivalent to a morphism concentrated in degrees \( \leq 0 \). Hence Proposition 3.2.4 implies that for \( WD(id_X) = (l, m) \) we can assume that \( t(m) = 0 \).

Then by assertion II we have \( WD(id_X^l) = (l^n, 0) \). Considering the distinguished triangle corresponding to \( WD(id_X^l) \) we obtain that \( id_X = id_X^l \) could be factorized through \( X^{w \leq 0} \). Hence \( X \) is a retract of \( X^{w \leq 0} \); since \( C_{w \leq 0} \) is Karoubi-closed in \( C \) we obtain that \( X \in C_{w \leq 0} \).

The case \( t(X) \in K_w(Hw)^{w \geq i} \) is considered similarly.

V Since \( t \) is weakly exact (see Definition 3.1.4), it suffices to check that \( t(X) = 0 \) implies \( X = 0 \). This is immediate from assertion IV.

VI Immediate from part 15 of Theorem 3.2.2.

VII We can assume that \( a = 0 \).

1. The proof is just a repetitive application of axioms (of weight structures).

Note first that \( t_* \) is bijective for \( X, X' \in C_{w=0} \). Next, for \( X \in C_{w=0} \) and any \( X' \) we consider the distinguished triangle \( X'^{w \leq 0} \to X^0 \to X' \to X'^{w \leq -1}[1] \). Then orthogonality yields that any \( h : C(X, X') \) gives a morphism \( X \to X' \); hence \( t \) is surjective in this case. We also can apply this
statement for $X'' = X^{t_w \leq -1}$. Hence considering the diagram

$$
\begin{array}{cccc}
C(X, X^{t_w \leq -1}) & \longrightarrow & C(X, X^0) & \longrightarrow & C(X, X') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_w(Hw)(t(X), t(X^{t_w \leq -1})) & \longrightarrow & K_w(Hw)(t(X), t(X^0)) & \longrightarrow & K_w(Hw)(t(X), t(X')) & \longrightarrow & 0
\end{array}
$$

induced by $t$ we obtain that $t_*$ is bijective in this case.

Now considering the distinguished triangle $X^{w \geq 1}[1] \to X \to X^0 \to X^{w \geq 1}$ and applying the dual argument one can easily obtain the claim.

2. Let $h \in K_w(Hw)(t(X), t(X'))$. By definition, we can "cut" $h$ to obtain a commutative diagram

$$
t(X^0) \xrightarrow{t(f_0)} t(X^{w \geq 1})
$$

By assertion VII, this diagram corresponds to some homomorphism $WD(X) \to WD(X')$. It remains to apply part I3 of Theorem 3.2.2.

**Remark 3.3.2.** Parts IV and V imply that $t$ it is always conservative and respect the filtrations on $C^b$; see part 3 of Proposition 1.3.4.

It seems probable that $t$ could be lifted to a certain "strong weight complex" functor.

**Conjecture 3.3.3.** $t$ could be lifted to an exact functor $t^*: C \to K(Hw)$.

**Remark 3.3.4.** 1. Let $Hw$ be (fully) embedded into the subcategory $B = \text{Proj} A$ of projective objects of an abelian category $A$ (probably, the most reasonable choice for $A$ is $Hw'$; cf. Lemma 5.4.3 below). Then we have a full embedding $K(Hw) \subset K(B) \subset D(A)$. Suppose now that $A$ is of projective dimension 1. Then any complex over $A$ is quasi-isomorphic to a complex with zero differentials; hence it could be presented in $D(A)$ as a direct sum of surjective morphisms in $B$ (placed in pairwise distinct dimensions). We check that $K_w(B) = K(B)$. Note that it suffices to prove the corresponding fact for $K^b$. Therefore it suffices to check that $K_w(B)(X, Y) = K(B)(X, Y)$ for $X, Y$ being surjective morphisms (as two-term complexes); let $X = X^{-1} \to X^0$. If $Y \in C^{[-1,0]}(B)$ then $K(B)(X, Y) = A(H^0(X), H^0(Y)) = K_w(B)(X, Y)$ (see part 2 of Remark 3.1.5). If $Y \in C^{[-2,-1]}(B)$ then the equality $K(B)(X, Y) = K_w(B)(X, Y)$ is obvious (cf. part VII of Theorem 3.3.1). For $Y$ placed in all other positions we have $K(B)(X, Y) = 0 = K_w(B)(X, Y)$.

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We conclude that $K_w(Hw) = K(Hw)$. Besides, any distinguished triangle in $K_w(Hw)$ is also distinguished in $K(Hw)$. Hence in this case our "weak" weight complex functor is simultaneously a "strong" one.

In particular, this reasoning could be applied if $Hw = Ab_{fin, fr}$ or $Hw = Ab_{fr}$. Hence this is the case for all categories of spectra considered in §4.5 below.

2. In §6.3 below we will also verify the conjecture in the case when $C$ has a differential graded enhancement.

3. Prof. A. Beilinson has kindly communicated to the author a proof of the conjecture in the case when $C$ has a filtered triangulated enhancement; see §8.2 below. Probably, a filtered triangulated enhancement exists for any "reasonable" triangulated category.

4 Connection between weight structures and $t$-structures; duality of hearts

In this section we prove that weight structures are closely related to $t$-structures.

In §4.1 we show that in many cases a weight structure could be described by specifying a negative $H \subset C$. In particular, this is the case for the category of finite spectra ($\subset SH$). In §4.2 we recall the definition of a $t$-structure. In §4.3 we define the notion of adjacent weight and $t$-structure for $C$; their hearts are "dual" in a very interesting sense (see Theorem 4.3.2). In §4.4 we study the conditions for adjacent weight and $t$-structures to exist. We only consider in detail the cases which are relevant for our main examples (motives and $SH$); other possibilities are described in Remark 4.4.3. Note also that the weight resolution construction used in the proof of Theorem 4.4.2 allows to construct Eilenberg-MacLane spectra in $SH$.

In 4.5 we apply the results of this section to the study of $SH$. In particular, we obtain a certain "weight filtration" on homotopy groups of spectra. In §7.1 below we will apply our results to $DM_{eff}$ (the category of motivic complexes of Voevodsky).

4.1 Recovering $w$ from $Hw$

In many cases instead of describing $C^{w \leq 0}$ and $C^{w \geq 0}$ it is easier to specify only $C^{w=0}$. We describe some conditions that ensure that $w$ could be recovered from $Hw$.

**Definition 4.1.1.** Let $H$ be a strict full additive subcategory of $C$. 
1. We will say that \( H \) is negative if for any \( X, Y \in \text{Obj}^\flat H \) and \( i > 0 \) we have \( \underline{C}(X, Y[i]) = 0 \).

2. We will say that \( H' \subset \underline{C} \) is the Karoubi-closure of \( H \) if the objects of \( H' \) are exactly all retracts of objects of \( H \) (in \( \underline{C} \)).

3. We define the small envelope of an additive category \( A \) as a category \( A' \) whose objects are \((X, p)\) for \( X \in \text{Obj}A \) and \( p \in A(X, X) : p^2 = p \) such that there exist \( Y \in \text{Obj}A \) and \( q \in A(Y, X) \), \( s \in A(Y, X) \) satisfying \( sq = 1 - p \), \( qs = id_Y \). We define

\[
A'((X, p), (X', p')) = \{ f \in A(X, X') : p' f = fp = f \}. \tag{12}
\]

Obviously, \( H' \) is exactly the smallest Karoubi-closed subcategory of \( \underline{C} \) containing \( H \).

The small envelope of \( A \) is (naturally) a full subcategory of the idempotent completion of \( A \) (cf. subsection 5.1 below). One should think of \( A' \) as of the category of \( X \odot Y \) for \( X, Y \in \text{Obj}A \), \( Y \) is a retract \( X \). Here \( X \odot Y \) is a certain "complement" of \( Y \) to \( X \).

It can be easily checked that the small envelope of an additive category is additive; \( X \to (X, id_X) \) gives and full embedding \( A \to A' \).

**Theorem 4.1.2.** I Let \( A \) be a full additive subcategory of some triangulated \( \underline{C} \). Then the embedding \( A \to \underline{C} \) could be extended to a full embedding of the small envelope of \( A \) into \( \underline{C} \).

II Let \( H \) be negative and generate \( \underline{C} \).

1. There exists a unique weight structure \( w \) for \( \underline{C} \) such that \( H \subset H_w \).

Moreover, it is bounded.

2. \( H_w \) equals the small envelope of \( H \).

III Let \( H \) weakly generate \( \underline{C} \), suppose that for any \( X \in \text{Obj}C \) there exists a \( j \in \mathbb{Z} \) such that

\[
\forall Y \in \text{Obj}H \text{ we have } \underline{C}(Y, X[i]) = 0 \ \forall i > j. \tag{13}
\]

Let \( H' \subset H \) be additive. Suppose that either

(i) There exists a cardinality \( c \) such that for any direct of sum of \( < c \) objects of \( H \) exists and belongs to \( H \), whence \( \text{Card} \) \( H' \subset c \). For any \( X \in \text{Obj}C \) and any \( Y \in \text{Obj}H' \) the group \( \underline{C}(Y, X) \) considered as a \( \underline{C}(Y, Y) \)-module can be generated by \( < c \) elements. Any object of \( H \) can be presented as \( \oplus_{i \in I} C_i \) for \( C_i \in \text{Obj}H' \), \( \text{Card} I < c \). For any \( I : \text{Card}(I) < c, Y \in \text{Obj}H', j \in \mathbb{Z}, \) and \( X_i \in \text{Obj}H, i \in I \), we have

\[
\underline{C}(Y, \oplus_{j \in I} X_j) = \oplus \underline{C}(Y, X_j) \tag{14}
\]

or

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(ii) Arbitrary direct sums exist in \( H \); all objects of \( H' \) are compact, \( \text{Obj} H' \) is a set.

Then there exists a weight structure \( w \) for \( \mathbb{C} \) such that \( H \subset H_w \). Moreover, it is non-degenerate and bounded above. In case (ii) it admits negative direct sums, in case (i) it admits negative direct sums of \( < c \) objects.

IV Suppose that all conditions of part III ((i) or (ii)) except (13) are fulfilled. Denote the set of objects of \( \mathbb{C} \) satisfying (13) for some \( j \in \mathbb{Z} \) by \( \mathbb{C}^j \); denote the class of objects of \( \mathbb{C} \) satisfying (13) for a fixed \( j \in \mathbb{Z} \) by \( \mathbb{C}^w \leq j \). Then the category \( \mathbb{C}^j \) is triangulated and satisfies all conditions of part III (we will identify the class \( \mathbb{C}^j \) with the corresponding full subcategory of \( \mathbb{C} \)).

Proof. I We map \((X, p)\) to (any choice of) \( \text{Cone}(q) \); we denote this object by \( Z \).

Now we define the embedding on morphisms. We note that in \( A \) the map \( q \) is a projection of \( X \) onto \( Y \). Hence \( A' \) we have \( X \cong (X, p) \oplus Y \), the isomorphism is given by \((p, q)\). Since \( q \) has a section in \( \mathbb{C} \), we have a distinguished triangle \( Z \rightarrow X \rightarrow Y \rightarrow Z[1] \) i.e. we also have a similar decomposition of \( X \) in \( \mathbb{C} \). It is easily seen that \( \mathbb{C}(Z, Z') \) is given exactly by the formula (12) if we assume that \( Z \) is a subobject of \( X \) i.e. if we fix the splitting of the projection \( X \rightarrow Z \). Hence if we fix the embedding \( Z \rightarrow X \) for each \((X, p)\) then (all possible choices) of objects \( \text{Cone}(q) \) would give a subcategory that is equivalent to the small envelope of \( A \); it is obviously additive.

II 1. We define \( \mathbb{C}^w \geq 0 \) as the smallest subset of \( \text{Obj} \mathbb{C} \) that contains \( \text{Obj} H[i] \) for \( i \leq 0 \) and satisfies the property 3 of Proposition 1.3.1; for \( \mathbb{C}^w \leq 0 \) we take a similar ‘closure’ of the set \( \cup \text{Obj} H[i] \) for \( i \geq 0 \).

Obviously, \( \mathbb{C}^w \geq 0 \) and \( \mathbb{C}^w \leq 0 \) satisfy property (ii) of Definition 1.1.1; we define \( \mathbb{C}^w \geq i \) and \( \mathbb{C}^w \leq i \) for \( i \in \mathbb{Z} \) in the usual way.

If we have a distinguished triangle \( X \rightarrow Y \rightarrow Z \rightarrow X[1] \) with \( \mathbb{C}(X, A) = \mathbb{C}(Z, A) = 0 \) for some \( X, Y, Z, A \in \text{Obj} \mathbb{C} \) then \( \mathbb{C}(Y, A) = 0 \); the same statement is valid for a functor of the type \( \mathbb{C}(B, -) \). Hence the equality \( \mathbb{C}(X[i], Y[j]) = 0 \) for \( i < 0 \leq j \), \( X, Y \in H \) easily implies (by induction) that \( \mathbb{C}(Z, T) = 0 \) for all \( Z \in C^w \geq 1 \), \( T \in C^w \leq 0 \).

Now we verify that any \( X \in \text{Obj} \mathbb{C} \) has a ‘weight decomposition’ (with respect to \( \mathbb{C}^w \geq 0 \) and \( \mathbb{C}^w \leq 0 \)).

We prove this by induction on the ‘complexity’ of \( X \) i.e. on the number of distinguished triangles that we have to consider to ‘generate’ \( X \) from objects of \( H[i] \), \( i \in \mathbb{Z} \). For \( X \) of ‘complexity one’ (i.e. for \( X \in \text{Obj} H[i] \)) we can take a ‘trivial’ weight decomposition i.e. define \( X^w \leq 0 \) as \( X \) for \( i \geq 0 \) and 0 otherwise; \( X^w \geq 0 \) will be 0 and \( X \), respectively.

Suppose now that \( X \cong \text{Cone}(Y \rightarrow Z) \) for \( Y, Z \) of ‘complexity less than
that of \( X' \). We proceed exactly as in the proof of Lemma 1.5.4. By the inductive assumption there exist 'weight decompositions' \( \text{of } Y \) and \( Z[-1] \) i.e. distinguished triangles \( Y \xrightarrow{a} A \to B \) and \( Z[-1] \xrightarrow{a'}[-1] A'[−1] \to B'[-1] \) for \( A \in C^u_{w<0}, A' \in C^u_{w<−1}, B \in C^u_{w≥1}, B' \in C^u_{w≥0} \).

By Proposition 1.4.1 \( d \) could be completed to a commutative square (this extension is unique but we don’t need this here)

\[
\begin{array}{ccc}
Y & \xrightarrow{a} & A \\
\downarrow{d} & & \downarrow{h} \\
Z' & \xrightarrow{a'} & A' \\
\end{array}
\]

Next we complete this square to a \( 3 \times 3 \)-diagram (see Proposition 1.4.2). We obtain that for some \( X_1, X_2 \in \text{Obj} C \) there exist distinguished triangles \( X[1] \to X_1[1] \to X_2[1], A'[1] \to X_1[1] \to A[1] \) and \( B'[1] \to X_2[1] \to B[1] \).

Hence \( X_1 \in C^u_{w<0}, X_2 \in B \in C^u_{w<1} \) and we obtain a weight decomposition of \( X \).

Now we take for \( C^u_{w≥0} \) and \( C^u_{w≤0} \) the Karoubi-closures of \( C^u_{w≥0} \) and \( C^u_{w≤0} \), respectively. Then we obtain a pair of Karoubi-closed subcategories of \( C \) that still satisfy properties (i), (ii), and (iv) of Definition 1.1.1. Moreover, if \( C(X, A) = 0 \) then \( C(X', A') = 0 \) for \( X' \) and \( A' \) being any retracts of \( X \) and \( A \), respectively. Hence \( C^u_{w≥0} \) and \( C^u_{w≤0} \) also satisfy the orthogonality property.

Now, since any object of \( C \) could be obtained by a finite sequence of considerations of cones of morphisms of objects of \( Hw \), we obtain that \( w \) is bounded.

It remains to check that \( w \) is the only weight structure such that \( H \subset Hw \). By part 3 of Proposition 1.3.1 for any weight structure \( u \) satisfying \( H \subset Hw \) we have \( C^u_{w≥0} \subset C^u_{w≥0} \) and \( C^u_{w≤0} \subset C^u_{w≤0} \). Since \( C^u_{w≥0} \) and \( C^u_{w≤0} \) are Karoubi-closed, we also have \( \overline{C^u_{w≥0}} \subset C^u_{w≥0} \) and \( \overline{C^u_{w≤0}} \subset C^u_{w≤0} \). Now Lemma 1.3.5 implies our claim immediately.

2. By assertion 1, \( C \) contains the small envelope of \( H \). To check that this envelope is actually contained in \( Hw \) it suffices to note that the object \( X \otimes Y \) could be presented both as a cone of the "embedding" \( Y \to X \) and of the "projection" \( X \to Y \).

To check the inverse inclusion we can assume that \( H \) equals its small envelope. Let \( X \in C^u_{w=0} \).

We apply the weight complex functor \( t \). We obtain that \( t(X) = X \) is a retract of two objects \( A, B \in \text{Obj} K^b_w(Hw); A \in \text{Obj} K^b_{w≥0}(Hw) \) and \( B \in \text{Obj} K^b_{w≤0}(Hw) \). Next, applying Lemma 3.1.2 we obtain that the same is true in \( K^b(Hw) \). This easily implies that \( t(X) = t(Z) \) for some \( Z \) that
could be presented as an object of a small envelope of $H$ in $C$. Hence the assertion follows from part VI of Theorem 3.3.1.

III 1. Again, for $C^w_{\leq 0}$ we take the smallest Karoubi-closed subset of $\text{Obj}C$ that contains $H[i]$ for $i \leq 0$ and satisfies the property 3 of Proposition 1.3.1.

For $C^w_{\leq 0}$ we take

$$\{X \in \text{Obj}C : \mathcal{C}(X, Y) = 0 \forall Y \in \text{Obj}H[i], \ i < 0\}.$$  

The proof of orthogonality is by induction on 'complexity' of $Y \in C^w_{\leq 0}$ as in the proof of part III. We have $\mathcal{C}(X, Y) = 0$ for any $X \in C^w_{\leq 0}$ and any $Y \in C^w_{\geq 0}$ of 'complexity one'. Now using the fact that all (co)representable functors are homological on $C$, we obtain that the same is true for objects of $C^w_{\geq 0}$ of arbitrary complexity. Obviously, the same is true for their direct sums i.e. for the whole $C^w_{\geq 0}$.

$C^w_{\leq 0}$ is obviously Karoubi-closed; $C^w_{\geq 0}$ is Karoubi-closed by definition. Both $C^w_{\leq 0}$ and $C^w_{\geq 0}$ are additive and strict. Moreover, property 3 of Proposition 1.3.1 shows that for any $w'$ satisfying the conditions we have $C^w_{\geq 0} \subseteq C^{w'}_{\geq 0}$, hence $C^w_{\leq 0} \supseteq C^{w'}_{\leq 0}$

Note also that $C^w_{\leq 0}$ also satisfies the property 3 of Proposition 1.3.1 i.e for any distinguished triangle $X \to Y \to Z$ we have $X, Z \in C^w_{\leq 0} \implies Y \in C^w_{\leq 0}$.

To prove that $w$ is a weight structure, it remains to prove the existence of weight decompositions. We will construct $X^{i \leq 0}$ and $X^{i \geq 0}$ for a fixed $X \in \text{Obj}C$ explicitly. The construction could be called the weight resolution, cf. the proof of Proposition 4.4.2 below and Proposition 7.1.2 of [19].

First we treat case (i). For each object $Y$ of $H'$ any $Z \in \text{Obj}C$ we choose some set of $f_i(Y, Z) \in \mathcal{C}(Y, Z)$ of cardinality < $c$ that $f_i(Y, Z)$ are $\mathcal{C}(Y, Y)$-generators of $\mathcal{C}(Y, Z)$. Let $j \in \mathbb{Z}$ satisfy (13).

Now we construct a certain sequence of $X_k$ for $k \leq j$ starting from $X_j = j$. For $k = j$ we take $P_j = \bigoplus_{Y \in \text{Obj}H, f_i(Y, X_j[j])} Y$. Note that the number of summands is < $c$, hence the sum exists and belongs to $ObjH$. Then we have a morphism $f_j : P_j \to X_j[j]$ given by $\prod f_i(Y, X[j])$. Let $X_{j-1}[j]$ denote a cone of $f_j$. Repeating the construction for $X_{j-1}$ instead of $X_j$ and with $k = j - 1$ we get an object $P_{j-1} \in \text{Obj}H'$, $f_{j-1} : P_{j-1} \to X_{j-1}[j - 1]$; we denote a cone of $f_1$ by $X_{j-2}[j - 1]$. Proceeding, we get an infinite sequence of $(P_i, f_i, X_i)$. Note that we have $P_i \in C^w_{\geq 0}$.

We denote the maps $X_i \to X_{i-1}$ given by the construction by $g_i$, $h_i = g_j \circ \cdots \circ g_{i+2} \circ g_{i+1} : X \to X_i$. We denote a cone of $h_i$ by $Y_i[-1]$; the map $Y \to X_i$ given by the corresponding distinguished triangle by $r_i$.

The octahedron axiom implies that the commutative triangle $X \xrightarrow{h_i} X_i \xrightarrow{g_i} X_{i-1}$ can be completed to an octahedron diagram (cf. §IV.1 of [12], or the
last paragraph of 1.4). Hence we obtain a distinguished triangle $P_i[-i] \rightarrow Y_i \rightarrow Y_{i-1} \rightarrow P_i[1-i]$. By induction on $i$ we obtain that $Y_i[i] \in C^w_{w\geq 0}$ for all $i \leq j$ (using the definition of $C^w_{w\geq 0}$).

Now we denote $Y_0$ by $Y$ and $X_0$ by $Z$. $Y, Z$ will be our candidates for $X^w_{w\geq 0}$ and $X^w_{w\leq 0}$.

It remains to prove that $Z \in C^w_{w\leq 0}$. We should check that $C(C, Z[k]) = 0$ for all $k > 0$, $C \in \text{Obj}H$. Since $C(-, Z)$ transforms arbitrary sums into products, it suffices to consider $C \in \text{Obj}H'$.

First we prove that $C(C, X_{k-1}[k]) = 0$ for all $k \leq j$.

We apply the distinguished triangle

$$V_k : P_k \rightarrow X_k[k] \rightarrow X_{k-1}[k] \rightarrow P_k[1].$$

Using (14), we obtain $C(C, P_k[k]) = \oplus_{Y \in \text{Obj}H', f_i(Y, X_k[k])} C(C, Y)$. By the definition of $f_i(C, Y)$ we obtain that this group surjects onto $C(C, X_k[k])$. Moreover, $C(C, P_k[1]) = \oplus_{Y \in \text{Obj}H', f_i(Y, X_k[k])} C(C, Y[1]) = 0$. We obtain $C(C, X_{k-1}[k]) = 0$.

Now we use distinguished triangles $V_i$ for all $l < k$. Again (14) yields $C(C, P_l[1]) = C(C, P_l[2]) = 0$. Hence $C(C, X_{l-1}[k]) = C(C, X_l[k]) = 0$ for all $l < k$.

Hence $C(C, Z[k]) = 0$ for all $j \leq k > 0$.

Lastly, the distinguished triangles $V_k$ easily give by induction that $C(C, X_l[k]) = 0$ for all $l \leq j$ and $k > j$.

The proof in case (ii) is almost the same; one should only always replace some choice of generators $f_i(Y, Z) \in C(Y, Z)$ by all elements of $C(Y, Z)$.

$(C, w)$ is obviously bounded above by (13).

Now we check that $(C, w)$ is non-degenerate. The condition (13) implies that $\cap C^w_{w\geq i} = 0$. Next, for any $X \in \text{Obj}C \setminus \{0\}$ there exists an $f \in C(Y[i], X)$ for some $Y \in H$ and $i \in \mathbb{Z}$ such that $f \neq 0$. Hence such an $X$ does not belong to $C^w_{w\leq i} = 0$ (see the definition of $C^w_{w\leq i} = 0$ in the proof of part III).

IV Everything is obvious except that a cone of a morphism of objects of $C^-$ belongs to $\text{Obj}C^-$. This fact is easy also since the functors $C(Y, -)$ are homological.

Note that in the proofs it was specified explicitly how to recover $w$ from $C^w_{w=0}$.

**Corollary 4.1.3.** It is well known that there are no morphisms of positive degrees between (copies) of the sphere spectrum $S^0$ in the stable homotopy category $SH$. Hence part II of Theorem 4.1.2 immediately implies that the category of finite spectra $SH_{\text{fin}}$ (i.e. the full subcategory of $SH$ generated

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by $S^0$) has a bounded weight structure $w$. Its heart could be described as a category $H$ of finite sums of (copies of) $S^0$ (since any retract of $S^0$ is trivial, no new objects appear in the small envelope of $H$). Since $SH(S^0, S^0) = \mathbb{Z}$, $Hw$ is equivalent to $Ab_{\text{fin,fr}}$ (the category of finitely generated free abelian groups).

This weight structure obtained is a certain 'dual' of the Postnikov $t$-structure for $SH$; cf. Theorem 4.3.2 and §4.5.

We will prove that the whole $SH$ satisfies the conditions of part IV2, while a certain category of quasi-finite spectra satisfies the conditions of part III for $c = \omega$, in §4.5 below.

Remark 4.1.4. 1. Note that in the proof of Theorem 4.1.2 it was specified explicitly how to recover $w$ from $\underline{C}_{w=0}$.

2. The conditions of parts III and IV could seem to be rather exotic. Yet they could be easily checked for a natural subcategory quasi-finite objects in $SH$, see §4.5.

3. Obviously, if for any $X \in \text{Obj}C$ and any $Y \in \text{Obj}H'$ the group $\underline{C}(Y, X)$ is generated by $< c$ elements as a group, then it is also generated by $< c$ elements as a $\underline{C}(Y, Y)$-module. In particular, this is the condition which we will actually check for the category $SH_{\text{fin}}$.

4. $\underline{C}_{w<0}$ described in the proof of part III of Theorem 4.1.2 is often a $\underline{C}_{\leq 0}$-part of a certain $t$-structure (cf. Definition 4.2.1 below); then this $t$-structure is left adjacent to $w$ (see Definition 4.3.1 below). Yet in order for the $t$-decompositions to exist when we take the only possible candidate for $\underline{C}_{w<0}$ (cf. Proposition 4.3.4) the projective limit of $Y_i$ defined as in the proof of part III for all $i < j$ should exist for all $X \in \text{Obj}C$ (cf. the proof of Proposition 4.4.2). Note that this is not true for the category $SH_{\text{fin}}$ (see Corollary 4.1.3). For example, one could note that Eilenberg-MacLane spectra do not belong to $SH_{\text{fin}}$.

This shows that weight structures 'exist more often than $t$-structures', while Theorem 4.3.2 below shows that they 'contain almost the same information' as the corresponding $t$-structures. This evidence (along with the corresponding results for Voevodsky's motives below, see Proposition 6.5.1) supports author's opinion that weight structures are more relevant for 'general' triangulated categories than $t$-structures.

Moreover, it could be easily seen that the natural "opposite" to the statement of part II (i.e. we take a positive generating subcategory $H$ and ask whether a $t$-structure with $H \subset Ht$ exists) is false.

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4.2 $t$-structures: reminder

To fix the notation we recall the definition of a $t$-structure.

**Definition 4.2.1.** A pair of subsets $C^t_{\geq 0}, C^t_{\leq 0} \subset \text{Obj} C$ for a triangulated category $C$ will be said to define a $t$-structure $t$ if $C^t_{\geq 0}, C^t_{\leq 0}$ satisfy the following conditions:

(i) $C^t_{\geq 0}, C^t_{\leq 0}$ are strict i.e. contain all objects of $C$ isomorphic to their elements.

(ii) $C^t_{\geq 0} \subset C^t_{\geq 0}[1], C^t_{\leq 0}[1] \subset C^t_{\leq 0}$.

(iii) **Orthogonality.** For any $X \in C^t_{\leq 0}[1], Y \in C^t_{\geq 0}$, we have $C(X, Y) = 0$.

(iv) **$t$-decomposition.** For any $X \in \text{Obj} C$ there exists a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow A[1]$$

such that $A \in C^t_{\leq 0}, B \in C^t_{\geq 0}[1]$. 

We will need some more notation for $t$-structures.

**Definition 4.2.2.** 1. A category $Ht$ whose objects are $C^t_{= 0} = C^t_{\geq 0} \cap C^t_{\leq 0}$, $Ht(X, Y) = C(X, Y)$ for $X, Y \in C^t_{= 0}$, will be called the heart of $t$. Recall (cf. Theorem 1.3.6 of [4]) that $Ht$ is abelian (short exact sequences are distinguished triangles in $C$).

2. $C^t_{\geq 1}$ (resp. $C^t_{\leq 1}$) will denote $C^t_{\geq 0}[-1]$ (resp. $C^t_{\leq 0}[-1]$).

Non-degenerate and bounded (above, below, or both) $t$-structures could be defined similarly to Definition 1.2.1.

Recall (cf. Lemma IV.4.5 in [12]) that (16) defines additive functors

$C \rightarrow C^t_{\leq 0}: X \rightarrow A$ and $C \rightarrow C^t_{\geq 1}: X \rightarrow B$. We will denote $A, B$ by $X^t_{\leq 0}$ and $X^t_{\geq 1}[-1]$, respectively.

(16) will be called the $t$-decomposition of $X$.

We denote by $H^0t$ the zeroth homology functor corresponding to $t$ (cf. part 10 of §IV.4 of [12]); $H^0t(X)$ is defined similarly to $X^{[0,1]}$ in part 1 of Proposition 1.5.5. Shifting the $t$-decomposition of $X^t_{\leq 0}[-1]$ by [1] we obtain a canonical and functorial (with respect to $X$) distinguished triangle

$X^t_{\leq 1}[1] \rightarrow X^t_{\leq 0} \rightarrow H^0t(X)$ for $X^t_{\leq -1} \in C^t_{\leq 0}$.

4.3 Adjacent weight and $t$-structures

**Definition 4.3.1.** We say that a weight structure $w$ is left (resp. right) adjacent to a $t$-structure $t$ if $C^w_{\leq 0} = C^t_{\leq 0}$ (resp. $C^w_{\geq 0} = C^t_{\geq 0}$).

In this situation we will also say that $t$ is right (resp. left) adjacent to $w$. 

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A simple example is obtained if one takes the canonical $t$-structure of (some version of) $D(A)$ for an abelian $A$. Then we have adjacent weight structures given by projective and injective resolutions in degrees $\geq 0$ and $\leq 0$ if such resolutions exist.

The following result shows that adjacent structures could be uniquely recovered from each other. It also shows that $Ht$ and $Hw$ are connected by a natural generalization of the relation between the categories $A$ and $Proj A$ for an abelian $A$. Note that in this case we also have $Hw \subset Ht$; this is a rather rare situation.

**Theorem 4.3.2 (Duality theorem).** Let $w$ be left adjacent to $t$. Then the following statements are fulfilled.

1. $\mathcal{C}^{w \geq 0} = \{ X \in Obj \mathcal{C} : \mathcal{C}(X,Y) = 0 \ \forall Y \in \mathcal{C}^{w \leq -1} = \mathcal{C}^{t \leq -1} \}.$
2. $\mathcal{C}^{t \geq 0} = \{ X \in Obj \mathcal{C} : \mathcal{C}(Y,X) = 0 \ \forall Y \in \mathcal{C}^{w \leq -1} = \mathcal{C}^{t \leq -1} \}.$
3. The functor $\mathcal{C}(-,Ht) : Hw \rightarrow Ht^*$ (see the Notation) that sends $X \in \mathcal{C}^{w=0}$ to $Y \rightarrow \mathcal{C}(X,Y)$, $Y \in \mathcal{C}^{t=0}$ is a full embedding of $Hw$ into the full subcategory $Ex(Ht,Ab) \subset Ht^*$ which consists of exact functors.
4. The functor $\mathcal{C}(-,Ht) : Ht \rightarrow Hw_*$ that sends $X \in \mathcal{C}^{t=0}$ to $Y \rightarrow \mathcal{C}(Y,X)$, $Y \in \mathcal{C}^{w=0}$ is a full exact embedding of $Ht$ into the abelian category $Hw_*.$
5. Let $t$ be non-degenerate. Then $\mathcal{C}^{t=0}$ equals the class

$$S = \{ X \in Obj \mathcal{C} : \mathcal{C}(Y,X[i]) = 0 \ \forall Y \in \mathcal{C}^{w=0}, \ i \neq 0 \}.$$

Proof. 1. Immediate from part 2 of Proposition 1.3.1 applied to $w$.
2. A well-known property of $t$-structures (note that one doesn’t have to consider $w$).
3. First we note that for any $X \in \mathcal{C}^{w=0}$ orthogonality for $w$ implies $\mathcal{C}(X,Y) = 0$ for all $Y \in \mathcal{C}^{w \leq -1} = \mathcal{C}^{t \leq -1}$, while orthogonality for $t$ gives $\mathcal{C}(X,Y) = 0$ for all $Y \in \mathcal{C}^{t \geq 1}$. In particular,

$$\mathcal{C}(X,Y[i]) = 0 \ \forall \ X \in \mathcal{C}^{w=0}, \ Y \in \mathcal{C}^{t=0}, \ i \neq 0. \quad (17)$$

Now, short exact sequences in $Ht$ give distinguished triangles in $\mathcal{C}$. Hence for any homological functor $F : \mathcal{C} \rightarrow Ab$ and for $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ being a short exact in $Ht$ we have a long exact sequence $\ldots \rightarrow F(C[-1]) \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow F(A[1]) \rightarrow \ldots$. If $F = \mathcal{C}(X,-)$ then $F(C[-1]) = F(A[1]) = 0$ (as was just noted). Hence objects of $Hw$ induce exact functors on $Ht$.

To prove that the restriction $Hw \rightarrow Ht^*$ is fully faithful functor it suffices to prove that the restriction of the functor $\mathcal{C}(X,-)$ to $Ht$ for $X \in Hw$ determines $X$ in a functorial way. Using Ionea’s lemma, we see that it suffices to recover $\mathcal{C}(-,X)$ from its restriction.
We prove that

$$C(X, Y) = C(X, H^0t(Y)) \quad \forall X \in C^{w=0}, \ Y \in \text{Obj}C.$$  \hspace{1cm} \text{(18)}

We apply the $t$-decomposition (i.e. (16)) twice.

We have a distinguished triangle $Y^t \geq 1[-1] \to Y^{t\leq 0} \to Y \to Y^{t\geq 1}$. Since $\underline{C}(X, Y^t \geq 1[-1]) = \underline{C}(X, Y^t \geq 1)$, we obtain $\underline{C}(X, Y) = \underline{C}(X, Y^{t\leq 0})$.

Next, we have a distinguished triangle $Y^{t\leq 1} \to Y^{t\leq 0} \to H^0t(Y) \to Y^{t\leq -1}[1]$. Since $\underline{C}(X, Y^{t\leq -1}) = \underline{C}(X, Y^{t\leq -1}[1]) = 0$, we obtain (18).

4. Again, it suffices to prove that the restriction of the functor $C(\cdot, X)$ to $Hw$ for $X \in Ht$ determines $X$ functorially.

We note that for any $X \in C^{t=0}$ orthogonality for $w$ implies $\underline{C}(Y, X) = 0$ for all $Y \in C^{t \leq -1} = C^{w \leq -1}$, while orthogonality for $t$ gives $\underline{C}(Y, X) = 0$ for all $Y \in C^{w \geq 1}$.

Let $\cdots \to Y^{-1} \to Y^0 \to Y^1 \to \cdots$ denote an arbitrary choice of the weight complex for $\underline{C}$, $w$ and $Y \in \text{Obj}C$. Then we prove that

$$\underline{C}(Y, X) = (\text{Ker}(\underline{C}(Y^0, X) \to \underline{C}(Y^{-1}, X)) / \text{Im}(\underline{C}(Y^1, X) \to \underline{C}(Y^0, X))).$$  \hspace{1cm} \text{(19)}

Indeed, consider the (infinite) weight decomposition of $Y$ that gives our choice of the weight complex and apply Theorem 2.4.1 to the functor $C(\cdot, X)$. The spectral sequence obtained converges since it satisfies condition II(ii) of Theorem 2.4.1 (it has only one non-zero column!). It remains to note that this only non-zero column of $(E^p_1(T, \underline{C}(\cdot, X), Y)) = (\underline{C}(Y^p, X[q]))$ is exactly $(E^p_1(T)) = \cdots \to \underline{C}(Y^1, X) \to \underline{C}(Y^0, X) \to \underline{C}(Y^{-1}, X) \to \cdots$

We obtain (19).

5. By assertion 4, an object of $Ht$ is non-zero if its represents a non-zero functor on $Hw$. Hence applying (18) we obtain that $S$ is exactly the class of objects that satisfy $H^it(X) = 0$ for all $i \neq 0$. It remains to note that for a non-degenerate $t$ this set is exactly $C^{t=0}$.

\[ \square \]

Remark 4.3.3. 1. Usually one can describe the images of embeddings in parts 3 and 4 more explicitly. If $C$ is 'large enough' then these embedding are equivalences; see below.

2. In fact, one can extend the notion of an adjacent structures to the case when there are two distinct triangulated categories $\underline{C}$ and $\underline{D}$ equipped with a duality $\Phi : \underline{C} \times \underline{D} \to \text{Ab}$ (that generalizes $\underline{C}(\cdot, \cdot) : \underline{C}^{\text{op}} \times \underline{C} \to \text{Ab}$). Then $\underline{D}^{t \geq 1}$ should be the dual of $\underline{C}^{w \leq 0}$ with respect to $\Phi$, while $\underline{D}^{t \leq -1}$ should be the dual of $\underline{C}^{w \geq 0}$. If $\Phi$ satisfies certain 'perfectness' conditions then we will obtain the analogues of parts 3,4 of Theorem 4.3.2
Theorem 4.3.2 gives a simple description of adjacent structures (of any type) when they exist.

**Proposition 4.3.4.** 1. Let \( w \) be a weight structure for \( C \). Then there exists a \( t \)-structure which is left (resp. right) adjacent to \( w \) iff for \( C^t \leq 0 = C^w \leq 0 \) and \( C^t \geq 0 = C^w \geq 0 \) = \( \{ X \in \text{Obj}C : C(X, Y) = 0 \ \forall Y \in C^w \leq -1 \} \) (resp. \( C^t \geq 0 = C^w \geq 0 \) = \( \{ X \in \text{Obj}C : C(X, Y) = 0 \ \forall Y \in C^w \geq 1 \} \), and any \( X \in \text{Obj}C \) there exists a \( t \)-decomposition (16) of \( X \). In this case our choice of \( C^t \leq 0 \) and \( C^t \geq 0 \) is the only one possible.

2. Let \( t \) be a \( t \)-structure for \( C \). Then there exists a weight structure which is left (resp. right) adjacent to \( w \) iff for \( C^w \leq 0 = C^t \leq 0 \) and \( C^w \geq 0 = C^t \geq 0 \) = \( \{ X \in \text{Obj}C : C(X, Y) = 0 \ \forall Y \in C^t \leq -1 \} \) (resp. \( C^w \geq 0 = C^t \geq 0 \) and \( C^w \leq 0 = C^t \leq 0 \) = \( \{ X \in \text{Obj}C : C(X, Y) = 0 \ \forall Y \in C^t \geq 1 \} \), and any \( X \in \text{Obj}C \) there exists a weight decomposition (1) of \( X \). In this case our choice of \( C^t \leq 0 \) and \( C^t \geq 0 \) is the only one possible.

**Proof.** First we note that by Theorem 4.3.2 our choices of the structures are the only one possible. Hence it suffices to check when these choices indeed give the corresponding structures.

1. We only consider the left adjacent structure case; the ‘right’ case is similar (and, in fact, dual).

The set \( C^t \geq 0 \) is automatically strict and since \( C^w \leq 0[1] \subset C^w \leq 0 \), we have \( C^t \geq 0 \subset C^t \geq 0[1] \).

Hence we obtain a \( t \)-structure if and only if there always exists a \( t \)-decompositions.

2. As in part 1, we consider only the ‘left’ case (for the same reason). It is well known that \( C^w \leq 0 \) is Karoubi-closed; hence both \( C^w \leq 0 \) and \( C^w \geq 0 \) are Karoubi-closed also. Again \( C^t \leq 0[1] \subset C^t \leq 0 \) implies \( C^t \geq 0 \subset C^t \geq 0[1] \).

Hence we obtain a weight structure if and only if there always exists a weight decompositions. \( \square \)

**4.4 Existence of adjacent structures**

Now we study certain sufficient conditions for adjacent weight and \( t \)-structures to exist.

We prove a statement that is relevant for Voevodsky’s \( DM^{eff} \) and \( SH \). We describe a certain version of the compactly generated category notion; \( DM^{eff} \) and \( SH \) will satisfy our conditions.

**Definition 4.4.1.** We will say that the a set of objects \( C_i \in \text{Obj}C \), \( i \in I \) (\( I \) is a set) negatively well-generate \( C \) if

(i) \( C_i \) are compact; they weakly generate \( C \) (cf. the Notation).
(ii) For all \( j > 0, i, i' \in I \) and \( j > 0 \) we have \( \mathcal{C}(C_i, C_{i'[j]} = 0 \) (i.e. the set \( \{C_i\} \) is negative).

(iii) \( \mathcal{C} \) contains the category \( H \) whose objects are arbitrary (small) direct sums of \( C_i \); \( \mathcal{C} \) also contains all direct limits of \( X_i \in \text{Obj} \mathcal{C} \) (see Lemma 1.4.3) such that \( X_{-1} = 0 \) and the cone of \( X_i \to X_{i+1} \in H[i] \).

**Theorem 4.4.2.** II. Suppose that \( C_i \in \text{Obj} \mathcal{C}, i \in I, \) negatively well-generate \( \mathcal{C} \). For \( H \) described in (iii) of Definition 4.4.1 we consider a full subcategory \( \mathcal{C}^- \subset \mathcal{C} \) whose objects are

\[
X \in \text{Obj} \mathcal{C}^- : \forall Y \in \text{Obj} H \text{ there exists } j \in \mathbb{Z} \text{ such that } \mathcal{C}(Y, X[i]) = 0 \forall i > j.
\]

(20)

Then there exist adjacent weight and t-structures \( w \) on \( \mathcal{C}^- \) and \( t \) on \( \mathcal{C} \) such that \( H \subset Hw, t \) restricts to a t-structure on \( \mathcal{C}^- \), and \( \mathcal{C}^{t \leq 0} = \mathcal{C}^{-, w \leq 0} = H \).

Note that \( w \) and \( t \) restricted to \( \mathcal{C}^- \) are adjacent by definition.

2. If \( \mathcal{C} \) also admits arbitrary countable direct sums, then \( w \) could be extended to the whole \( \mathcal{C} \).

If \( \mathcal{C}, \mathcal{C}, w, t \) be either as in part 12 or as in 11 with the additional condition \( \mathcal{C} = \mathcal{C}^- \) (i.e. \( w \) is defined on \( \mathcal{C} \)) fulfilled. Then the following statements are valid.

1. \( Hw \) is the small envelope of the category \( H \) (whose objects are direct sums of \( C_i \) in \( \mathcal{C} \).

2. Restrict the functors from \( Ht \) (considered as a subset of \( Hw \), by part 4 of Theorem 4.3.2) to the full additive subcategory \( \mathcal{C} \subset Hw \) consisting of finite direct sums of \( C_i \). Then this restriction functor gives an equivalence of \( Ht \) with \( C \).

3. For any object of \( Y \in \mathcal{C}^{t = 0} \) and any \( X \in \text{Obj} \mathcal{C} \) we have \( \mathcal{C}(X, Y) = (\text{Ker}(\mathcal{C}(X^0, Y) \to \mathcal{C}(X^{-1}, Y))/\text{Im}(\mathcal{C}(X^1, Y) \to \mathcal{C}(X^0, Y)) \) where \( \cdots \to X^{-1} \to X^0 \to X^1 \to \cdots \) is an arbitrary choice of the weight complex for \( X \).

**Proof.** II. The existence of \( w \) on \( \mathcal{C}^- \) is immediate from part III (version (ii)) of Theorem 4.1.2. We define \( \mathcal{C}^{t \geq 0} = \{X \in \text{Obj} \mathcal{C} : \mathcal{C}(Y, X) = 0 \forall Y \in \mathcal{C}^{w \leq -1}\} \). Then to prove that \( t \) is a t-structure it suffices (cf. Proposition 4.3.4) to check that for any \( X \in \text{Obj} \mathcal{C} \) there exists a t-decomposition (16).

We will construct \( X^{t \leq 0} \) and \( X^{t \geq 1} \) explicitly. Our construction is uses almost the same argument as in the proof of part III version (ii) of Theorem 4.1.2. It could also be thought about as of a 'triangulated version of the construction of Eilenberg-MacLane spaces' (this construction really allows to construct Eilenberg-MacLane spectra from \( S^0 \) in \( SH \), see 4.5 below!).
We take $P_0 = \bigoplus_{i \in I, s \in C(C_i, X)} C_i$. Then we have a morphism $f_0 : P_0 \to X$ whose component that corresponds to $(C_i, s)$ is given by $s$. Let $X_0$ denote a cone of $f_0$. Repeating the construction for $X_0[-1]$ instead of $X$ we get an object $P_1$ being a direct sum of certain $C_i$, $f_1 : P_1 \to X_0[-1]$; we denote a cone of $f_1$ by $X_1[-1]$. Proceeding (with $X_i[-1 - i]$), we get an infinite sequence of $(P_i, f_i, X_i)$. We denote the map $X \to X_0$ given by the construction by $g_0$, $g_i : X_{i-1} \to X_i, h_i = g_i \circ \cdots \circ g_1 \circ g_0 : X \to X_i$. We denote a cone of $h_i$ by $Y_i[1]$; the map $Y_i \to X[1]$ given by the corresponding distinguished triangle by $r_i$. We have $P_i \in \mathbb{C}^{w=0}$ by the definition.

We have $Y_0 = P_0$. The octahedron axiom implies that the commutative triangle $X \xrightarrow{h_{i-1}} X_{i-1} \xrightarrow{g_i} X_i$ could be completed to an octahedron diagram. This yields a distinguished triangle $Y_i \to P_i[i] \to Y_{i-1}[1] \to Y_i[1]$, we denote the map $Y_i-1 \to Y_i$ by $\phi_{i-1}$. The octahedron diagram (cf. §IV.1 of [12]) also gives $r_{i-1} = r_i \circ \phi_{i-1}$. Hence $Y_i \in \mathbb{C}^{w=0}$ by definition.

Now we consider the limit of $Y_i$ as in Lemma 1.4.3. Then for any $C \in \text{Obj} \mathbb{C}$ we have $\mathbb{C}(Y, C) = \varprojlim \mathbb{C}(Y_i, C)$. In particular, the sequence $r_i$ gives a canonical $f : Y \to X$. We denote its cone as $Z$.

Since $(\mathbb{C}, w)$ admit negative direct sums, we have $D \in \mathbb{C}^{w=0}$, hence $Y \in \mathbb{C}^{w \leq 0}$. $Y, Z$ will be our candidates for $X^{t \leq 0}$ and $X^{t \geq 1}$.

We verify that $Z \in \mathbb{C}^{t \geq 1}$. First we check that $\mathbb{C}(C_i[j], Z) = 0$ for all $i \in I, j \geq 0$. This is equivalent to the fact that the map $f_*: \mathbb{C}(C_i[j], Y) \to \mathbb{C}(C_i[j], X)$ is an isomorphism for all $i \in I, j \geq 0$ and is injective for $j = -1$ (cf. part 1 of Remark 1.3.2).

We compute $\mathbb{C}(C, Y)$ for any compact $C$. We have $\mathbb{C}(C, D) = \bigoplus \mathbb{C}(C, Y_i)$. Hence it is easily seen that the map $a_* : \mathbb{C}(C, D) \to \mathbb{C}(C, D)$ is surjective, while its cokernel is $\varprojlim \mathbb{C}(C, Y_i)$. Moreover, we have $\mathbb{C}(C_i[-1], Y) = 0$ since $Y \in \mathbb{C}^{w=0}$ and $C_i[-1] \in \mathbb{C}^{w=1}$. Hence it suffices to verify that $\mathbb{C}(C_i[j], X_i) = 0$ for $l > j \geq 0$ (this gives $\mathbb{C}(C_i[j], Y_i) \cong \mathbb{C}(C_i[j], X_i)$).

We apply the distinguished triangle $P_j[j] \to X_{j-1} \to X_j \to P_j[j + 1]$. Since $C_i[j]$ is compact, we easily obtain

$$\mathbb{C}(C_i[j], P_j[j]) = \bigoplus_{m \in I, s \in \mathbb{C}(C_m[j], X_{j-1})} \mathbb{C}(C_i, C_m).$$

Hence this group has an element for each morphism $C_m[j] \to X_{j-1}$; it follows that the map $\mathbb{C}(C_i[j], P_j[j]) \to \mathbb{C}(C_i[j], X_{j-1})$ is surjective. Next, since $C_i[j]$ is compact and $C_i[j] \in \mathbb{C}^{w=j}$, $\mathbb{C}(C_i[j], P_j[j + 1]) = \mathbb{C}(C_i, P_j[1])$ equals the direct sum of corresponding $\mathbb{C}(C_i, C_m[1])$; hence it is zero by the orthogonality property for $w$ (cf. Definition 1.1.1). We obtain $\mathbb{C}(C_i[j], X_{j}) = 0$.

Now we use distinguished triangle $P_l[l] \to X_{l-1} \to X_l \to P_l[l+1]$ for $l > j$. Again compactness of $C_i[j]$ yields $\mathbb{C}(C_i[j], P_l[l + 1]) = \mathbb{C}(C_i[j], P_l[l]) = 0$. 47
Hence \( C(C_i[j], X_i) = C(C_i[j], X_{l-1}) = 0 \) for all \( l > j \).

It remains to check that for any \( T \in \text{Obj} \mathcal{C} \) the condition \( C(C_i[j], T) = 0 \) for all \( i \in I, \ j \geq 0 \) implies that \( C(C, T) = 0 \) for all \( C \in C^{w \leq 0} \). This follows immediately from part III version (ii) of Theorem 4.1.2.

Moreover, part III version (ii) of Theorem 4.1.2 implies that if on \( C^- \) (defined as in part IV of Theorem 4.1.2) there exists a weight structure such that \( H \subset Hw \). Note that \( C^- \) also satisfies the conditions of the theorem. The description of \( C^{w \leq 0} \) in the proof of Theorem 4.1.2 shows that \( w \) is left adjacent to \( t \) on \( C^- \).

2. We should check that \( w \) could be extended to the whole \( C \). We define \( X^{w \geq 0} \) using the orthogonality axiom (of weight structures).

For any \( X \in \text{Obj} \mathcal{C} \) we denote \( X_i \leq t \) by \( X_i \) and take \( Y = \lim Y_i \) for \( Y_i = X_i^{w \geq 0} \). Here the maps \( s_{i,j+1} : Y_i \to Y_{i+1} \) are obtained by applying part 1 of Lemma 1.5.1 to the natural morphisms \( X_i \to X_{i+1} \), other \( s_{ij} \) are defined as compositions. Note that the limit of \( Y_i \) exists by Lemma 1.4.3. It belongs to \( C^{w \leq 0} \) since for \( Z \in C^{w \leq -1} \) we have

\[
C(Y, Z) = \lim C(Y_i, Z) = \lim \{0\} = \{0\}.
\]

\( Y \) will be our candidate for \( X^{w \geq 0} \) (cf. the proof of part II). We have composition maps \( Y_i \leq t \to X_i \to X \) which induce some \( f \in C(Y \leq t, X) \).

Now we show that \( f \) extends to a weight decomposition of \( X \) using part 1 of Remark 1.3.2. We should check that \( C(C_k[j], \text{Cone}(f)) = 0 \) for all \( k \in I \) and \( j < 0 \) (see the description of \( C^{w \leq 0} = C^{w \leq -0} \) in the proof of part III of Theorem 4.1.2). Since all \( C_k \) are compact, as in the proof of part II we obtain that \( C(C_k[j], Y) = \lim C(C_k[j], Y_i) \). Moreover, \( C(C_k[j], X_i) = C(C_k[j], X) \) for \( i > -j \). Hence it suffices to note that the direct limit of isomorphisms is an isomorphism, while a direct limit of surjections is surjective if the targets stabilize (obvious!).

III. Obviously, \( Hw \) contains \( \text{Obj} H \). Since \( Hw \) is Karoubi-closed in \( \mathcal{C} \), it also contains all retracts of objects of \( H \). Hence it suffices that any object of \( Hw \) is such a retract.

We consider the "weight resolution" of \( X \in C^{w = 0} \) (in fact, it suffices to consider first few terms). We obtain that the weight complex of \( X \) can be presented by \( \cdots \to P_1 \to P_0 \). Since it is homotopy equivalent to \( X \), we obtain that \( X \) is a retract of \( P_0 \). The assertion is proved.

2. By assertion III, the restriction of representable functors to the category of all direct sums of \( C_i \) is fully faithful on \( Ht \) (see part 4 of Theorem 4.3.2). Since \( C_i \) are compact, we can fully faithfully restrict these functors further to \( C \). So it remains to compute the categorical image of this restriction.
Since \((C, w)\) admits negative direct sums, \(C\) contains all direct sums of \(C(-, C_i)\). Since \(C_i\) are compact, these sums represent functors \(\oplus C(-, C_i)\) on \(C\). Since \(Ht\) is abelian, its image also contains all cokernels of morphisms of objects that could be presented as \(\oplus C(-, C_i)\).

It remains to note that cokernels of morphisms of objects of the type \(\oplus C(-, C_i)\) give the whole \(C_+\). This fact was mentioned in the Notation, see also Lemma 8.1. of [22]. In fact, this is very easy: every \(F : C \to Ab^{op}\) can be presented as a factor of the natural

\[ h : \sum_{i \in I, x \in F(C_i)} C_i \to F, \]

and the same could be said about the kernel of \(h\).

3. This is just the formula (19).

\[ \square \]

**Remark 4.4.3.** 1. Dualizing part 12, one obtains certain sufficient conditions for the right adjoint weight and \(t\)-structures to exist. Unfortunately, this requires "positive products" and cocompact weak cogenerators which do not usually exist.

2. Suppose now that \(C\) is endowed with a \(t\)-structure. Then if a left adjacent weight structure exists then any \(C \in C^{w=0}\) co-represents \(c \circ H^{0W}\) for a covariant exact functor \(c : Ht \to Ab\) (see (18)).

Conversely, let a set of objects \(C_i\) corepresent a set of \(c_i \circ H^{0W}\) (here \(c_i\) are necessarily exact). It is easily seen that there cannot be any non-zero morphisms of positive degrees between such \(C_i\) (i.e. \(\{C_i\}\) is negative). Now let \(t\) be non-degenerate, \(C_i\) be compact, let the set \(c_i\) be conservative on \(Ht\). Then to any \(C\) that contains all direct sums and limits mentioned in part (iii) of Definition 4.4.1 we can apply Theorem 4.4.2.

Unfortunately, it seems that no "classical" version of the Brown representability condition yields the existence of \(C_i\) that induce a conservative family of \(c_i\) (in the general case). For example, this is the reason why \(DM^{eff}\) doesn't have a "Gersten" weight structure (see Remark 7.1.1 below). The problem here is similar to those mentioned in part 1 of this remark. Yet for \(C = SH\) one could take \(C = S^0\) (see below); for the derived category of modules over a ring \(R\) (resp. coherent sheaves over a scheme \(S\)) one could take \(C = R\) (resp. \(C = \mathcal{O}_S\)).

Moreover, the standard Brown representability condition is sufficient for the right adjacent weight structure to exist.
4.5 The 'spherical' weight structure for the stable homotopy category

We consider the stable homotopy category $SH$. The objects of $SH$ are called spectra. $SH$ contains the sphere spectrum $S^0$ that weakly generates it.

The groups $A_i = SH(S^0[i], S^0)$ are called the stable homotopy groups of spheres. We have $A_i = 0$ for $i < 0$, $A_i = \mathbb{Z}$ for $i = 0$; $A_i$ are finite for $i > 0$. For an arbitrary $A \in \text{Obj} SH$ the groups $SH(S^0[i], A)$ are called the homotopy groups of $A$ (they are denoted by $\pi_i(A)$).

The category $SH_{\text{fin}} \subset SH$ of finite spectra was defined in Corollary 4.1.3. We will also consider the category $SH_{q\text{fin}} \subset SH$ of quasi-finite spectra. Its objects are described by the following conditions; all $\pi_i(A)$ are finitely generated and $\pi_i(A) = 0$ for all $i > j$ for some $j \in \mathbb{Z}$. Lastly, we will also mention the full subcategory $SH^- \subset SH$ whose objects are spectra with homotopy groups that are zero for $i > j$ (for some $j$ that depends on the spectrum chosen). Obviously, all categories mentioned are triangulated subcategories of $SH$.

We see that $SH_{\text{fin}}$ and $SH_{q\text{fin}}$ satisfy the conditions of part III (version (i)) of Theorem 4.1.2 if we take $H = H'$ equal to the category of finite direct sums of $S^0$ and $c = \omega$. Indeed, in this case we only need finite sums and their properties which are valid for arbitrary $C$.

Hence we obtain certain non-degenerate weight structures on $SH_{\text{fin}} \subset SH_{q\text{fin}}$. It is bounded above for $SH_{q\text{fin}}$, whence $SH_{\text{fin}}$ is bounded since it is generated by $S^0$. Note that $S^0$ are compact hence all objects of $H'$ also are. Hence using part III version (ii) of Theorem 4.1.2 we can extend $w$ to $SH^-$. 

Now we describe the hearts of weight structures obtained. Since $SH(S^0, S^0) = \mathbb{Z}$, we obtain that $H' \cong Ab_{\text{fin}, fr}$ (the category of finitely generated free abelian groups); note that $H = H'$ in this case. Since $H$ is idempotent complete, part III (i) of Theorem 4.1.2 implies that $Hw_{SH_{\text{fin}}} = Hw_{SH_{q\text{fin}}} \cong Ab_{\text{fin}, fr}$.

In $SH^-$ we have $H \cong Ab_{fr}$ (the category of all free abelian groups). Since $Ab_{fr}$ is idempotent complete, we obtain $Hw_{SH^-} \cong Ab_{fr}$.

Now note that $SH$ admits countable (and also, in fact, arbitrary) direct sums. Hence by part I2 of Theorem 4.4.2 we can extend $w$ to the whole $SH$. This certainly means that $Hw_{SH} \cong Ab_{fr}$. Hence the functor $t$ is "strong" for all categories of spectra mentioned, see part 1 of Remark 3.3.4. Besides we obtain a certain "weight filtration" on homotopy groups of spectra. By definition, it is trivial (i.e. "canonical") on the homotopy of $S^0$.

Now we describe the connection of the weight complex functor for this weight structure with singular homology and cohomology of spectra.

To this end we recall that $SH$ supports a non-degenerate Postnikov $t$-
structure \( t_{\text{post}} \); the corresponding cohomology functor is given by \( SH(S^0, -) \). We obtain that \( SH^{-w \leq 0} = SH^{t_{\text{post}} \leq 0} \). Hence \( t_{\text{post}} \) is exactly the \( t \)-structure described in part II of Theorem 4.4.2. Besides by part 5 of Theorem 4.3.2, any Eilenberg-MacLane spectrum belongs to \( SH^{t_{\text{post}} = 0} \). Recall that singular cohomology of spectra are represented by the Eilenberg-MacLane spectrum \( HZ \) that corresponds to \( \mathbb{Z} \), while the singular homology of \( X \) can be calculated as \( SH(S^0, HZ \wedge X) \); we will denote it by \( H^i_{\text{sing}}(X) \).

We identify \( Hw = H \) with \( Ab_{fr} \) using the functor \( H(S^0, -) \).

**Proposition 4.5.1.** 1. \( H^i(t(X)) \cong H^i_{\text{sing}}(X) \).
2. \( H^0(\text{Ab}(X^{-i}, \mathbb{Z})) \cong H^0(X) \).

**Proof.** 1. We apply Theorem 2.3.1 to the functor \( H^i_{\text{sing}} \). We have \( E^q_2 = H^q_{\text{sing}}(X^p) \) while each \( X^p \) is a (possibly, infinite) direct sum of copies of \( S^0 \). Now, the only non-zero homology group of \( S^0 \) is \( \mathbb{Z} \) placed in dimension 0; the functor \( Y \rightarrow HZ \wedge Y \) commutes with (arbitrary) direct limits and sums.

Hence the spectral sequence \( T(H^i_{\text{sing}}, X) \) degenerates to the weight complex of \( X \). By the convergence condition II(ii) of loc. cit. we have \( T(H^i_{\text{sing}}, X) \implies H^i_{\text{sing}}(X) \).

2. Part II3 of Theorem 4.4.2 calculates \( SH(X, Y) \) for any Eilenberg-MacLane spectrum \( Y \). In particular, taking \( Y = HZ \) we obtain the claim.

**Remark 4.5.2.** 1. Alternatively, if we take an Eilenberg-MacLane spectrum \( HI \) corresponding to some injective group \( I \) instead, we will get \( SH(X, HI) = \text{Ab}(H^0(t(X)), I) \).

2. Note also that \( S^0[-1]^{t \geq 1} \) is exactly \( HZ[-1] \). Hence \( HZ[-1] \) could be obtained by applying the construction described in the proof of Theorem 4.4.2 to \( S^0 \).

3. The proof of part 1 of Proposition 4.5.1 shows that the weight filtration given by the spherical weight structure on singular homology coincides with the canonical filtration. This is not the case for homotopy groups of spectra.

5 *Idempotent completions; \( K_0 \) of categories with bounded weight structures*

In §5.1 we recall that an idempotent completion of a triangulated category is triangulated. In §5.2 we prove that a bounded \( C \) is idempotent complete iff \( Hw \) is; in general, the idempotent completion of a bounded \( C \) has a weight structure whose heart is the idempotent completion of \( Hw \).
In §5.3 we prove that if $C$ is bounded and idempotent complete then the embedding $Hw \to C$ induces an isomorphism $K_0(C) \cong K_0(Hw)$. It is a ring isomorphism if $Hw \subset C$ are endowed with compatible tensor structures. In §5.4 we study a certain Grothendieck group of endomorphisms in $C$. Unfortunately, it is not always isomorphic to $K_0(\text{End} \ Hw)$; yet it is if $Hw$ is regular; see Definition 5.4.2. Besides, we can still say something about it in other cases. In particular, this allows us to generalize Theorem 3.3 of [7] to arbitrary endomorphisms of motives (in Corollary 5.4.6); see also §8.4 of [9].

In 5.5 we calculate explicitly the groups $K_0(\text{SH}_{fin})$ and $K_0(\text{End} \, \text{SH}_{fin})$. It turns out that the classes of $[X]$ and $[g : X \to X]$ are easily recovered from the rational singular homology of $X$; see Proposition 5.5.1. More generally, one could calculate certain groups $K_0(\text{End}^{n} \, \text{SH}_{fin})$ for $n \in \mathbb{N}$ in a similar way, see Remarks 5.5.2 and 5.4.7.

5.1 Idempotent completions: reminder

We recall that an additive category $A$ is said to be idempotent complete if for any $X \in \text{Obj} \, A$ and any idempotent $p \in A(X, X)$ there exists an image of $p$ in $A$.

Any additive $A$ can be canonically idempotent completed. Its idempotent completion is (by definition) the category $A'$ whose objects are $(X, p)$ for $X \in \text{Obj} \, A$ and $p \in A(X, X) : p^2 = p$; we define

$$A'((X, p), (X', p')) = \{ f \in A(X, X') : p' f = f p = f \}.$$  

It can be easily checked that this category is additive and idempotent complete, and for any idempotent complete $B \supset A$ we have a natural unique embedding $A' \to B$.

The main result of [3] (Theorem 1.5) states that an idempotent completion of a triangulated category $C$ has a natural triangulation (with distinguished triangles being all direct summands of distinguished triangles of $C$).

In this section $C'$ will denote the idempotent completion of $C$, $Hw'$ will denote the idempotent completion of $Hw$.

Note that if $C$ is idempotent complete then $Hw$ is also, since $Hw \subset C$ and $Hw$ is Karoubi-closed.

5.2 Idempotent completion of a triangulated category with a weight structure

We prove that $C^{\text{sh}}$ is idempotent complete if $Hw$ is.
Lemma 5.2.1. If $w$ is bounded, $Hw$ is idempotent complete, then $\mathcal{C}$ also is.

Proof. We prove that all $C^{[i,j]}$ are idempotent complete by induction on $j - i$. The base is: $C^{[i,i]} = C_{w=0}^{[i]}$ is idempotent complete.

To make the inductive step it suffices to prove that $C^{[-i,1]}$ if idempotent complete if $C^{[-i,0]}$ is (for $i > 0$). For $X \in C^{[-i,1]}$ and an idempotent $p \in C(X,X)$ we consider the functor $WD$ (see part I of Theorem 3.2.2). We obtain an idempotent $q = WD(p) \in K^{[0,1]}(\mathcal{C})(WD(X),WD(X))$ whence $Y = WD(X)$ has the form $(Z \rightarrow T)$ for $Z,T \in ObjK^{[0,1]}(\mathcal{C}^{[-i,0]})$. Since $\mathcal{C}^{[-i,0]}$ is idempotent complete, $K^{[0,1]}(\mathcal{C}^{[-i,0]})$ also is by part 2 of Proposition 3.1.6. Hence there exists a $Z' \rightarrow T'$ and idempotent endomorphisms $r,s$ of $Z'$ and $T'$, respectively, such that $(Y,q)$ could be presented by the diagram

\[
\begin{array}{ccc}
Z' & \longrightarrow & T' \\
\downarrow r & & \downarrow s \\
Z' & \longrightarrow & T'
\end{array}
\]

(in $K^{[0,1]}(\mathcal{C}^{[-i,0]})$).

By Part I5 of Theorem 3.2.2, $Z',T'$ come from a certain weight decomposition of $X$. Then any corresponding weight decomposition of $p$ is homotopy equivalent to $(r,s)$. Then part I2 of Theorem 3.2.2 yields that $(r,s)$ also give a weight decomposition of $p$. Hence the object $(X,p) \in Obj\mathcal{C'}$ (see §5.1) could be presented as a cone of a certain map $(Z',r) \rightarrow (T',s)$ in $\mathcal{C'}$; whence $(Z',r),(T',s) \in Obj\mathcal{C}$ by the inductive assumption.

Now we prove that in the general (bounded) case a weight structure could be extended from $\mathcal{C}$ to its idempotent completion $\mathcal{C'}$.

Proposition 5.2.2. Let $w$ be bounded. Then the following statements are valid.

1. $w$ extends to a weight structure $w'$ for $\mathcal{C'}$.
2. The heart of $\mathcal{C'}$ equals $Hw'$ (the idempotent completion of $Hw$).

Proof. 1. By Part III of Theorem 4.1.2, we have a weight structure that extends $w$ on the subcategory $D \subset C'$ generated by $Hw'$. Hence it suffices to recall that $D$ is idempotent complete; see Lemma 5.2.1.

2. Since $Hw'$ is idempotent complete, the assertion follows from part II2 of Theorem 4.1.2.

Corollary 5.2.3. If $(\mathcal{C},w)$ is bounded and non-degenerate then $Hw'$ generates $\mathcal{C'}$.
Proof. By Proposition 5.2.2, $Hw'$ is the heart of a bounded non-degenerate weight structure for $C'$. Now Corollary 1.5.6 yields the result.

\[\square\]

Remark 5.2.4. It seems possible that the boundedness condition for $w$ is in Proposition 5.2.2 is too strong. Yet this does not seem to make much difference since in all "natural" cases either $(C, w)$ is bounded or $C$ admits countable direct sums. In the latter case $C$ is idempotent complete, see Proposition 1.6.8 of [20].

5.3 $K_0$ of a triangulated category with a bounded weight structure

We recall some standard definitions (cf. 3.2.1 of [13]). We define the Grothendieck group of an additive category $A$ as a group whose generators are of the form $[X], X \in \text{Obj} A$; the relations are $[X \oplus Y] = [X] + [Y]$ for $X, Y \in \text{Obj} A$. The $K_0$-group of a triangulated category $T$ is defined as the group whose generators are $[t], t \in \text{Obj} T$; if $A \to B \to C \to A[1]$ is a distinguished triangle then $[B] = [A] + [C]$. Note that $X \oplus 0 \cong X$ implies that $[X] = [Y]$ if $X \cong Y$ in $A$ or in $T$.

For an additive $A$ we define $K_0(K^b_w(A))$ similarly to $K_0(K^b(A))$; hence it equals $K_0(K^b(A))$ (see Definition 3.1.4).

The existence of a bounded $w$ allows to calculate $K_0(C)$ easily.

Theorem 5.3.1. Let $(C, w)$ be bounded, let $Hw$ be idempotent complete. Then the inclusion $i : Hw \to C$ induces an isomorphism $K_0(Hw) \to K_0(C)$.

Proof. Since $t$ is an weakly exact functor (see Definition 3.1.4), it gives an abelian group homomorphism $a : K_0(C) \to K_0(K^b_w(Hw)) = K_0(K^b(Hw))$.

By Lemma 3 of 3.2.1 of [13], there is a natural isomorphism $b : K_0(K^b(Hw)) \to K_0(Hw)$. The embedding $Hw \to C$ gives a homomorphism $c : K_0(Hw) \to K_0(C)$. The definitions of $a, b, c$ imply immediately that $b \circ a \circ c = \text{id}_{K_0(Hw)}$.

Hence $a$ is surjective, $c$ is injective.

It remains to verify that $c$ is surjective. It follows immediately from the fact that $Hw$ generates $C$, see Corollary 1.5.6.

\[\square\]

Remark 5.3.2. Obviously, if $C$ is a tensor triangulated category then $K_0(C)$ is a ring. If the tensor structure on $C$ induces a tensor structure on $Hw$ then $K_0(Hw)$ is a ring also and $c$ is a ring isomorphism.
5.4 $K_0$ for categories of endomorphisms

Now we define various Grothendieck groups of endomorphisms of an additive category $A$. Our definitions are similar to those of [1].

**Definition 5.4.1.** 1. The generators of $K_0^{add}(\text{End} A)$ are endomorphism of objects of $A$; the relations are of the form $[g] = [f] + [h]$ if $(f, g, h)$ give an endomorphism of a split short exact sequence.

2. If $A$ is also abelian then we also consider the group $K_0^{ab}(\text{End} A)$. Its generators again are endomorphism of objects of $A$; the relations are of the form $[g] = [f] + [h]$ if $(f, g, h)$ give an endomorphism of an arbitrary short exact sequence.

3. If $A$ is triangulated then we consider the group $K_0^{tr}(\text{End} A)$.

The generators of $K_0^{tr}(\text{End} A)$ are endomorphism of objects of $A$ again; the relations are $[g] = [f] + [h]$ if $(f, g, h)$ give an endomorphism of a distinguished triangle in $A$.

Note that $K_0^{ab}(\text{End} A)$ and $K_0^{tr}(\text{End} A)$ are natural factors of $K_0^{add}(\text{End} A)$ when these groups are defined. Indeed, $K_0^{ab}(\text{End} A)$ and $K_0^{tr}(\text{End} A)$ have the same generators as $K_0^{add}(\text{End} A)$ and more relations.

Let $C$ be bounded. We provide some sufficient conditions for $K_0(\text{End} C)$ to be isomorphic to $K_0(\text{End} Hw)$. We need a notion of a regular additive category $A$. Recall that $A'_e$ is the full abelian subcategory of $A$, generated by $A$.

**Definition 5.4.2.** An additive category $A$ will be called regular if it satisfies the following conditions.

1. $A$ equals its small envelope (see part 3 of Definition 4.1.1) i.e. if $X, Y \in \text{Obj} A$, $X$ is a retract of $Y$, is then $X$ has a complement to $Y$ (in $A$).

2. Every object of $A'_e$ has a finite resolution by objects of $A$.

The most simple examples of regular categories are abelian semisimple categories and the category of finitely generated projective modules over a noetherian (commutative) local ring all of whose localizations are regular local; cf. the end of §1 of [1].

We will need the following technical statement. Let $R$ be an associative ring with a unit.

**Lemma 5.4.3.** 1. If $A$ is regular then $K_0^{add}(\text{End} A) \cong K_0^{ab}(\text{End} A'_e)$.

2. If $A$ is the category of finitely generated projective modules over $R$ then $A'_e$ is the category of all (left) modules over $R$. 

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Proof. 1. We apply the method of the proof of Proposition 5.2 of [2]. First we consider the obvious category End $Hw'$ and note that it is abelian. Next, the objects of $Hw$ become projective in $Hw'$. Hence all 3-term complexes in $Hw$ that become exact in $Hw'$ do split in $Hw$. Therefore we can define $K_0^{ad}(\text{End } Hw')$ as the Grothendieck group of an exact subcategory of End $Hw'$.

Condition 1 of Definition 5.4.2 ensures that for any short exact sequence $0 \to G' \to G \to G'' \to 0$ in $\text{End } Hw'$ if $G, G'' \in \text{End } Hw$ then $G' \in \text{End } Hw$. Lastly, condition 2 of Definition 5.4.2 easily implies that any $G \in \text{End } Hw'$ has a finite resolution by objects of $\text{End } Hw$ (again note that objects of $Hw$ become projective in $Hw'$!). Hence applying Theorem 16.12 of [23] (page 235) we obtain the result.

2. The equivalence is given by sending a functor $F$ to $F(R)$ and a module $Q/R$ to $P \to \text{Hom}_R(P, Q)$ (here $R$ is also considered as right $R$-module). Note that all $F(P)$ could be uniquely recovered from $F(R)$ since all finitely generated projective modules are direct summands of $R^m$, $m \in \mathbb{N}$.

\begin{proposition}
1. There exist natural homomorphisms $K_0(\text{End } Hw) \to K_0^b(\text{End } C)$, $d \to K_0^b(\text{End } Hw')$; $c$ is a surjection.

2. $c$ is an isomorphism if $Hw$ is regular.
\end{proposition}

Proof. 1. $c$ is induced by $i : Hw \to C$. For $g : X \to X$ we define

$$d(g) = \sum (-1)^i [g_{i*} : H^i(t(X)) \to H^i(t(X))].$$

Here $H^i(t(X)) \in \text{Obj } Hw'$ are the cohomology of the weight complex; see part 2 of Remark 3.1.5. We obtain a well-defined homomorphism since $t$ is a weakly exact functor (see Definition 3.1.4); see part 3 of Remark 3.1.5.

$c$ is surjective since for $g : X \to X$ we have the equality $[g] = \sum (-1)^i [g^i : X^i \to X^i]$. This equality follows easily from the fact that multiple application of the (single) weight decomposition functor to a morphism yields its infinite weight decomposition (see Theorem 3.2.2; note that $X$ is bounded).

2. In the case when $Hw$ is abelian semi-simple we have $Hw = Hw'$. Hence the equality $d \circ c = id_{K_0(\text{End } C)}$ yields the assertion (in this case).

Now, in the general (regular) case it suffices to apply the equality $K_0(\text{End } C) = K_0^b(\text{End } Hw')$ (this is part 1 of Lemma 5.4.3).

\begin{remark}
1. Unfortunately, $c$ is not an isomorphism in the general case. To see this it suffice to consider the example described in part 2 of Remark 1.5.2 for $C = K^b(Z)$ where $Z$ in the category of free $\mathbb{Z}/8\mathbb{Z}$-modules. This
\end{remark}
fact is also related to the observation in the end of §1 of [1]. Certainly, \( Z \) is not regular.

2. Certainly, if \( i : Hw \to \mathcal{C} \) is a tensor functor then \( c, d \) are ring homomorphisms, cf. Remark 5.3.2.

The surjectivity of \( c \) immediately implies the following fact.

**Corollary 5.4.6.** Let \( r : \mathcal{C} \to D^b(R) \) and \( s : \mathcal{C} \to D^b(S) \) be exact functors for an abelian \( R, S \); let \( r_+ : K_0(\mathcal{C}) \to K_0(D^b(R)) \) and \( s_+ : K_0(\mathcal{C}) \to K_0(D^b(S)) \) be the induced homomorphisms. Let \( u : K_0(\text{End } D^b(R)) \to K_0(\text{End } R) \) and \( v : K_0(\text{End } D^b(S)) \to K_0(\text{End } S) \) be defined as \( (g : X \to X) \to [g_+ : H^i(X) \to H^i(X)] \). Let \( T \) be an abelian group; \( x : K_0(\text{End } R) \to T \) and \( y : K_0(\text{End } S) \to T \) be group homomorphisms. Then the equality \( x \circ u \circ r_+ \circ c = y \circ v \circ s_+ \circ c \) implies \( x \circ u \circ r_+ = y \circ v \circ s_+ \).

In particular, one could take \( \mathcal{C} = DM^{eff}_{gm} \), \( Hw = Chow^{eff} \) (see §6 below), \( r, s \) be given by \( l \)-adic cohomology realizations (for two different \( l \)), \( x, y \) be given by traces of endomorphisms. It follows that the alternated sum of traces of maps induced by \( g \in DM^{eff}_{gm}(X, X) \) on the cohomology of \( X \) does not depend on \( l \). We also obtain the independence from \( l \) of \( n_\lambda(H) = (-1)^{i}n_\lambda g^i_{H^i(X)} \); here \( n_\lambda g^i_{H^i(X)} \) for a fixed algebraic \( \lambda \) denotes the algebraic multiplicity of the eigenvalue \( \lambda \) for the operator \( g^i_{H^i(X)} \).

This generalizes Theorem 3.3 of [7] to arbitrary correspondences of motives; see §8.4 of [9] for more details.

Lastly, we consider some more general \( K_0 \)-groups.

**Remark 5.4.7.** 1. For an additive \( A \) instead of \( \text{End } A \) one could for any \( n \geq 0 \) consider the category \( \text{End}^n A \) whose objects are the following \( n + 1 \)-tuples: \( (X \in \text{Obj } A; g_1, \ldots, g_n \in A(X, X)) \). We have \( \text{End}^0(A) = A \), \( \text{End}^1(A) = \text{End } A \). Generalizing Definition 5.4.1 in an obvious way one defines \( K^0_{\text{ad}}(\text{End}^n A), K^0_{\text{ab}}(\text{End}^n A), K^n_{\text{tr}}(\text{End}^n A) \) (for \( A \) additive, abelian or triangulated, respectively). Next, one can define \( c, d \) as in Proposition 5.4.4; exactly the same argument as in the proof of the Proposition shows that \( c \) is always surjective and it is also injective if \( Hw \) is regular. In particular, this is true for \( \mathcal{C} = SH_{\text{fin}} \); see Proposition 5.5.1 below.

2. Even more generally, for any ring \( R \) one could consider the category \( \text{End}(R, A) \) of \( R \)-representations in \( A \) i.e. of pairs \( (X, H : R \to A(X, X)) \); here \( X \in \text{Obj } A, H \) is a unital homomorphism of rings. In particular, we have \( \text{End}(R, A) = A \) for \( R = \mathbb{Z} \), \( \text{End } A \) for \( R = \mathbb{Z}[t] \), and \( \text{End}^n A \) for \( R = \mathbb{Z}[t_1, \ldots, t_n] \) (the algebra of non-commutative polynomials). Again one defines \( K^0_{\text{ad}}(\text{End}(R, A)), K^0_{\text{ab}}(\text{End}(R, A)), K^n_{\text{tr}}(\text{End}(R, A)) \), \( c \) and \( d \). Yet the method of the proof of Proposition 5.4.4 fails for a general \( R \); one could only note that \( d \circ c \) is an isomorphism if \( Hw \) is abelian semi-simple.

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5.5 An application: calculation of \(K_0(SH_{fin})\) and \(K_0(\text{End } SH_{fin})\)

Now we calculate explicitly the groups \(K_0(SH_{fin})\) and \(K_0^{tr}(\text{End } SH_{fin})\). The author doesn’t think that (all of) these results are new; yet they illustrate our methods very well.

We will need the following simple observation: \(K_0(A)\) is naturally a direct summand of \(K_0(\text{End } A)\) (both in the "triangulated" and in the "additive" case). The splitting is induced by \([f : X \to X] \to [X] \to [0 : X \to X]\); see §1 of [1].

We define the splitting group \(\Lambda\) as a subgroup of the multiplicative group \(\Lambda(\mathbb{Z}) = \{1 + t\mathbb{Z}[[t]]\}\) that is generated by polynomials (with constant term 1). \(\Lambda\) and \(\Lambda(\mathbb{Z})\) are also rings; see Proposition 3.4 of [2] for \(\Lambda\) and [15] for \(\Lambda(\mathbb{Z})\).

**Proposition 5.5.1.**

1. \(K_0(SH_{fin}) \cong \mathbb{Z}\) with the isomorphism sending \(X \in \text{Obj } SH_{fin}\) to \([X] = \sum (-1)^i \dim_{\mathbb{Q}}(H^i_{\text{sing}}(X) \otimes \mathbb{Q})\) (the rational singular homology of \(X\)).

2. \(K_0^{tr}(\text{End } SH_{fin}) \cong \mathbb{Z} \oplus \Lambda\) with the isomorphism sending \(g : X \to X\) to \([X] \oplus \prod_i (\det_{\mathbb{Q}[t]}(id - g_i t \otimes \mathbb{Q}))^{-1}^i;\) here \(g_i t \otimes \mathbb{Q}\) is the map induced by \(g \otimes t\) on \(H^i_{\text{sing}}(X) \otimes \mathbb{Z}[t]\).

**Proof.**

1. We have \(Hw = Ab_{fin, fr}\) for the spherical weight structure \(w\) on \(SH_{fin}\); see §4.5. Hence \(K_0(SH_{fin}) \cong K_0(Ab_{fin, fr}) = K_0(\mathbb{Z}) = \mathbb{Z}\).

The second assertion could easily be deduced from part 1 of Proposition 4.5.1. Note that \(K_0(SH_{fin})\) is a direct summand of \(K_0^{tr}(\text{End } SH_{fin})\); hence

\([X] = \sum (-1)^i [H^i(t(X))] = \sum (-1)^i [H^i_{\text{sing}}(X)]\)

by (21). We also use the fact that \(K_0(\mathbb{Z})\) injects into \(K_0(\mathbb{Q})\), so \([H^i_{\text{sing}}(X)]\) could be computed rationally.

2. By part 2 of Lemma 5.4.3 we have \(Hw_\ast \cong Ab_{fin, fr}\) (the category of finitely generated abelian groups). Hence \(Hw\) is regular (see Definition 5.4.2). Therefore by part 2 of Proposition 5.4.1 we have \(K_0^{tr}(\text{End } SH_{fin}) \cong \mathbb{Z} \oplus \Lambda\). Then the Main Theorem in §1 of [1] implies that \(K_0^{tr}(\text{End } SH_{fin}) \cong \mathbb{Z} \oplus \Lambda\).

Next, (21) implies \([g] = (-1)^i [g_\ast]\). Now note that \(\Lambda(\mathbb{Q}) \to \Lambda(\mathbb{Z})\) is injective; so it suffices to calculate \([g_\ast]\) rationally. Lastly, the equality

\([g_\ast \otimes \mathbb{Q}] = \dim_{\mathbb{Q}}(H^i_{\text{sing}}(X) \otimes \mathbb{Q}) + \det_{\mathbb{Q}[t]}(id - g_i t \otimes \mathbb{Q})\)

follows from the formula at the bottom of p. 376 of [1].

\(\Box\)

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Remark 5.5.2. 1. Note that the isomorphisms described are compatible with the natural ring structures of $K_0$-groups involved.

2. Assertion 1 doesn’t seem to be new; yet the author doesn’t now of any paper that contains assertion 2 in its current form.

3. One also has $K_0^0(\text{End}^n SH_{\text{fin}}) \cong K_0^{\text{adm}}(\text{End}^n Ab_{\text{fin}, fr})$; see Remark 5.4.7.

6 Twisted complexes over a negative differential graded category; Voevodsky’s motives

The goal of this section is to compare our theory with those of [9]; this will allow to apply it to motives.

In §6.1 we recall the definitions of differential graded categories and twisted complexes over them. In 6.2 we consider negative differential graded categories; we obtain a weight structure on the category of twisted complexes (over it). In §6.3 we construct the so-called truncation functors $t_N$; $t_0$ is the strong weight complex functor, see Conjecture 3.3.3. In §6.4 we recall the spectral sequence $S(H, X)$ constructed in §7 of [9] for $H$ having a differential graded enhancement and prove that it could be obtained from $T(H, X)$ by decalage. In particular, this shows that $S$ does not depend on the choice of enhancements. In §6.5 we apply our theory to Voevodsky’s motivic categories $DM^{eff}_{gm}$ and $DM_{gm}$; we calculate the heart of the corresponding Chow weight structures obtained in §6.6.

6.1 Basic definitions

We recall the basic definitions and results of the theory as they were presented in [9]; cf. also [5], [8], and [11].

Categories of twisted complexes were first considered in [8]. Yet our notation differs slightly from those of [8]; some of the signs are also different.

An additive category $C$ is called graded if for any $P, Q \in \text{Obj}C$ there is a canonical decomposition $C(P, Q) \cong \oplus_i C_i(P, Q)$ defined; this decomposition satisfies $C_i(\ast, \ast) \circ C_j(\ast, \ast) \subset C_{i+j}(\ast, \ast)$. A differential graded category (cf. [8] or [11]) is a graded category endowed with an additive operator $\delta : C_i(P, Q) \to C_{i+1}(P, Q)$ for all $i \in \mathbb{Z}, P, Q \in \text{Obj}C$. $\delta$ should satisfy the equalities $\delta^2 = 0$ (so $C(P, Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in \text{Obj}C$, $f \in C_i(P, Q)$, $g \in C(Q, R)$. In particular, $\delta(id_P) = 0$.

We denote $\delta$ restricted to morphisms of degree $i$ by $\delta^i$. 59
For an additive category $A$ one can construct the following differential graded categories.

We denote the first one by $S(A)$. We set $\text{Obj} S(A) = \text{Obj} A$; $S(A)_i(P, Q) = A(P, Q)$ for $i = 0$; $S(A)_i(P, Q) = 0$ for $i \neq 0$. We take $\delta = 0$.

We also consider the category $B^b(A)$ whose objects are the same as for $C^b(A)$ whence for $P = (P_i)$, $Q = (Q_i)$ we define $B^{-}(A)(P, Q)_i = \oplus_{j \in \mathbb{Z}} A(P_j, Q_{i+j})$. Obviously $B^b(A)$ is a graded category. $B(A)$ will denote the unbounded analogue of $B^b(A)$.

We set $\delta f = d_Q \circ f - (-1)^i f \circ d_P$, where $f \in B_i(P, Q)$, $d_P$ and $d_Q$ are the differentials in $P$ and $Q$. Note that the kernel of $\delta^0(P, Q)$ coincides with $C(A)(P, Q)$ (the morphisms of complexes); the image of $\delta^{-1}$ are the morphisms homotopic to 0.

$B^b(A)$ can be obtained from $S(A)$ by means of the category functor $\text{Pre-Tr}$ described below.

For any differential graded $C$ we define a category $H(C)$; its objects are the same as for $C$; its morphisms are defined as

$$H(C)(P, Q) = \text{Ker} \delta^0_C(P, Q)/\text{Im} \delta^1_C(P, Q).$$

Having a differential graded category $C$ one can construct another differential graded category $\text{Pre-Tr}(C)$ as well as a triangulated category $Tr(C)$. The simplest example of these constructions is $\text{Pre-Tr}(S(A)) = B^b(A)$.

**Definition 6.1.1.** The objects of $\text{Pre-Tr}(C)$ are

$$\{(P_i), \ P_i \in \text{Obj} C, i \in \mathbb{Z}, q_{ij} \in C_{i-j+1}(P_i, P_j)\};$$

here almost all $P_i$ are 0; for any $i, j \in \mathbb{Z}$ we have

$$\delta q_{ij} + \sum_l q_{il} \circ q_{lj} = 0 \quad \quad (22)$$

We call $q_{ij}$ *arrows* of degree $i - j + 1$. For $P = \{(P_i), q_{ij}\}$, $P' = \{(P'_i), q'_{ij}\}$ we set

$$\text{Pre-Tr}_l(P, P') = \bigoplus_{i, j \in \mathbb{Z}} C_{l+i-j}(P_i, P'_j).$$

For $f \in C_{l+i-j}(P_i, P'_j)$ (an arrow of degree $l + i - j$) we define the differential of the corresponding morphism in $\text{Pre-Tr}(C)$ as

$$\delta_{\text{Pre-Tr}(C)}f = \delta_C f + \sum_m (q'_{jm} \circ f - (-1)^{(i-m)}f \circ q_{jm}).$$
It can be easily seen that $\text{Pre-Tr}(C)$ is a differential graded category (see [8]). There is also an obvious translation functor on $\text{Pre-Tr}(C)$. Note also that the terms of the complex $\text{Pre-Tr}(C)(P, P')$ do not depend on $q_{ij}$ and $q'_{ij}$ whence the differentials certainly do.

We denote by $Q[j]$ the object of $\text{Pre-Tr}(C)$ that is obtained by putting $P_i = Q$ for $i = -j$, all other $P_j = 0$, all $q_{ij} = 0$. We will write $[Q]$ instead of $Q[0]$.

Immediately from the definition we have $\text{Pre-Tr}(S(A)) \cong B^b(A)$.

A morphism $h \in \text{Ker} \delta^0$ (a closed morphism of degree 0) is called a twisted morphism. For a twisted morphism $h = (h_{ij}) \in \text{Pre-Tr}((P_i, q_{ij}), (P'_i, q'_{ij}))$, $h_{ij} \in C(P_i, P'_j)$ we define $\text{Cone}(h) = P''_i, q''_{ij}$, where $P''_i = P_{i+1} \oplus P'_i$,

$$q''_{ij} = \begin{pmatrix} q_{i+1,j+1} \\ h_{i+1,j} \\ q'_{ij} \end{pmatrix}$$

We have a natural triangle of twisted morphisms

$$P \xrightarrow{f} P' \xrightarrow{\text{Cone}(f)} P[1],$$

the components of the second map are $(0, \text{id}_{P'})$ for $i = j$ and 0 otherwise. This triangle induces a triangle in the category $H(\text{Pre-Tr}(C))$.

**Definition 6.1.2.** For distinguished triangles in $Tr(C)$ we take the triangles isomorphic to those that come from the diagram (23) for $P, P' \in \text{Pre-Tr}(C)$.

We summarize the properties of $\text{Pre-Tr}$ and $Tr$ of [8] that are most relevant for the current paper. See [9] and [8] for the proofs.

**Proposition 6.1.3.** I $Tr(C)$ is a triangulated category.

II For any additive category $A$ there are natural isomorphisms

1. $\text{Pre-Tr}(B(A)) \cong B(A)$.
2. $Tr(B(A)) \cong K(A)$.
3. $Tr(S(A)) \cong B^b(A)$

III 1. There are natural embeddings of categories $i : C \to \text{Pre-Tr}(C)$ and $H(C) \to \text{Tr}(C)$ sending $P$ to $[P]$.

2. $\text{Pre-Tr}$ and $Tr$ are functors on the category of differential graded categories i.e. any differential category functor $F : C \to C'$ naturally induces functors $\text{Pre-Tr}F$ and $TrF$.

3. Let $F : \text{Pre-Tr}(C) \to D$ be a differential graded functor. Then the restriction of $F$ to $C \subset \text{Pre-Tr}(C)$ gives a differential graded functor $FC : C \to D$. Moreover, since $FC = F \circ i$, we have $\text{Pre-Tr}(FC) = \text{Pre-Tr}(F) \circ \text{Pre-Tr}(i)$; therefore $\text{Pre-Tr}(FC) \cong \text{Pre-Tr}(F)$. 

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For example, for \( X = (P_i, q_{ij}) \in \text{Obj} \text{Pre-Tr}(C) \) we have \( \text{Pre-Tr}F(X) = (F(P_i), F(q_{ij})) \); for a morphism \( h = (h_{ij}) \) of \( \text{Pre-Tr}(C) \) we have \( \text{Pre-Tr}F(h) = (F(h_{ij})) \). Note that the definition of \( \text{Pre-Tr}F \) on morphisms does not involve \( q_{ij} \); yet \( \text{Pre-Tr}F \) certainly respects differentials for morphisms.

**Remark 6.1.4.** 1. By definition, any morphism \( g : A = (P_i, f_{ij}) \to B = (P'_i, f'_{ij}) \) can be described as sets \((g_{ij}) \in \text{C}_{i-j}(P'_i)(P_i), i, j \in \mathbb{Z}\), where \( g_{ij} \) satisfy

\[
\delta_C g_{ij-1} + \sum_m (f'_{mj} \circ g_{im} - g_{mj} \circ f_{im}) = 0 \quad \forall i, j \in \mathbb{Z}. \tag{24}
\]

Moreover, two sets of \( g_{ij} \) give the same morphism whenever they are 'homotopy equivalent' i.e. there exist \( h_{ij} \in \text{C}_{i-j-1}(P'_j)(P_i), i \leq j \), such that

\[
\delta_C h_{ij-1} + \sum_m (f'_{mj} \circ h_{im} - (-1)^{i-m} h_{mj} \circ f_{im}) = g_{ij} \quad \forall i, j \in \mathbb{Z}. 
\]

2. In [9], following [8] and [11], the author also defined a certain differential graded subcategory \( \text{Pre-Tr}^+(C) \subset \text{Pre-Tr}(C) \). \( \text{Tr}^+(C) \subset \text{Tr}(C) \) was defined as \( H(\text{Pre-Tr}^+(C)) \). The main virtue of \( \text{Tr}^+(C) \) is that it is always generated by the image of the natural map \( \text{Obj} \text{C} \to \text{Obj} \text{Tr}^+(C) : P \to [P] \). We don’t need this definition here since in the case when \( C \) is negative (see below) we have \( \text{Pre-Tr}^+(C) = \text{Pre-Tr}(C) \).

### 6.2 Negative differential graded categories; a weight structure for \( \text{Tr}(C) \)

Suppose now that a differential graded category \( C \) is negative i.e. for any \( X, Y \in \text{Obj} \text{C} \) we have \( i > 0 \implies C^i(X, Y) = 0 \) (cf. Definition 4.1.1).

For \( C = \text{Tr}(C) \) we define \( C^{w \leq 0} \) as a set that contains all objects that are isomorphic to those that satisfy \( P_i = 0 \) for \( i > 0 \). \( C^{w \geq 0} \) is defined similarly by the condition \( P_i = 0 \) for \( i < 0 \).

**Proposition 6.2.1.** 1. \( C^{w \leq 0} \) and \( C^{w \geq 0} \) give a non-degenerate weight structure for \( C \).

2. \( Hw \) is isomorphic to the small envelope of \( HC \) (cf. Definition 4.1.1).

**Proof.** The definition of morphisms in \( C \) immediately yields that \( C(C^{w \geq 0}, C^{w \leq 0}) = 0 \). The verification of the fact that \( C^{w \leq 0} \) and \( C^{w \geq 0} \) are Karoubi-closed in \( C \) is straightforward; yet we will never actually use this statement below.

It remains to check that any object \( X \) of \( C \) admits a weight decomposition. We follow the proof of Proposition 2.6.1 of [9].

We take \( (P_i, f_{ij}, i, j \leq 0) \) as \( X^{w \leq 0} \) and \( (P_i, f_{ij}, i, j \geq 1)[1] \) as \( X^{w \geq 1} \). We should verify that \( X^{w \leq 0} \) and \( X^{w \geq 1} \) are objects of \( C \).
We have to check that the equality (22) is valid for $X^{w \leq 0}$ (resp. $X^{w \geq 1}$). Yet all terms of (22) are zero unless $i \leq j \leq 0$ (resp. $1 \leq i \leq j$). Moreover, in the case $i \leq j \leq 0$ (resp. $1 \leq i \leq j$) the terms of (22) are the same as for $X$. Both of these facts follow immediately from the negativity of $C$.

Now we verify that $(id_{P_i}, i \leq 0)$ gives a morphism $X \to X^{w \leq 0}$ and $(id_{P_i}, i \leq 1)$ gives a morphism $X^{w \geq 1}[-1] \to X$. The condition (24) for these cases is obvious by the negativity of $C$.

Next we should check that $X \to X^{w \leq 0}$ is the second morphism of the triangle corresponding to $X^{w \leq 1}[1] \to X$; this easily follows from (23).

Lastly, $w$ is non-degenerate since any object of $C$ is bounded from both sides (in the obvious sense).

2. Obviously, the objects of $HC$ belong to $C^{w=0}$. Next, the definition of $C$ easily yields that $Hw(X, Y) \cong HC(X, Y)$ for $X, Y \in HC$.

Moreover, part 1 implies that any object of $C$ has a "filtration" by sub-objects whose "successive factors" come from $HC$. By part II2 of Theorem 4.1.2 we obtain that $Hw$ is isomorphic to the small envelope of $HC$.  

Obviously, the same construction also gives weight structures for all unbounded versions of $Tr(C)$.

Remark 6.2.2. Alternatively, Proposition 6.2.1 could be deduced from part II of Theorem 4.1.2. In particular, this method easily deduces the fact that $C^{w \leq 0}$ and $C^{w \geq 0}$ are Karoubi-closed from the assertion that the small envelope of $HC$ lies in both of them (cf. the beginning of the proof of Part II2 loc. cit.).

6.3 Truncation functors; comparison of weight complexes

For $N \geq 0$, $P, Q \in ObjC$ we denote the $-N$-th canonical filtration of $C(P, Q)$ (i.e. $C_{-N}(P, Q)/d_P C_{-N-1}(P, Q) \to C_{-N+1}(P, Q) \to \cdots \to C_0(P, Q) \to 0$) by $C^N(P, Q)$.

We denote by $C_N$ the following differential graded category. Its objects are the same as for $C$ whence $C_N(P, Q)_i = C^N_i(P, Q)$. The composition of morphisms is induced by those in $C$. For morphisms in $C_N$ presented by $g \in C_i(P, Q)$, $h \in C_j(Q, R)$, we define their composition as the morphism represented by $h \circ g$ for $i + j \geq -N$ and zero for $i + j < -N$. Note that for $i + j = -N$ we take the class of $h \circ g \mod d_R C_{-N-1}(P, R)$; for $i = -N$, $j = 0$, and vice versa, $g$ is only defined up to an element of $d_Q C_{-N-1}(P, Q)$ (resp. $h$ is defined up to an element of $d_R C_{-N-1}(Q, R)$) yet the composition is well-defined. Certainly, all $C_N$ are negative (i.e. there are no morphisms of degree $> 0$).
We have an obvious functor $C \to C_N$. As noted in Proposition 6.1.3, this gives canonically a functor $t_N : C \to Tr(C_N)$. We denote $Tr(C_N)$ by $\underline{C}_N$; note that $\underline{C}_0$ is precisely $K^b(HC)$. For any $m \leq N$ we also have an obvious functor $C_N \to C_m$. It induces a functor $t_{Nm} : \underline{C}_N \to \underline{C}_m$ such that $t_m = t_{Nm} \circ t_N$.

Obviously, objects of $\mathcal{N}_N$ could be represented as certain $(P_i, f_{ij} \in C_{i-j+1}^{N}(P_i, P_j), i < j \leq i + N + 1)$, the morphisms between $(P_i, f_{ij})$ and $(P'_i, f'_{ij})$ are represented by certain $g_{ij} \in C_{i-j}^{N}(P_i, P_j), i \leq j \leq i + N$, etc. The functor $t_N$ 'forgets' all elements of $C_m([P], [Q])$ for $P, Q \in SmPrVar$, $m < -N$, and factorizes $C_{-N}([P], [Q])$ modulo coboundaries. In particular, for $N = 0$ we get ordinary complexes over $HC$.

$t_0$ will be called the strong weight complex.

One could easily verify that the strong weight complex functor constructed is a lift of the weight complex functor $t$ corresponding to the weight structure $w$ to an exact functor $t^w$ (as in Conjecture 3.3.3). This follows immediately from the explicit description of $X^w \leq 0$ and $X^w \geq 1$ for any $X \in Obj\underline{C}$ (in the proof of Proposition 6.2.1).

### 6.4 The weight spectral sequence for enhanced realizations

The method of construction of the weight spectral sequences in [9] was somewhat distinct from the method we use here. In [9] we used a certain filtration on the complex that computes cohomology; that filtration could be obtained from the filtration corresponding to our current method by Deligne’s décalage (see §1.3 of [10]). So the spectral sequence there was ‘shifted one level down’ (in particular, it was functorial starting from $E_1$). We compare the methods here.

Let $J$ be some negative differential graded category, let $\mathcal{F} = Tr(J)$, $J' = \text{Pre-}Tr(J)$. Below we will use the same notation for Voevodsky’s motives (which are the most important example of this situation).

In [9] weights were constructed only for (co)homological functors that admit an enhancement i.e. those that could be factorized through $Tr(F)$ for a differential graded functor $F : J \to C$. Here we consider only $C = B(A)$ for an abelian $A$ and homological functors of the form $H_{K_A} \circ Tr(F)$ (here $H_{K_A}$ denotes the zeroth cohomology functor for $C(A)$). The cohomological functor case was considered in §7.3 of [9] (certainly, reversing the arrows is no problem). Note for those realization for which $C \neq B(A)$ one can reduce the situation to the case $C = B(A')$ for a large $A'$ (for example, for the Hodge realization, see §7.3 of [9]). Alternatively, one could apply décalage
to the filtration of (Hodge) complexes corresponding to our method.

Now we recall the formalism of [9] (modified for the homological functor case).

We denote the functor $\text{Pre-Tr}(F) : J' \to B(A)$ by $G$, denote $\text{Tr}(F) : \mathcal{F} \to K(A)$ by $E$.

We recall that for a complex $Z$ over $A$, $b \in \mathbb{Z}$, its $b$-th canonical truncation from above is the complex \ldots $Z_{b-1} \to \text{Ker}(Z_b \to Z_{b+1})$, here $\text{Ker}(Z_b \to Z_{b+1})$ is put in degree $b$.

For any $b \geq a \in \mathbb{Z}$ we consider the following functors. By $F_{\tau \leq b}$ we denote the functor that sends $[P]$ to $\tau_{\leq b}(F([P]))$. These functors are differential graded; hence they extend to $G_{\tau} = \text{Pre-Tr}(F_{\tau \leq b}) : J' \to B(A)$. Note that we consider the $-i$-th filtration here in order to make the filtration decreasing (which is usual when the decalage is applied); this is another minor distinction of the current exposition from those of [9].

Let $X = (P_i, q_{ij}) \in \text{Obj}J'$. The complexes $G_b(X)$ give a filtration of $G(X)$; one could also consider $G_{a,b}(X) = G_b(X)/G_{a-1}(X)$.

Let $X = (P_i, q_{ij}) \in \text{Obj}J' = \text{Obj}\mathcal{F}$. We obtain the spectral sequence of a filtered complex

$$S : E_0^{ij}(S) = F_i([P_i]) \to H^{i+j}(G(X)).$$

Note that $H^{i+j}(G(X)) = H^{i+j}(E(X))$, in the right hand side we consider $X$ as an object of $\mathcal{F}$.

All $G_b(X)$ are $J'$-contravariantly functorial with respect to $X$. Besides, starting from $E_1$ the terms of $S$ depend only on the homotopy classes of $G_b(X)$. Hence starting from $E_1$ the terms of $S$ are functorial with respect to $X$ (considered as an object of $\mathcal{F}$).

Now we compare the spectral sequences obtained using this method with the ones provided by Theorem 2.4.1.

To this end we compare the filtrations of $G(X)$ corresponding to $T$ and $S$. Fortunately, we don’t have to write down the differential in $G$; it suffices to recall that $G_j(X) = \bigoplus_{k+l=j} F_k(P_l)$.

The method of Theorem 2.4.1 gives the following filtration on $G(X)$:

$$Q_iG_j(X) = \bigoplus_{k+l=j, l \geq i} F_k(P_l).$$

Now we apply decalage to this filtration. It is easily seen that we obtain the filtration given by $G_i$ i.e.

$$(\text{Dec}Q)_i(G_j(X)) = \bigoplus_{k+l=j, l \geq j+i+1} F_k(P_l) \oplus \text{Ker}(F_{-b}(P_{i+j}) \to F_{-b+1}(P_{i+j+1})).$$

Hence $T^{pq}_{n+1} = S^{-q,p+2q}_n$ for all integral $i, j$ and $n > 0$; the corresponding filtrations on the limit (i.e. on $H^{i+j}(E(X))$) coincide up to a certain shift of indices.

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Lastly we note that the formula (15) of [9] computes all $E^n_{ij}(S)$ for $n \geq 1$ in terms of the weight filtration of so-called truncated realizations of $X$; this description is $\mathcal{F}$-functorial.

6.5 \textbf{SmCor}, $J$, $\mathcal{F}$, $DM_{gm}^{eff}$ and $DM_{gm}$; "Chow" weight structure

We recall some definitions of [25].

$k$ will denote the ground field; we will mostly assume that the characteristic of $k$ is zero. $pt$ is a point, $\mathbb{A}^n$ is the $n$-dimensional affine space (over $k$), $x_1, \ldots, x_n$ are the coordinates, $\mathbb{P}^n$ is the projective space of dimension $n$.

$Var \supset SmVar \supset SmPrVar$ will denote the class of all varieties over $k$, resp. of smooth varieties, resp. of smooth projective varieties.

We define the category of smooth correspondences: $\text{ObjSmCor} = SmVar$, $SmCor(X, Y) = \sum_U \mathbb{Z}$ for all $U \subset X \times Y$ that are integral closed finite sub-schemes which are dominant (over a connected component of) $X$.

$SmCor$ is additive: the addition of objects is given by the disjoint union operation for varieties.

$Shv(\text{SmCor})$ is the abelian category of additive cofunctors $\text{SmCor} \to Ab$ that are sheaves in the Nisnevich topology.

$DM_{gm}^{eff} \subset D^-(Shv(\text{SmCor}))$ is defined as the subcategory defined by the condition that the cohomology sheaves are homotopy invariant (i.e. $S(X) \cong S(X \times \mathbb{A}^1)$ for any $S \in SmVar$).

There is a natural functor $RC \circ L : K^b(\text{SmCor}) \to DM_{gm}^{eff}$ (cf. Theorem 3.2.6 of [25]) given by Suslin complexes (see below); that could be factorized as a composition of the 'localization by homotopy invariance and Mayer-Vietoris' and a full embedding; it categorical image will be denoted by $DM^*$. $DM_{gm}^{eff}$ is idempotent complete; hence it contains the idempotent completion of $DM^*$ which is Voevodsky’s $DM_{gm}^{eff}$ (by definition; see [25]).

Now we define a differential category $J$ with $\text{Obj}J = SmPrVar$ (the addition of objects is the same as for $SmCor$. The morphisms of $J$ are given by cubical Suslin complexes $J_i(Y, P) \subset SmCor(\mathbb{A}^{-i} \times Y, P)$ consisting of correspondences that are zero if one of the coordinates is zero'. Being more precise, we define $C'_i(P, Y) = SmCor(\mathbb{A}^{-i} \times Y, P)$ for all $P, Y \in SmVar$; note that $C'_i$ are zero for positive $i$. For all $1 \leq j \leq -i$, $x \in k$, we define $d_{ijx} = d_{jx} : C'_i \to C'_{i+1}$ as $d_{jx}(f) = f \circ g_{jx}$, where $g_{jx} : \mathbb{A}^{-i-j} \times Y \to \mathbb{A}^{-i} \times Y$ is induced by the map $(x_1, \ldots, x_{-1-i}) \to (x_1, \ldots, x_{j-1}, x, x_{j}, \ldots, x_{-1-i})$. We define $J_i(Y, P)$ as $\cap_{1 \leq j \leq -i} \text{Ker} d_{j0}$. The boundary maps $\delta^i : J_i(-, -) \to J_{i+1}(-, -)$ are defined as $\sum_{1 \leq j \leq -i} (-1)^j d_{j1}$.

The composition of morphisms in $J$ is induced by the obvious composition.
$C'_i(Y \times \mathbb{A}^{-j}, X \times \mathbb{A}^{-j}) \times C'_j(Z, Y) \rightarrow C'_{i+j}(Z, X)$ combined with the embedding of $C'_i(Y, X)$ into $C'_i(Y \times \mathbb{A}^{-j}, X \times \mathbb{A}^{-j}$ via 'tensoring' its elements by $id_{\mathbb{A}^{-j}}$; here $X, Y, Z \in SmPrVar$, $i, j \leq 0$.

It was checked in §2 of [9] that $J$ is a differential graded category. It is negative by definition.

We denote $Tr(J)$ by $\mathcal{H}$. $\mathcal{H}$ is equivalent to $DM^s$ by Theorem 3.1.1 of [9].

By Proposition 6.2.1 we obtain, that there exists a weight structure $w$ in $\mathcal{H}$; hence it also gives a weight structure for $DM^s$. We have $Hw = J'_0$ where $J'_0$ is the small envelope of $J_0 = HJ$ (cf. Definition 4.1.1 and part II2 of Theorem 4.1.2).

In $DM^s_{gm}$ we have a decomposition $[P^1] = [pt] \oplus \mathbb{Z}(1)[2]$ for $\mathbb{Z}(1)$ being the Tate motif. Moreover, $DM^s_{gm}$ is a tensor category with $\otimes \mathbb{Z}(1)$ being a full embedding of $DM^s_{gm}$ into itself (the Cancellation Theorem, see [25] and [27]). Hence one could define Voevodsky’s $DM_{gm}$ as the direct limit of $DM^s_{gm}$ with respect to tensoring by $\mathbb{Z}(1)$; it also could be described as the ‘union’ of $DM^s_{gm}(-i)$ (whence each $DM^s_{gm}(-i)$ is isomorphic to $DM^{eff}_{gm}$).

**Proposition 6.5.1.** $w$ extends to a weight structure for $DM^{eff}_{gm}$ and $DM_{gm}$.

**Proof.** I Extending $w$ to $DM^{eff}_{gm}$.

We define $DM^{eff}_{gm} w \leq 0$ as the set of retracts of $DM^s w \leq 0$ in $DM^{eff}_{gm}$; the same for $DM^{eff}_{gm} w \geq 0$. By Proposition 5.2.2, this gives a weight structure on $DM^{eff}_{gm}$.

II Extending $w$ to $DM_{gm}$.

We not that $\otimes \mathbb{Z}(1)[2]$ sends $[P]$ to a retract of $[P \times \mathbb{P}^1]$. Hence $\otimes \mathbb{Z}(1)$ maps $DM^{eff}_{gm} w \leq 0$ and $DM^{eff}_{gm} w \geq 0$ into themselves. It follows that one can define $DM^w_{gm} w \leq 0$ and $DM^w_{gm} w \geq 0$ as $\cup DM^{eff}_{gm} w \leq 0(-i)[-2i]$ and $DM^{eff}_{gm} w \geq 0(-i)[-2i]$ respectively. Indeed, the Cancellation Theorem gives us orthogonality; since each object of $DM_{gm}$ belongs to $DM_{gm}^{eff}(-i) = DM_{gm}^{eff}(-i)[2i]$ for some $i \in \mathbb{Z}$, we also have the weight decomposition property.

We call the weight structure constructed the Chow weight structure (for any of $\mathcal{H}, DM^s, DM^{eff}_{gm}$, $DM_{gm}$, and also for $DM^{eff}$ considered below).

Note that the same arguments prove the existence of weight structures on rational hulls of $DM^s$, $DM^{eff}_{gm}$ and $DM_{gm}$ (i.e. we tensor the groups of morphisms by $\mathbb{Q}$) as well as on their idempotent completions (which do not coincide with $DM^{eff}_{gm} \otimes \mathbb{Q}$ and $DM_{gm} \otimes \mathbb{Q}$).

### 6.6 The heart of the Chow weight structure

Now we calculate the hearts of $w$ in each of the categories constructed.
For our choice of $J$ we have $\text{Obj} J = [P], P \in SmPrVar$, while $J_0([P], [Q]) = DM^{eff}_{gm}([P], [Q]) = \text{Chow}([P], [Q])$ (cf. section §4.2 of [25]). Hence the heart of $DM^s$ is the small envelope of the category $\text{Corr}_{rat}$ of effective rational correspondences (see Definition 4.1.1 and part II of Theorem 4.1.2). Note that the small envelope of $\text{Corr}_{rat}$ contains $\mathbb{Z}(1)[2]$ while $\text{Corr}_{rat}$ does not. Now Proposition 5.2.2 implies that the heart of $DM^{eff}_{gm}$ is the idempotent completion of $\text{Corr}_{rat}$ i.e. the whole category $\text{Chow}^{eff}$. Lastly, this easily implies that the heart of $DM^{gm} \cong \text{Chow}^{eff}$.

We obtain that for any (co)homological functor of $DM^{eff}_{gm}$ (or $DM^{eff}_{gm}$) there exist weight spectral sequences and weight filtrations. Note that we don't need any enhancements here (as we did in [9])! Moreover, the weight spectral sequences are functorial with respect to natural transformations of (co)homological functors (we don't need transformations for enhancements).

Lastly we recall (from [9]) that the results obtained also concern motivic cohomology.

## 7 New facts on motives

The first subsection is dedicated to the study of $DM^{eff}$. We prove that the Chow weight structure extends to it; moreover $DM^{eff}$ supports a (right) adjacent Chow $t$-structure. In §7.2 we note that (any) "mixed motivic" $t$-structure induces a canonical "weight filtration" on the values of the corresponding homological functor $DM^{eff}_{gm} \to MM$. In §7.3 we prove that a certain (possibly, "infinite") weight complex functor could be defined for motives over any perfect field (without any resolution of singularities assumptions).

### 7.1 "Chow" weight and $t$-structures for $DM^{eff}$

We recall (see §3 of [25]) that for any $S \in DM^{eff}$ and $X \in SmVar$ we have $DM^{eff}(M_{gm}(X), S) = \mathbb{H}^0(S)(X)$ (here $S$ is considered as a complex of sheaves). It follows (cf. 6.5) that $M_{gm}(X)$ for $X \in SmPrVar$ weakly generate $DM^{eff}$.

Now we take $\{C_i\} = \text{ObjChow} \subset DM^{eff}$ (we can assume that $\text{ObjChow}$ is a set). We obtain that $(DM^{eff}, \{C_i\})$ satisfy the conditions of part II of Theorem 4.4.2. Hence it has is $t$-structure whose heart is $\text{Chow}$. Unfortunately, it seems that this $t$-structure cannot be restricted to $DM^{eff}_{gm}$ (i.e. it is not "geometric").

Now we check that the Chow weight structure of $DM^{eff}_{gm}$ could be extended to $DM^{eff}$. Let $t$ denote the homotopy $t$-structure of $DM^{eff}$ (i.e. the
one given by Nisnevich hypercohomology as in [25]).

We define $DM_{\leq 0}$ as the "closure" of $DM_{gm, w \leq 0}$ in $DM_-$ with respect to arbitrary direct sums and to "taking middle terms of distinguished triangles" (as in the proof of part III of Theorem 4.1.2). Note that $DM_{\leq 0} \subset DM_{\leq 0}$. We recover $DM_{w \geq 0}$ from $DM_{w \leq 0}$ from the orthogonality condition in the usual way (see part 2 of Proposition 1.3.1). Certainly, $DM_{w \geq 0}$ also satisfies property 3 of Proposition 1.3.1. Besides, it contains arbitrary direct sums of objects of $DM_{gm, w = 0}$ (here we apply the compactness of objects of $DM_{gm}$ in $\mathcal{O}$).

As usual, the only non-trivial axiom check here is the verification of the existence of weight decompositions. Recall that any object of $\text{Shv} (\text{SmCor})$ has a "canonical resolution" by direct sums of $L(X) = \text{SmCor}(-, Y)$ for $Y \in \text{SmVar}$ (placed in degrees $\leq 0$; see §3.2 of [25]). Hence any object $X$ of $DM_{\leq 0}$ is a direct limit of certain $X_i$ for the cone of $X_i \to X_{i+1}$ being a direct sum of some $L(Y_{ij})[i]$.

We construct $X \to Y$ as a direct limit of $X_i \to Y$. By Lemma 1.4.3, it suffices to prove the following statement: for $Y \in \text{SmVar}$, we have $(M_{gm}(Y)[i])^w \geq 1 \in DM_{w \leq 0}$ (for any choice of $(M_{gm}(Y)[i])^w \geq 1$). This is easy since $(M_{gm}(Y)[i])^w \geq 1$ and $(M_{gm}(Y)[i])^w \leq 0 \in DM_{w \leq 0} < DM_{w \leq 0}$. $\text{Shv} (\text{SmCor})$ (see [25]). Unfortunately, one cannot define a weight structure that would be left adjacent to it. Yet one could define the corresponding weight structure in a certain "closure" of $DM_{gm}$ with respect to (certain) inverse limits; see Remark 4.4.3. Note that this weight structure would be closely connected with the Gersten resolutions of homotopy invariant pretheories; cf. §4.6 of [26].

7.2 Weight filtration for (conjectural) mixed motives

Suppose now that there exists so-called 'mixed motivic' $t$-structure on $DM_{gm}$ or $DM_{gm, \mathbb{Q}}$ (then one could extend it to $DM_{gm}$ and $DM_{gm, \mathbb{Q}}$, respectively). We will not discuss any of its properties here; yet it would automatically induce a homological functor $H_{MM} : DM_{gm} \to MM$ for some abelian category $MM$ (of so-called mixed motives) that is the heart of the $t$-structure. Hence for any $X \in DM_{gm}$ there will be a certain 'weight filtration' on $H_{MM}(X)$ (cf. Remark 2.4.2). This filtration would be trivial when $X$ is smooth projective. It could be easily checked that there could exist only one filtration on $H_{MM}(X)$ which is $DM_{gm}$-functorial and satisfies this property.

Moreover, any transformation $H_{MM} \to H$ for $H$ being a realization (of
$DM^{eff}_\text{gm}$) with values in an abelian category would induce the transformation of the weight filtration for $H^{i}_{MM}$ to the weight filtration of $H$. Here the weight filtration of $H$ is defined by the weight structure method, yet it coincides with the 'classical one' (cf. part 2 of Remark 2.4.2).

Therefore we obtain that our results will give the weight filtration for $H^{i}_{MM}(X)$ (an the corresponding weight spectral sequence) automatically when $H^{i}_{MM}$ will be defined. Amazingly, we don’t need any information on $H^{i}_{MM}$ for this! Yet it seems difficult to prove that the filtration on $H^{i}_{MM}(X)$ depends only on the object $H^{i}_{MM}(X)$ and does not depend on the choice of possible $X$. For this one possibly needs the degeneration of the weight spectral sequence for $H^{i}_{MM}$ and all $X$; this is probably the most important property of the motivic $t$-structure.

### 7.3 Motives over perfect fields of finite characteristic

In our study of motives (here and in [9]) we applied several results of [25] that use resolution of singularities. So we had assume that the characteristic of the ground field $k$ is $0$. In §8.3 of [9] it was shown that using de Jong’s alterations one could extend most of our results to motives with rational coefficients over an arbitrary perfect $k$.

In this subsection we consider motives with integral coefficients over a perfect field $k$ of characteristic $0$. Our goal is to justify a certain claim made in §8.3.2 of [9].

In [5] it was proved unconditionally that $DM^s$ has a differential graded enhancement. In fact, this fact could be easily obtained by applying Drinfeld’s description of localizations of enhanced triangulated categories. Moreover, Proposition 5.6 of [5] extends the Poincare duality for Voevodsky motives to our case. Therefore for $P, Q \in SmPrVar$ we obtain

$$DM^s(M_{gm}(P), M_{gm}(Q)[i]) = Corr_{rat}(\langle P \rangle, \langle Q \rangle)$$

for $i = 0$; $0$ for $i > 0$.

Hence the triangulated subcategory $DM_{pr}$ of $DM^s$ generated by $[P], P \in SmPrVar$ could be described as $Tr(I)$ for a certain negative differential graded $I$. In particular, we obtain the existence of a conservative weight complex functor $t_0 : DM_{pr} \to K^b(Corr_{rat})$. Moreover, for any realization of $DM_{pr}$ and any $X \in Obj DM_{pr}$ one has the weight spectral sequence $T$.

The problem is that (to the knowledge of the author) at this moment there is no way to prove that $DM_{pr}$ contains the motives of all varieties (though it contains the motives of varieties that have "smooth projective stratification").

Instead we will prove that the weight structure on $DM_{pr}$ could be extended to a weight structure on a larger category containing all $M_{gm}(X)$.
Recall that $M_{gm}$ is a full embedding of $DM_{gm}^{eff} \subseteq DM_{pr}^{eff}$, while $DM_{gm}^{eff} \subseteq D(Shv(SmCor)) (M_{gm} \text{ is denoted by } i \text{ in Theorem 2.3.6 of [25]). We denote by } D \subseteq D(Shv(SmCor)) \text{ the full category of complexes with homotopy invariant hypercohomology. We have a full embedding } DM_{gm}^{eff} \to D.

We can extend to $D \subseteq D(Shv(SmCor))$ the assertion of Proposition 3.2.3 of [25] (i.e. construct a projection $D(Shv(SmCor)) \to D$ which is left adjoint to the embedding) using the fact that

$$D(Shv(SmCor))(D(Shv(SmCor))^{t<0}, D(Shv(SmCor))^{t\geq 1}) = 0.$$ 

Here $t$ denotes the usual i.e. the Nisnevich $t$-structure of $D(Shv(SmCor))$. It follows that all objects of $M_{gm}(DM_{gm}^{eff})$ are compact. Indeed, it is sufficient to prove this for $DM_{gm}^{eff}([X])$ where $X \in SmVar$; Proposition 3.2.3 of [25] implies that $D(DM_{gm}^{eff}([X]), -)$ is the corresponding hypercohomology functor which commutes with arbitrary direct sums.

Consider $S = \{ X \in ObjD : D(Y, X) = \{0\} \ \forall Y \in ObjDM_{pr} \}$. Note that in the definition of $S$ it suffices to consider $Y = M_{gm}(P)[i], \ P \in SmPrVar, \ i \in \mathbb{Z}$, since $[P]$ generate $DM_{pr}$. Obviously, $S$ is the class of objects for a certain full triangulated subcategory of $D(Shv(SmCor))$. We denote the localization of $D$ by $S$ by $D_{S}$. By definition of $S$, the set $H = \{ |P|, \ P \in SmPrVar \}$ weakly generates $D_{S}$. Since objects of $DM_{pr}$ are compact, $S$ is closed with respect to arbitrary direct sums. It follows that $D_{S}$ admits arbitrary direct sums. Note that $DM_{pr} \subseteq D_{S}$ by Proposition III.2.10 of [12]; hence we have a full embedding $Chow^{eff} \to D$.

By part II of Theorem 4.4.2 we obtain that $D_{S}$ supports adjacent weight and $t$-structures which we will call Chow ones. By part II of Theorem 4.4.2 we have $Ht_{Chow} = Chow^{eff}_{S}$. Moreover, $Hw$ is the category $Chow^{eff}_{S}$ of arbitrary direct sums of effective Chow motives since $Chow^{eff}_{S}$ is idempotent complete.

Note that the definition of $w_{Chow}$ is compatible with the definition of the Chow weight structure on $DM_{pr}$. In particular, this reasoning extends the weight complex functor to a functor $D \to K_{w}(Chow^{eff}_{S})$. This would give a (possibly, infinite) weight complex for any $X \in ObjDM_{gm}^{eff}$. Note that (by the results of §8.3.1 of [9]) $t(X)$ becomes (homotopy equivalent to) a finite complex after tensoring the coefficients by $\mathbb{Q}$. This weight complex functor could be "strengthened" (see Remark 3.3.4) since $D(Shv(SmCor))$ has a differential graded enhancement.
8 Supplements

In §8.1 we show that a weight structure \( \mathcal{W} \) on \( \mathcal{C} \) which induces a weight structure on a triangulated \( \mathcal{D} \subset \mathcal{C} \) yields also a weight structures on the localization \( \mathcal{C}/\mathcal{D} \). In §8.2 we prove (using an argument due to A. Beilinson) that any \( f \)-category enhancement of \( \mathcal{C} \) yields a lift of \( t \) to a "strong" weight complex functor \( \mathcal{C} \to K(H_w) \); cf. Remark 3.3.4. In §8.3 we describe our ideas on the (possible) "higher truncation functors" for the general weight structure setting (related to the functors \( t_N \) described in §6.3). In §8.4 we discuss other possible sources of conservative "weight complex-like" functors.

8.1 Weight structures in localizations

We call a category a localization of an additive category \( A \) by its full additive subcategory \( B \) (we denote it by \( \mathcal{D} \)) if \( \text{Obj}(\mathcal{D}) = \text{Obj}A \) and \( \text{Obj}\,(\mathcal{D})(X,Y) = A(X,Y)/(\sum_{Z \in \text{Obj}B} A(Z,Y) \circ A(X,Z)) \).

**Proposition 8.1.1.** 1. Let \( \mathcal{D} \subset \mathcal{C} \) be a strict triangulated subcategory of \( \mathcal{C} \); suppose that \( \mathcal{W} \) induces a \( t \)-structure on \( \mathcal{D} \) i.e. \( \text{Obj}\,(\mathcal{D})_0^{w \leq 0} \) and \( \text{Obj}\,(\mathcal{D})_0^{w \geq 0} \) give a weight structure for \( \mathcal{D} \). We denote the heart of the weight structure of \( \mathcal{D} \) by \( HD \).

Then \( \mathcal{W} \) induces a weight structure on \( \mathcal{C}/\mathcal{D} \) (the localization of \( \mathcal{C} \) with respect to \( \mathcal{D} \)). This means that the Karoubi-closures of \( \mathcal{C}^{w \leq 0} \) and \( \mathcal{C}^{w \geq 0} \) (in \( \mathcal{C}/\mathcal{D} \)) give a weight structure for \( \mathcal{C}/\mathcal{D} \) (note that \( \text{Obj}\,(\mathcal{C}) = \text{Obj}\,(\mathcal{C}/\mathcal{D}) \)).

2. \( H(\mathcal{C}/\mathcal{D}) \) is the Karoubi-closure of \( \mathcal{H}^{w \leq 0}_\mathcal{D} \) in \( \mathcal{C}/\mathcal{D} \).

3. If \( \mathcal{C}/\mathcal{W} \) is bounded (above, below, or both), then \( \mathcal{C}/\mathcal{D} \) also is.

**Proof.** 1. It clearly suffices to prove that for any \( X \in (\mathcal{C}/\mathcal{D})^{w \geq 0} \) and \( Y \in (\mathcal{C}/\mathcal{D})^{w \leq -1} \) we have \( (\mathcal{C}/\mathcal{D})(X,Y) = 0 \); all other axioms of Definition 1.1.1 are fulfilled automatically since \( \mathcal{C}/\mathcal{D} \) is a localization of \( \mathcal{C} \).

Recall now (see Lemma III.2.8 of [12]) that any morphism in \( (\mathcal{C}/\mathcal{D})(X,Y) \) can be presented as \( fs^{-1} \) where \( f \in \mathcal{C}(T,Y) \) for some \( T \in \text{Obj}\,(\mathcal{C}) \), \( s \in \mathcal{C}(T,X) \), \( \text{Cone}(s) = Z \in \text{Obj}\,(\mathcal{D}) \).

By our assertion, there exists a choice of \( Z^{w \geq 0} \) that belongs to \( \text{Obj}\,(\mathcal{D}) \). Since \( \mathcal{C}(X,Z^{w \leq -1}[1]) = 0 \) we can factorize the morphism \( X \to Z \) (induced by \( s \)) through \( Z^{w \geq 0} \).

Hence (applying the octahedron axiom) we obtain that there exist \( T' \in \text{Obj}\,(\mathcal{C}) \), a morphism \( d : T' \to T \), such that \( \text{Cone} \, d = Z^{w \leq -1}[1] \in \text{Obj}\,(\mathcal{D}) \), whence a cone of the composite morphism \( s' : T' \to X \) equals \( Z^{w \geq 0} \). It follows that \( fs^{-1} = (fd)s'^{-1} \in \mathcal{C}/\mathcal{D} \). Now note that \( T' \in \mathcal{C}^{w \geq 0} \) by part 3 of Proposition 1.3.1). Hence \( \mathcal{C}(T,Y) = 0 \), which yields \( fd = 0 \).

2. By construction, \( \mathcal{C}^{w = 0} \subset (\mathcal{C}/\mathcal{D})^{w = 0} \).
Now we prove that any object of $H(C/D)$ is a retract of an object of $Hw$ (in $C/D$).

Let $Z \in C/D_{w=0} \subset Obj_C$. We consider a weight decomposition $Z_{w \geq 1}[-1] \to Z \to Z_{w \leq 0}$ of $Z$ in $C$. In $C/D$ we have $Z_{w \geq 1} \in C/D_{w \geq 1}$, hence $C/D(Z_{w \geq 1}[-1], Z) = 0$. Hence $Z$ in $C/D$ is a retract of $Z_{w \leq 0}$. Moreover, $Z_{w \geq 1} \in C/D_{w=0}$ since it is a retract of $Z_{w \leq 0} \in C/D_{w \leq 0}$; therefore $Z_{w \leq 0} \in C/D_{w=0}$. Now applying the dual argument to $Z_{w \leq 0}$, we obtain that $Z$ in $C/D$ is a retract of some $Z^0 \in Obj_C^{w=0}$.

To conclude the proof it suffice to check that the natural functor $i : Hw/HD \to H(C/D)$ is a full embedding. We consider the composition $C \xrightarrow{1} K_w(Hw) \to K_w(Hw/HD)$. Obviously, it maps all objects of $D$ to 0. Hence $i$ is injective on morphisms.

It remains to prove that any morphism $g : X \to Y$ in $C/D$ comes from $C(X, Y)$. Applying the same argument as in the proof of assertion 1 we obtain that $g$ could be presented as $fs^{-1}$ where $f \in C(T, Y)$ for some $T \in Obj_C$, $s \in C(T, X)$, $\text{Cone} \ s = Z \in D_{w \geq 0}$. Then $C(X, Y)$ surjects onto $C(T, Y)$. Now the "calculus of fractions" yields the result.

3. Since $Obj_C/C = Obj_C$, we obtain the claim.

$\square$

**Corollary 8.1.2.** Let $E \subset Hw$ be an additive subcategory. If $X$ belongs to the Karoubi-closure $Obj \langle E \rangle$ then $t(X)$ is a retract of some object of $K_w^b(E)$ (here we mean that $K_w^b(E) \subset K_w(HC)$).

If $(C, w)$ is bounded then the converse implication also holds.

**Proof.** We can assume that $X \in Obj \langle E \rangle$. Then $X$ could be obtained from objects of $E$ by repetitive consideration of cones of morphisms. Since $t(Obj E) \subset Obj K_w(E)$ and $t$ is a weakly exact functor in the sense of Definition 3.1.4 we obtain that $t(X) \in Obj K_w^b(E)$.

Conversely, let $t(X)$ be a retract of $Y \in Obj K_w^b(E) \subset Obj K_w^b(Hw)$. By Proposition 8.1.1 we obtain that $C/\langle E \rangle$ possesses a bounded weight structure whose heart contains $D_{w}^E$ as a full subcategory. Hence, by part V of Theorem 3.3.1 we obtain that $t_{C/\langle E \rangle}$ is conservative. $Y \in Obj K_w^b(E)$ gives $t_{C/\langle E \rangle}(Y) = 0$, hence $X$ and $Y$ belong to the Karoubi-closure of $\langle E \rangle$.

$\square$

**Remark 8.1.3.** 1. Note that (in general) one cannot be sure that the 'factor weight structure' on $C/D$ is non-degenerate. In fact, it is non-degenerate iff $D$ satisfies the following 'closedness condition': for $X \in Obj_C$ we have $t(X) \in Obj K_w(HD) \implies X \in Obj_D$. In particular, for any degenerate $t$-structure one can consider its maximal non-degenerate factor. Note that $t(X) \in Obj K_w(HD)$ iff for any $N > 0$ there exist weight decompositions of
\(X[N]\) and \(X[N]^{w \leq 0}[-2N]\) (cf. Remark 1.2.2) such that \(X[N]^{\leq 0}[-2N]^{w \geq 0} \in \text{Obj} \bar{D}\).

2. Corollary 8.1.2 is parallel to part 3 of Proposition 8.2.1 of [9]. In particular, it could be used to prove that a motif of a smooth variety is mixed Tate whenever its weight complex (defined in [13]) is (that is Corollary 8.2.2 of [9]).

3. Adding certain additional restrictions, one could also formulate a criterion for \(t(X)\) to belong to the Karoubi-closure of \(\text{Obj} K_w(E)\) (instead of \(\text{Obj} K^{\text{tr}}_w(E)\)).

8.2 A strong weight complex functor for triangulated categories that admit an \(f\)-triangulated enhancement

Now we check that the strong weight complex functor \(t\) exists if there exists an \(f\)-category enhancement of our category (we will define this notion very soon); see part 3 of Remark 3.3.4. The argument below was kindly communicated to the author by prof. A. Beilinson. To make our notation compatible with those of [6] we will denote our basic triangulated category (which is usually \(\mathcal{C}\)) by \(D\). As usual, \(D\) is endowed with a weight structure \(w\).

The plan of the construction is the following one. Suppose that there exists an \(f\)-category \(DF\) over \(D\). In particular, this yields the existence of the "forgetting of filtration" functor \(\omega : DF \to D\). We describe a class of objects \(DF^* \subseteq \text{Obj} DF\) such that:

(i) any object of \(X \in \text{Obj} D\) could be "lifted" to an element of \(X^* \in DF^*\);

(ii) For every \(M, N \in DF^*\) the map \(DF(N, M) \to D(\omega(N), \omega(M))\) is surjective;

(iii) There exists a functor \(e : DF \to C_b(D)\) such that \(e(DF^* \subseteq \text{Obj}(C_b(Hw))\)

and for any \(M, N \in DF^*\) the functor \(e\) maps \(\text{Ker} \ DF(N, M) \to D(\omega(N), \omega(M))\)
ton morphisms that are homotopic to 0.

We will denote the induced functor \(DF \to K_b(D)\) by \(e'\).

Then \(X \to e(X^*)\) yields an additive functor \(T : D \to K\) where \(K\) is a certain triangulated category isomorphic to \(K^b(Hw)\). Indeed, by (ii) any two choices of \(X^*\) are connected by (possibly, non-unique) morphisms. By (iii) these morphisms become canonical isomorphisms after the application of \(e'\). Hence it suffices to take \(K\) being the category obtained from \(K_b(Hw)\) by factorizing by these isomorphisms. Indeed, this family respects direct sums since \(\omega\) and \(e\) do.

Now we recall the relevant definitions of the Appendix of [6].
Definition 8.2.1. A triangulated category $DF$ will be called a filtered triangulated one if it is endowed with strict triangulated subcategories $DF(\leq 0)$ and $DF(\geq 0)$; an exact autoequivalence $s : DF \to DF$; and a morphism of functors $\alpha : id_{DF} \to s$, such that the following axioms hold (for $DF(\leq n) = s^n(DF(\leq 0))$ and $DF(\geq n) = s^n(DF(\geq 0))$).

(i) $DF(\geq 1) \subset DF(\geq 0)$; $DF(\leq 1) \supset DF(\leq 0)$; $\cup_{n \in \mathbb{Z}}DF(\geq n) = \cup_{n \in \mathbb{Z}}DF(\leq n) = DF$.

(ii) For any $X \in \text{Obj}DF$ we have $\alpha_X = s(\alpha_{s^{-1}X})$.

(iii) For any $X \in \text{Obj}DF(\geq 1)$ and $Y \in \text{Obj}DF(\leq 0)$ we have $DF(X, Y) = 0$; whence $\alpha$ induces an isomorphism $DF(Y, s^{-1}X) \cong DF(sY, X) \to DF(Y, X)$.

(iv) Any $X \in \text{Obj}DF$ could be completed to a distinguished triangle $A \to X \to B$ with $A \in \text{Obj}DF(\geq 1)$ and $B \in \text{Obj}DF(\leq 0)$.

II $DF$ is called an f-category over $D$ if $D \subset DF$; $\text{Obj}D = \text{Obj}DF(\leq 0) \cap \text{Obj}DF(\geq 0)$.

III We will denote by $\omega$ (see Proposition A3 of [6]) the only exact functor $DF \to D$ such that:

(i) Its restrictions are right adjoint to the inclusion $D \to DF(\leq 0)$ and left adjoint to the inclusion $D \to DF(\geq 0)$ respectively.

(ii) $\omega(\alpha_X)$ is an isomorphism.

(iii) $DF(X, Y) = D(\omega_X, \omega_Y)$ for any $X \in \text{Obj}DF(\leq 0), Y \in \text{Obj}DF(\geq 0)$.

A simple example of this axiomatics is described in Example A2 loc. cit.

By Proposition A3 loc. cit. there also exist exact functors $\sigma_{\geq n} : DF \to DF(\geq n)$, and $\sigma_{\leq n} : DF \to DF(\leq n)$ that are respectively right and left adjoint to the corresponding inclusions. We denote $gr_F^{[a,b]} := \sigma_{\leq b} \sigma_{\geq a}$, $gr_F^a := gr_F^{[a,a]}$. Note that these exist canonical and functorial (in $X$) morphisms $d : \sigma_{\leq 0} X \to \sigma_{\geq 1} X[1]$ that could be completed to a distinguished triangle in I(iv) of Definition 8.2.1.

Now we define $e$. For $M \in \text{Obj}DF$ the complex $e(M)$ has components equal to $s^{-a}gr_F^a M[a]$ (this lies in $\text{Obj}D \subset \text{Obj}DF$), the differential will be equal to $s^{-a-1}(s(d') \circ \alpha_{gr_F^a})[a]$; here $d'$ is the boundary map of the canonical triangle $gr_F^{a+1} \to gr_F^{[a,a+1]} \to gr_F^a d' \to gr_F^{a+1}[1]$. $M$ is a complex indeed by the axiom I(ii). We have $e(s(X)) \cong e(X)$.

Now for a weight structure $w$ on $D$ we define $DB^w = e^{-1}(C^b(Hw))$ i.e. we demand $gr_F^a(X) \in s^a D^{w=a}$.

We will use the following statement.

Lemma 8.2.2. For every $M, N \in DF^w$ the map $\alpha_sDF(N, M) \to DF(N, s(M))$ is surjective; all $DF(N, s^a(M)) \to DF(N, s^{a+1}(M))$ for $a > 0$ are bijective.
Proof. Set \( P = \text{Cone}(\alpha_M : M \to s(M)) \). By the long exact sequence for \( DF(N, -) \), it suffices to show that \( DF(N, s^a(P)[b]) = 0 \) for \( a + b \geq 0 \). Since \( s^aM[-a] \in DF^s \), it suffices to show that \( DF(N, P[b]) = 0 \) for \( b \geq 0 \).

By devisor, we can assume that \( gr^b_M \) and \( gr^K_N \) vanish for \( a \neq m, b \neq n, m, n \in \mathbb{Z} \). In other words, \( M = s^m(K)[-m] \), \( N = s^n(L)[-n] \) for some \( K, L \in \mathcal{C}_{w=0}^w \subset \text{Obj}D \subset \text{Obj}DF \).

One has \( DF(N, P[b]) = D(L[-n], \omega(\sigma_{\geq n}P)[b]) \). To see that this group vanishes, consider 3 cases.

(a) Suppose that \( n > m + 1 \). Then \( \sigma_{\geq n}P = 0 \).

(b) Suppose \( n \leq m \). Then \( \sigma_{\geq n}P = P \), so \( \omega(\sigma_{\geq n}P) = \omega(P) = 0 \).

(c) Suppose \( n = m + 1 \). Then \( \sigma_{\geq n}P = s(M) \), so \( \omega(\sigma_{\geq n}P) = K[-m] \) and \( D(N, P[b]) = D(L[-n], K[-m+b]) = D(L, K[b+1]) = 0 \) since \( w \) is a weight structure.

Now (ii) follows from Lemma 8.2.2 immediately since for any \( X, Y \in \text{Obj}DF \) we have \( DF(X, s(Y)) \cong D(\omega(X), \omega(s^aY)) \cong D(\omega(X), \omega(Y)) \) for \( n \) large enough by parts III(ii), III(iii) of Definition 8.2.1.

(ii) easily yields (i). Indeed, we can prove the statement for \( X \in D^{[i,j]} \) by the induction on \( j - i \). We have obvious inclusions \( D^{w=i} \to DF^s \) (that split \( \omega \)).

To make the inductive step it suffices to consider \( X \in D^{[0,m]} \) for \( m > 0 \). Then \( X^{w=0} \) and \( X^{\geq 1} \) could be lifted to \( DF^s \) by the inductive assumption. The map \( X^{w=0} \to X^{\geq 1} \) lifts to \( DF^s \) by Lemma 8.2.2; its cone will belong to \( DF^s \) and so will be a lift of \( X \).

Now we verify (iii). By Lemma 8.2.2, for \( M, N \in DF^s \) we have

\[ \text{Ker} DF(N, M) \to D(\omega(N), \omega(M)) = \text{Ker}(\alpha, DF(N, M) \to DF(N, s(M))). \]

Since \( \omega(M) \to \omega(s(M)) \) is an isomorphism, we obtain (iii). Hence \( T \) is a well-defined functor.

Now we need to check that \( T \) is an "enhancement" of our "weak weight complex functor" \( t \). Indeed, \( T \) and \( t \) coincide on \( Hw \); both of them respect weight decompositions of objects and morphisms in a compatible way.

Lastly we check that \( T \) is an exact functor. As in the proof of part I of Theorem 3.3.1, it suffices to lift any distinguished triangle \( C \to X \to X' \) so that the sequence \( e(C^*) \to e(X^*) \to e(X'^*) \) splits termwisely (in \( C^b(Hw) \)). Indeed, this would yield that any distinguished triangle is mapped by \( T \) to a triangle \( Tr \) any of whose two sides are two sides of a of a distinguished triangle in \( K^b(Hw) \); hence \( Tr \) is distinguished in \( K^b(Hw) \).

Now, to find such lifting it suffices to choose the weight decompositions of \( X \) and \( X' \) arbitrarily; choose a weight decomposition of \( C \) as in the proof of part I of Theorem 3.3.1; and lift to \( DB^s \) the map \( t(C) \to t(X) \) as in the
proof of (i). Then the map $a^* : C^* \rightarrow X^*$ will become split surjective after the application of each $gr^F_i$; hence we can choose $\text{Cone } a^* \in DF^*$ as a lift for $X'$. This yields the lift desired.

Hence $T$ is a strong weight complex functor for $D, w$. This argument is a certain weight structure counterpart of Proposition A5 of [6].

It also seems possible that an $f$-category enhancement of $D$ would allow to define certain higher truncation functors; see below.

8.3 Notes on higher truncated categories and truncated (co)homological functors

The differential graded case suggests that there should exist exact 'higher truncation functors' $t_N$ such that $t_0$ is the 'strong' weight complex functor; cf. Conjecture 3.3.3. Their targets $C_N$ should satisfy: for if $X, Y \in C^{w=0}$ then $C_N(t_N(X), t_N(Y)[-i]) = C(X, Y)$ for $0 \leq i \leq N$ and $= 0$ otherwise. These categories should admit full embeddings $t_N : C^{[0,N]} \rightarrow C_N$; distinguished triangles of $C$ consisting of elements of $C^{[0,N]}$ should be mapped to distinguished triangles by $t_N$.

It seems possible to construct these categories and functors defining the objects of $C_N$ as 'admissible' sequences of objects of $C^{[0,N]}$. Yet checking the axioms of triangulated categories for such a construction could be (technically) difficult.

The author didn’t put much effort into defining higher truncation functors because in his study of Voevodsky’s motives only $t_0$ was shown to be really useful.

We also describe the conjecture that certain exact functors from $C$ could be factorized through $t_N$.

**Conjecture 8.3.1.** Let $I : C \rightarrow D(A)$ be an exact functor, where $C, w$ is a triangulated category with a weight structure, $A$ is an abelian category. If $I(C_*^{w=0}) \subset D_{[0,N]}(A)$ then $I$ can be canonically factorized through $t_N$.

**Remark 8.3.2.** Unfortunately, there is no way to define truncation functors if $D(A)$ is replaced by a general triangulated category $D$ with an arbitrary $t$-structure. Indeed, even in the case when $H(C_*^{w=0}) \subset \text{Obj} A$ (here $A$ is the heart of the $t$-structure on $D$) there is no general way to define a functor $K(Hw) \rightarrow D$; see Remark IV.4.15 in [12].

Yet in some cases one could modify the conditions of Conjecture 8.3.1. For example, it could be valid in the cases when $D \subset D(A')$ for some abelian $A'$ and the $t$-structure on $D$ is induced by those of $D(A')$. In particular, this is the case for the Hodge realization of motives.
8.4 Possible variations of the weight complex functor

Now we try to answer the question: could the main results of this paper be generalized to a more general setting. We cannot prove any if and only if conditions; yet we try to clarify the picture. Since we include this subsection only to explain our choice of definitions, it is rather sketchy.

First we study the question where do exact conservative functors come from.

Suppose that \( f : \mathcal{C} \to \mathcal{C}' \) is an exact functor (here \( \mathcal{C}, \mathcal{C}' \) are triangulated categories). We denote by \( \text{Ker} f \) the set of morphisms that are mapped to 0 by \( f \). \( \text{Ker} f \) satisfies a certain set of obvious properties; hence it could be called a \textit{triangulated ideal} of \( \text{Mor}\mathcal{C} \).

It is easily seen that if \( f \) is conservative if and only if \( \text{id}_X \notin \text{Ker} f \) for any \( X \in \text{Obj}\mathcal{C} \). Note that in this case \( \text{Ker} f \) could be called a radical ideal since for any \( X \in \text{Obj}\mathcal{C}, s \in \mathcal{C}(X, X) \cap \text{Ker} f \), \( \text{id}_X + s \) will be an automorphism.

Now we study an inverse problem: which triangulated ideals can correspond to conservative exact functors. Unfortunately, it seems that there does not exist a nice way to kill morphisms in an arbitrary \( I \) unless \( \mathcal{C} \) has a differential graded enhancement. So we suppose that \( \mathcal{C} = \text{Tr}^+(D) \) for a dg-category \( D \); a triangulated ideal \( I \subset \text{Mor}\mathcal{C} \) comes from a differential graded nilpotent (or formally nilpotent in an appropriate sense) ideal \( I' \) of \( \text{Pre-}\text{Tr}^+ D \). Then one can form a category \( \mathcal{C}' = \text{Tr}^+(D/I') \); using a certain spectral sequence argument for representable functors \( X \), for \( X \in \text{Obj}\mathcal{C} \) similar to those described below (for realizations) one can verify that the natural differential graded functor \( \mathcal{C} \to \mathcal{C}' \) is conservative. Yet one cannot hope for a spectral sequence for a realization \( H \) unless \( H(I) \) belongs to some nice radical ideal (probably more conditions are needed). Note that this is obviously the case for representable functors.

We describe one of the cases when it makes sense to construct such a theory (and which does not come from a weight structure). Let \( \mathcal{C}, D \) be pro-
-\( p \)-categories (i.e. the set of morphisms is an abelian profinite \( p \)-group for any pair of objects) and \( I' = \text{pMor}(\mathcal{C}) \). Let \( H = \text{Tr}^+(E) \) for a dg-functor \( E : D \to B(\text{pro-}p - Ab) \), where \( B(\text{pro-}p - Ab) \) is the 'big' category of complexes of abelian profinite \( p \)-groups (see subsection 6.4). Then the complex that computes \( H(X) \) for \( X \in \text{Obj}\mathcal{C} \) has a natural filtration by subcomplexes given by \( p^i E \). These subcomplexes correspond to the functors \( \text{Tr}^+(p^i E) \) and the factors of the filtration are quasi-isomorphic to those calculating the functors \( F_i = \text{Tr}^+(p^i E/p^{i+1} E) \). It remains to note that \( F_i \) can be factorized through the natural functor \( \mathcal{C} \to \text{Tr}^+(D/p) \). Hence in this case the spectral sequence of a filtered complex has properties similar to those of the spectral sequence \( T \) in 2.3; \( F_i \) are similar to truncation functors.
References


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