TWISTED *K*-THEORY

OLD AND NEW

by

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Some history and motivation about this paper

The subject "*K*-theory with local coefficients", now called "twisted *K*-theory", was introduced by P. Donovan and the author in [DK] 37 years ago. It associates to a compact space X and a "local coefficient system"

$$\alpha \in GBr(X) = \mathbf{Z}/2 \times H^1(X ; \mathbf{Z}/2) \times Tors(H^3(X ; \mathbf{Z}))$$

an abelian group $K^\alpha(X)$ which generalizes the usual Grothendieck-Atiyah-Hirzebruch *K*-theory of X when we restrict α being in $\mathbf{Z}/2$ (cf. [A1]). This "graded Brauer group" $GBr(X)$ has the following group structure : if $\alpha = (\varepsilon, w_1, W_3)$ and $\alpha' = (\varepsilon', w_1', W_3')$ are two elements, one defines the sum $\alpha + \alpha'$ as $(\varepsilon + \varepsilon', w_1 + w_1', W_3 + W_3' + \beta(w_1 w_1'))$, where $\beta : H^2(X ; \mathbf{Z}/2) \to H^3(X ; \mathbf{Z})$ is the Bockstein homomorphism. With this definition, one has a generalized cup-product

$$K^\alpha(X) \times K^{\alpha'}(X) \to K^{\alpha+\alpha'}(X)$$

The motivation for this definition is to give in *K*-theory a satisfactory Thom isomorphism and Poincaré duality pairing which are analogous to the usual ones in cohomology with local coefficients. More precisely, as proved in [K1] (a precursor to twisted *K*-theory), if V is a real vector bundle on a compact space X with a positive metric, the *K*-theory of the Thom space of V is isomorphic to a certain group $K^{C(V)}(X)$ associated to the Clifford bundle $C(V)$, viewed as a bundle of $\mathbf{Z}/2$-graded algebras. A careful analysis of this group shows that it depends only on the class of $C(V)$ in $GBr(X)$, the three invariants being respectively the rank of V mod. 2, $w_1(V)$ and $\beta(w_2(V)) = W_3(V)$, where $w_1(V)$, $w_2(V)$ are the first two Stiefel-Whitney classes of V. In particular, if V is a ^{c}spinorial bundle of even rank, one recovers a well-known theorem of Atiyah and Hirzebruch [AH]. On the other hand, if X is a compact connected manifold, it is well-known that such a Thom isomorphism theorem induces a pairing between K-groups

$$K^\alpha(X) \times K^{\alpha'}(X) \to \mathbf{Z}$$

if $\alpha + \alpha'$ is the class of $- C(V)$ in $GBr(X)$, where V is the tangent bundle of X.

The necessity to revisit these ideas comes from a new interest in the subject because of its relations with Physics [Wi], as shown by the number of recent publications. However, for these applications, the first definition recalled above is not complete since the coefficient system is restricted to the torsion elements of $H^3(X ; \mathbf{Z})$. As it was pointed out by J. Rosenberg [R] and later on by C. Laurent, J.-L. Tu, P. Xu [LTX], M. F. Atiyah and G. Segal [AS], this restriction is in fact not necessary. In order to avoid it, one may use for instance the Atiyah-Janich theorem [A1][J] about the representability of *K*-theory by the space of Fredholm operators (already mentioned in [DK], but not for this purpose).

In the present paper, the road we choose is to give a synthesis between different viewpoints on the subject : [DK], [R], [CTX] and [AS] (partially of course)[1]. We hope we have been "pedagogical" in some sense to the non experts.

However, we dont pretend to be exhaustive. For instance, one should add to the previous references [BCMMS] for the relations with the theory of gerbes and D-branes in Physics ;V. Mathai and D. Stevenson [MS] who use the Chern character defined by A. Connes and M. Karoubi in order to prove an isomorphism between twisted *K*-theory and a computable "twisted cohomology" (at least rationally) ; D. Freed, M. Hopkins and C. Teleman [FHT] about the relation between loop groups and twisted K-theory and many other works which are mentioned inside the paper and in the list of references at the end.

We would like to point out also that in a recent paper M. F. Atiyah and M. Hopkins [AHo] introduced another type of *K*-theory, denoted by them $K_{\pm}(X)$, which is in fact a particular case of twisted equivariant *K*-theory. More precisely, this définition of $K_{\pm}(X)$ was already given in [K1] § 3.3, 40 years ago, in terms of Clifford bundles with a group action (see 6. 16 and also [K4] for the details).

This paper is not just historical. It presents the theory with a new point of view and contains some new results. We extend the Thom isomorphism to this more general setting (see also [Ca]), which is important in order to relate the "ungraded" and "graded" twisted *K*-theories. We compute many interesting equivariant twisted K-groups, complementing the basic papers [LTX], [CTX] and [AS]. For this purpose, we use the "Chern character" for finite group actions, as defined by Baum, Connes, Kuhn and Slominska [BC]Kuh][S], together with our generalized Thom isomorphism. These last computations are related to some previous ones [K4] and to the work of many authors : A. Adem and Y. Ruan [AR], P. Hu and I. Kriz [HK], J.-L. Tu and P. Xu [TX], among others. Finally, we introduce new cohomology operations which are complementary to those defined in [DK] and [AS2]

[1] As it was pointed out to me by J. Rosenberg, one should also add the following reference, in the spirit of [DD]: Ellen Maycock Parker, The Brauer group of graded continuous trace C*-algebras, Trans. Amer. Math. Soc. 308 (1988), no. 1, 115--132.

General plan of the paper

Let us first recall the point of view developed in [DK], in order to describe the background material. We consider a locally trivial bundle of graded central simple complex algebras $\mathcal{A}$, i.e. modelled on $M_{2n}(\mathbf{C})$ or $M_n(\mathbf{C})$ x $M_n(\mathbf{C})$, with the obvious gradings[2]. Then $\mathcal{A}$ has a well-defined class α in the group $GBr(X)$ (as introduced above). On the other hand, one may consider the category of "$\mathcal{A}$-bundles" , whose objects are vector bundles provided with an $\mathcal{A}$-module structure (fibrewise). We call this category $E^{\mathcal{A}}(X)$; the graded objects of this category are vector bundles which are modules over $\mathcal{A} \hat{\otimes} C^{0,1}$, where $C^{0,1}$ is the Clifford algebra $\mathbf{C}$ x $\mathbf{C} = \mathbf{C}[x]/(x^2 - 1)$. The group $K^{\alpha}(X)$ is now defined as the Grothendieck group of the forgetful functor

$$E^{\mathcal{A} \hat{\otimes} C^{0,1}}(X) \to E^{\mathcal{A}}(X)$$

We refer the reader to [K1] for the definition of this group which generalizes the usual Grothendieck group of a category. For our purpose, we make it quite explicit at the end of § 1, using the concept of "grading".

Despite its algebraic simplicity, the previous definition of $K^{\alpha}(X)$ is not quite satisfactory for various reasons. For instance, it is not clear how to define in a simple way a "cup-product"

$$K^{\alpha}(X) \times K^{\alpha'}(X) \to K^{\alpha+\alpha'}(X)$$

as mentioned earlier (even if α and α' are in the much smaller group $\mathbf{Z}/2$).

To correct this defect, a second definition may be given in terms of Fredholm operators in a Hilbert space. More precisely, we consider graded Hilbert bundles E which are also graded $\mathcal{A}$-modules in an obvious sense, together with a continuous family of Fredholm operators

$$D : E \to E$$

with the following properties :

1) D is self-adjoint of degree one
2) D commutes with the action of $\mathcal{A}$ (in the graded sense)

One gets an abelian semi-group from the homotopy classes of such couples (E, D), with the addition rule

$$(E, D) + (E', D') = (E \oplus E', D \oplus D')$$

The associated group gives the second definition of $K^{\alpha}(X)$ which is equivalent to the first one (see [DK] p. 18 and [K2] p. 88).

With this new viewpoint, the cup-product alluded to above becomes obvious. It is defined by the following formula :

$$(E, D) \cup (E', D') = (E \hat{\otimes} E', D \hat{\otimes} 1 + 1 \hat{\otimes} D')$$

[2] up to graded Morita equivalence.

where the symbol $\hat{\otimes}$ denotes the graded tensor product of bundles or morphisms.

For simplicity's sake, we have only considered complex *K*-theory. We could as well study the real case : one has to replace GBr(X) by

$$GBrO(X) = \mathbf{Z}/8 \times H^1(X\,;\mathbf{Z}/2) \times H^2(X\,;\mathbf{Z}/2)$$

If we take for the coefficient system $\alpha = n$ to be in $\mathbf{Z}/8$, we get the usual groups $KO^n(X)$ as defined using Clifford algebras in [K1] and [K2] p. 88 (these groups being written $\overline{K}^n$ in the later reference).

In this paper, we essentially follow the plan above, but with bundles of "infinite dimension" as we shall explain later on. As a matter of fact, all the technical tools are already present in [DK], [K2] and [R], for instance the Fredholm operator machinery. However, this paper is not a rewriting of these papers, since we take a more synthetic view point and have other applications in mind. For instance, the *K*-theory of Banach algebras $K_n(A)$ and its graded version, denoted here by $GrK_n(A)$, are more systematically used. On the other hand, since the equivariant *K*-theory has been studied carefully in [CKRW], [LTX] and [AS], we limit ourselves to the applications in this case. One of them is the definition of operations in twisted *K*-theory in the graded and ungraded situations.

Here are the contents of the paper :

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1. *K*-theory of Z/2-graded Banach algebras.

1.1. Higher *K*-theory of real or complex Banach algebras *A* is well-known (cf. [K1] or [B] for instance). Starting from the usual Grothendieck group $K(A) = K_0(A)$, there are many equivalent ways to define "derived functors" $K_n(A)$, for $n \in \mathbf{Z}$, such that any exact sequence of Banach algebras

$$0 \to A' \to A \to A'' \to 0$$

induces an exact sequence of abelian groups

$$\ldots \to K_{n+1}(A) \to K_{n+1}(A'') \to K_n(A') \to K_n(A) \to K_n(A'') \to \ldots$$

Moreover, by Bott periodicity, these groups are periodic of period 2 in the complex case and 8 in the real case.

1.2. The *K*-theory of **Z**/2-graded Banach algebras *A* (in the real or complex case) is less well-known[3] and for the purpose of this paper we shall recall its definition which is already present but not systematically used in [K1] and [DK]. We first introduce $C^{p,q}$ as the Clifford algebra of $\mathbf{R}^{p+q}$ with the quadratic form

$$-(x_1)^2 - \ldots - (x_p)^2 + (x_{p+1})^2 + \ldots + (x_{p+q})^2$$

It is naturally **Z**/2-graded. If *A* is an arbitrary **Z**/2-graded Banach algebra, $A^{p,q}$ is the graded tensor product $A \hat{\otimes} C^{p,q}$. For *A* unital, we now define the graded *K*-theory of *A* (denoted by $GrK(A)$) as the *K*-theory of the forgetful functor :

$$\phi : \mathcal{P}(A^{0,1}) \to \mathcal{P}(A)$$

(see [K1] or 1.4 for a concrete definition). Here $\mathcal{P}(A)$ denotes in general the category of finitely generated projective left *A*-modules. One may remark that the objects of $\mathcal{P}(A^{0,1})$ are graded objects of the category $\mathcal{P}(A)$. The functor ϕ simply "forgets" the grading. One should also notice that $GrK(A^{p,q})$ is naturally isomorphic to $GrK(A^{p+1,\, q+1})$, since $A^{p,q}$ is Morita equivalent to $A^{p+1,\, q+1}$) (in the graded sense).

If *A* is not unital, we define $GrK(A)$ by the usual method, as the kernel of the augmentation map

$$GrK(A^+) \to GrK(k)$$

where A^+ is the *k*-algebra *A* with a unit added ($k = \mathbf{R}$ or $\mathbf{C}$, according to the theory, with the trivial grading).

1.3. There is a "suspension functor" on the category of graded algebras, associating to *A* the graded tensor product $A^{0,1} = A \hat{\otimes} C^{0,1}$. One of the fundamental results[4] in [K1] p. 210 is the fact that this suspension functor is compatible with the usual one : in other words, we have a well-defined isomorphism

[3] This is of course included in the general KK-theory of Kasparov which was introduced later than our basic references [K1] and [DK].

[4] Strictly speaking, one has to replace the category $C^{p,q}$ in [K1] with an arbitrary graded category. However, the proof of theorem 2.2.2 easily extends to this case (see also [W]).

$$GrK(A^{0,1}) \cong GrK(A(\mathbf{R}))$$

where $A(T)$ denotes in general the algebra of continuous maps $f(t)$ on the locally compact space T with values in A, such that $f(t) \to 0$ when t goes to infinity. As a consequence, we have an isomorphism between the following groups (for $n \geq 0$) :

$$GrK(A^{0,n}) \cong GrK(A(\mathbf{R}^n))$$

which we call $GrK_n(A)$. More generally, we put $GrK_n(A) = GrK(A^{p,q})$ for $q - p = n \in \mathbf{Z}$. These groups GrK_n satisfy the same exactness property as the groups $K_n(A)$ above, from which they are naturally derived. They are of course linked with them by the following exact sequence (for all $n \in \mathbf{Z}$) :

$$K_{n+1}(A \hat{\otimes} C^{0,1}) \to K_{n+1}(A) \to GrK_n(\mathrm{A}) \to K_n(A \hat{\otimes} C^{0,1}) \to K_n(A)$$

In particular, if we start with an ungraded Banach algebra A, we see that Bott periodicity follows from these previous considerations, thanks to the periodicity of Clifford algebras up to graded Morita equivalence : this was the main theme developed in [K1], in order to give a more conceptual proof of the periodicity theorems.

1.4. When A is unital, it is technically important to describe the group $GrK(A)$ in a concrete way. If E is an object of $\mathcal{P}(A)$, a grading of E is given by an involution ε which commutes (resp. anticommutes) with the action of the elements of degree 0 (resp. 1) in A. In this way, E with a grading ε may be viewed as a module over the algebra $A \hat{\otimes} C^{0,1}$. We now consider triples $(E, \varepsilon^1, \varepsilon^2)$, where ε^1 and ε^2 are two gradings of E. The homotopy classes of such triples obviously form a semi-group. Its quotient by the semi-group of "elementary" triples (i.e. such that $\varepsilon^1 = \varepsilon^2$) is isomorphic to $GrK(A)$.

2. Ungraded twisted K-theory in the finite and infinite-dimensional cases.

2.1. Let X be a compact space and let us consider bundles of algebras $\mathcal{A}$ with fiber $M_n(\mathbf{C})$. As was shown by Serre [G], such bundles are classified by Cech cocycles (up to Cech coboundaries) :

$$g_{ji} : U_i \cap U_j \to PU(n)$$

where $PU(n)$ is $U(n)/S^1$, the projective unitary group. In other words, the bundle of algebras $\mathcal{A}$ may be obtained by glueing together the bundles of C*-algebras (U_i x $M_n(\mathbf{C})$), using the transition functions g_{ji} [Note that $U(n)$ acts on $M_n(\mathbf{C})$ by inner automorphisms and therefore induces an action of $PU(n)$ on the algebra $M_n(\mathbf{C})$]. The Brauer group of X denoted by $Br(X)$ is obtained as the quotient of this semi-group of bundles (via the tensor product) by the following equivalence relation : $\mathcal{A}$ is equivalent to $\mathcal{A}'$ iff there exist vector bundles V and V' such that the bundles of algebras $\mathcal{A} \otimes END(V)$ and $\mathcal{A}' \otimes END(V')$ are isomorphic. It was proved by Serre [G] that $Br(X)$ is naturally isomorphic to the torsion subgroup of $H^3(X ; \mathbf{Z})$.

The Serre-Swan theorem (cf. [K1] for instance) may be easily translated in this situation to show that the category of finitely generated $\mathcal{A}$-module bundles E (as in [DK]) is equivalent to the category $\mathcal{P}(A)$ of finitely generated projective modules over $A = \Gamma(X , \mathcal{A})$, the algebra of continuous sections of the bundle $\mathcal{A}$. The key observation for the proof is that E is a direct

factor of a "trivial" $\mathcal{A}$-bundle ; this is easily seen with finite partitions of unity, since X is compact. One should notice that if $\mathcal{A}$ is equivalent to $\mathcal{A}$' the associated categories $\mathcal{P}(A)$ and $\mathcal{P}(A')$ are equivalent.

2.2. DEFINITION. *The ungraded* [5] *twisted K-theory $K^{(\mathcal{A})}(X)$ is by definition the K-theory of the ring A (which is the same as the K-theory of the category $E^{\mathcal{A}}(X)$ mentioned in the introduction). By abuse of notation, we shall simply call it $K(\mathcal{A})$. We also define $K_n(\mathcal{A})$ as the K_n-group of the Banach algebra $\Gamma(X, \mathcal{A})$. It only depends on the class of $\mathcal{A}$ in $Br(X) = Tors(H^3(X; \mathbf{Z}))$.*

2.3. The key observation made by J. Rosenberg [R] is the following : we can "stabilize" the situation (in the C*-algebra sense), i.e. embed $M_n(\mathbf{C})$ into the algebra of compact operators $\mathcal{K}$ in a separable Hilbert space H, thanks to the split inclusion of $\mathbf{C}^n$ in $l^2(\mathbf{N})$. Now, a bigger group $PU(H) = U(H)/S^1$ is acting on $\mathcal{K}$ by inner automorphisms. If we take a Cech cocycle

$$g_{ji} : U_i \cap U_j \to PU(H)$$

we may use it to construct a bundle $\mathcal{A}$ of (non unital) C*-algebras with fiber $\mathcal{K}$.

Let us now consider the commutative diagram

$$\begin{array}{ccc} S^1 & \to & S^1 \\ \downarrow & & \downarrow \\ U(n) & \to & U(H) \\ \downarrow & & \downarrow \\ PU(n) & \to & PU(H) \end{array}$$

Thanks to Kuiper's theorem [Ku], we remark that the classifying space of $U(H)$ is contractible. Therefore, the classifying space $BPU(H)$ of the topological group $PU(H)$, is a nice model of the Eilenberg-Mac Lane space $K(\mathbf{Z}, 3)$ (compare with the well-known paper of Dixmier and Douady [DD]). Moreover, if we start with a finite-dimensional algebra bundle $\mathcal{A}$ over X with fiber $M_n(\mathbf{C})$, the diagram above shows how to associate to $\mathcal{A}$ another bundle of algebras $\mathcal{A}$' with fiber $\mathcal{K}$, together with a C*-inclusion from $\mathcal{A}$ to $\mathcal{A}$'. We note that the invariant $W_3(\mathcal{A})$ in $Br(X) = Tors(H^3(X ; \mathbf{Z}))$ defined in [G] is simply induced by the classifying map from X to $BPU(H)$ (which factors through $BPU(n)$). In this finite example, it is an n-torsion class since one has another commutative diagram

$$\begin{array}{ccc} \mu_n & \to & S^1 \\ \downarrow & & \downarrow \\ SU(n) & \to & U(n) \\ \downarrow & & \downarrow \\ PU(n) & \to & PU(n) \end{array}$$

[5] We use the notation $K^{(\mathcal{A})}(X)$, not to be confused with the graded twisted K- group $K^{\mathcal{A}}(X)$ which will be defined in the next section.

2.4. THEOREM. *The inclusion from $\mathcal{A}$ to $\mathcal{A}'$ induces an isomorphism*

$$K_r(\mathcal{A}) = K_r(\Gamma(X, \mathcal{A})) \to K_r(\Gamma(X, \mathcal{A}')) = K_r(\mathcal{A}')$$

where K_r is the classical topological K-theory of C-algebras.*

Proof. The proof is classical for a trivial algebra bundle, since **C** is Morita equivalent to $\mathcal{K}$ (in the C*-algebra sense). It extends to the general case by a no less classical Mayer-Vietoris argument.

2.5. DEFINITION. *Let now $\mathcal{A}$ be an algebra bundle with fiber $\mathcal{K}$ on a compact space X* <u>*with structural group*</u> *PU(H). We define $K^{(\mathcal{A})}(X)$ (also denoted by $K(\mathcal{A})$) as the K-theory of the (non unital) Banach algebra $\Gamma(X, \mathcal{A})$. This K-theory only depends of the class α of $\mathcal{A}$ in $H^3(X\,;\mathbf{Z})$ and we shall also call it $K^{(\alpha)}(X)$* [Due to 2.4, this is a genereralization of definition 2.2].

2.6. Before treating the graded case in the next section, we would like to give an equivalent definition of $K(\mathcal{A})$ in terms of Fredholm operators, as was done in [DK] for the torsion elements in $H^3(X\,;\mathbf{Z})$ and, later on, in [AS] for the general case. The basic idea is to remark that $PU(H)$ acts not only on the C*-algebra of compact operators in H, but also on the ring of bounded operators $End(H)$ and on the Calkin algebra $End(H)/\mathcal{K}$ (with the norm topology). Let us call $\mathcal{B}$ the algebra bundle with fiber $End(H)$ associated to the cocycle defined in 2.1 and $\mathcal{B}/\mathcal{A}$ the quotient algebra bundle. Therefore, we have an exact sequence of C*-algebra bundles

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0$$

which induces an exact sequence for the associated rings of sections (thanks to a partition of unity again)

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B}) \xrightarrow{\pi} \Gamma(X, \mathcal{B}/\mathcal{A})) \to 0$$

If $\mathcal{B}$ is trivial, it is well-known that the algebra of continuous maps from X to $End(H)$ has trivial K_n-groups because this algebra is flabby[6]. By a Mayer-Vietoris argument, it follows that $K_n(\Gamma(X, \mathcal{B}))$ is also trivial. Therefore the connecting homomorphism

$$K_1(\mathcal{B}/\mathcal{A}) = K_1(\Gamma(X, \mathcal{B}/\mathcal{A})) \to K_0(\Gamma(X, \mathcal{A})) \cong K^{(\mathcal{A})}(X) = K(\mathcal{A})$$

is an isomorphism, a well-known observation in index theory.

2.7. Let us now consider the elements of $\mathcal{B}$ which map onto $(\mathcal{B}/\mathcal{A})^*$ via the map π. These elements form a bundle of Fredholm operators on H (the twist comes from the action of $PU(H)$). This subbundle of $\mathcal{B}$ will be denoted by $\mathcal{Fredh}(H)$. Therefore, we have a principal fibration

$$\Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{Fredh}(H)) \to \Gamma(X, (\mathcal{B}/\mathcal{A})^*)$$

[6] A unital Banach algebra Λ is called flabby if there exists a continuous functor τ from $\mathcal{P}(A)$ to itself such that τ + Id is isomorphic to τ. For instance, $A = End(H)$ is flabby since $\mathcal{P}(A)$ is equivalent to the category C of Hilbert spaces which are isomorphic to direct factors in H ; τ is then defined by the infinite Hilbert sum $\tau(E) = E \oplus \ldots \oplus E \oplus \ldots$.

with contractible fiber the Banach space $\Gamma(X, \mathcal{A})$ (this fibration admits a local section thanks to Michael's theorem [Mi]). Therefore, the space of sections of $\mathcal{F}redh(H)$ has the homotopy type of $\Gamma(X, (\mathcal{B}/\mathcal{A})^*)$. In particular, the path components are in bijective correspondence via the map π. The following theorem is a generalization of a well-known theorem of Atiyah and Janich [A1] [J] :

2.8. THEOREM [AS]. *The set of homotopy classes of continuous sections of the fibration*

$$\mathcal{F}redh(H) \to X$$

is naturally isomorphic to $K^{(\mathcal{A})}(X)$.

Proof. As we have seen above, the two spaces $\Gamma(X, \mathcal{F}redh(H))$ and $\Gamma(X, (\mathcal{B}/\mathcal{A})^*)$ have the same homotopy type. On the other hand, it is a well-known consequence of Kuiper's theorem [Ku] than the (non unital) ring map $\Gamma(X, \mathcal{B}/\mathcal{A}) \to \Gamma(X, \mathrm{M}_r(\mathcal{B}/\mathcal{A}))$ induces a bijection between the path components of the associated groups of invertible elements (see for instance [K5] p. 93). Therefore, $\pi_0(\Gamma(X, \mathcal{F}redh(H)))$ may be identified with

$$\lim_{\substack{\rightarrow \\ r}} \pi_0(\Gamma(X, GL_r(\mathcal{B}/\mathcal{A}))) = K_1(\mathcal{B}/\mathcal{A})$$

and therefore with $K(\mathcal{A})$, as we already mentioned in 2.6.

2.9. Remark. We may also consider the following "stabilized" bundle

$$\mathcal{F}redh_s(H) = \lim_{\substack{\rightarrow \\ n}} \mathcal{F}redh(H^n)$$

and, without Kuiper's theorem, prove almost in the same way that the connected component of the space of sections of this bundle is also isomorphic to $K^{(\mathcal{A})}(X)$.

2.10. There is an obvious ring homomorphism (since the Hilbert tensor product $H \otimes H$ is isomorphic to H)

$$\mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$$

If $\mathcal{A}$ and $\mathcal{A}'$ are bundles of algebras on X modelled on $\mathcal{K}$, we may use this homomorphism to get a new algebra bundle on X, which we denote by $\mathcal{A} \otimes \mathcal{A}'$ From the cocycle point of view, we have a commutative diagram, where the top arrow is induced by complex multiplication

$$\begin{array}{ccc} S^1 \times S^1 & \to & S^1 \\ \downarrow & & \downarrow \\ U(H) \times U(H) & \to & U(H \otimes H) \\ \downarrow & & \downarrow \\ PU(H) \times PU(H) & \to & PU(H \otimes H) \end{array}$$

It follows that $W_3(\mathcal{A} \otimes \mathcal{A}') = W_3(\mathcal{A}) + W_3(\mathcal{A}')$ in $H^3(X ; \mathbf{Z})$ and one gets a "cup-product"

$$K^{(\alpha)}(X) \times K^{(\alpha')}(X) \to K^{(\alpha + \alpha')}(X)$$

This is a particular case of a "graded cup-product" which will be introduced in the next section.

3. Graded twisted K-theory in the finite and infinite-dimensional cases.

3.1. We are going to change our point of view and now consider $\mathbf{Z}/2$-graded finite-dimensional algebras which are central and simple (in the graded sense). We are only interested in the complex case. The real case is treated with great details in [DK] and does not seem to generalize to the infinite-dimensional framework[7].

In the complex case, there are just two "types" of graded algebras (up to Morita equivalence[8]) which are $R = \mathbf{C}$ and $\mathbf{C} \times \mathbf{C} = \mathbf{C}[x]/(x^2 - 1)$. For a type R of algebra, the graded inner automorphisms of $A = R \hat{\otimes} End(V \oplus W)$ may be given by either an element of degree 0 or an element of degree 1 in A^*. This gives us an augmentation (whose kernel is denoted by $Aut^0(A)$) :

$$Aut\,(A) \rightarrow \mathbf{Z}/2$$

Therefore, for bundles of $\mathbf{Z}/2$-graded algebras modelled on A, we already get an invariant in $H^1(X\,;\mathbf{Z}/2)$, called the "orientation" of A and which may be represented by a line bundle. A typical example is the (complexified) Clifford bundle $C(V)$, associated to a real vector bundle V of rank n. Its orientation invariant is the first Stiefel-Whitney class associated to V (cf. [DK]). Note that the type R of $C(V)$ is $\mathbf{C}$ if n is even and $\mathbf{C} \times \mathbf{C}$ if n is odd[9].

3.2. For the second invariant, let us start with $M_{2n}(\mathbf{C})$ as a basic graded algebra to fix the ideas and let us put a C*-algebra metric on A. We have an exact sequence of groups as in the ungraded case (where $PU^0(2n) = PU(2n) \cap Aut^0(A)$)

$$1 \rightarrow S^1 \rightarrow U(n) \times U(n) \rightarrow PU^0(2n) \rightarrow 1$$

Therefore, a bundle with structural group $PU^0(2n)$ also has a class in $H^2(X\,;\boldsymbol{S}^1) = H^3(X\,;\mathbf{Z})$ which is easily seen to have order n as in the ungraded case. A similar argument holds if the graded algebra is $M_n(\boldsymbol{C}) \times M_n(\boldsymbol{C})$. It follows that the "graded Brauer group" $GBr(X)$ is

$$GBr(X) = \mathbf{Z}/2 \times H^1(X\,;\mathbf{Z}/2) \times Tors(H^3(X\,;\mathbf{Z}))$$

as already quoted in the introduction (see [DK] for the explicit group law on $GBr(X)$). If $\mathcal{A}$ is a bundle of $\mathbf{Z}/2$-graded finite-dimensional algebras, we define the graded twisted K-theory $K^{\mathcal{A}}(X)$ as the graded K-theory of the graded algebra $\Gamma(X, \mathcal{A})$ as recalled in Section 1. This definition only depends on the class of $\mathcal{A}$ in $GBr(X)$. We recover the definition in [DK] by using again the Serre-Swan theorem as in § 2. For instance, if we consider the bundle of (complex) Clifford algebras $C(V)$, associated to a real vector bundle V, the invariants we get are $w_1(V)$ and $W_3(V)$, as quoted in the introduction. If these invariants are trivial, the bundle V is ^{c}spinorial of even rank and $C(V)$ may be identified with the bundle of endomorphisms $End(S^+ \oplus S^-)$, where S^+ and S^- are the even and odd "spinors" associated to the $Spin^c$-structure.

[7] This is not quite true if we work in the context of "Real" K-theory in the sense of Atiyah [A1]. We shall not consider this generalization here, although it looks interesting in the light of "equivariant twisted K-theory" as we shall show in § 5.

[8] This means that we are allowed to take the graded tensor product with $End(V \oplus W)$ with the obvious grading.

[9] If V is oriented and even dimensional for instance, this does not imply that $C(V)$ is a bundle of graded algebras of type $M_2(\Lambda)$ for a certain bundle of ungraded algebras Λ. However, for suitable vector bundles V and W, this is the case for the graded tensor product $C(V) \hat{\otimes} End(V \oplus W)$.

3.3. In order to define graded twisted K-theory in the infinite-dimensional case, we follow the same pattern as in § 2. For instance, let us take a graded bundle of algebras $\mathcal{A}$ modelled on $M_2(\mathcal{K})$: it has 2 invariants, one in $H^1(X\ ;\ \mathbf{Z}/2)$, the other in $H^3(X\ ;\ \mathbf{Z})$ (and not just in the torsion of this group). We then define $K^{\mathcal{A}}(X)$ as the graded K-theory of the graded algebra $\Gamma(X, \mathcal{A})$, according to § 1. The same definition holds for bundles of graded algebras modelled on $\mathcal{K} \times \mathcal{K} = \mathcal{K}[x]/(x^2 - 1) = \mathcal{K} \oplus \mathcal{K}x$.
If $C^{0,1}$ is the Clifford algebra $\mathbf{C} \times \mathbf{C}$ with its usual graded structure, the general results of § 1 show that the group $K^{\mathcal{A}}(X)$ fits into an exact sequence :

$$K_1^{(\mathcal{A}\, \hat{\otimes}\, C^{0,1})}(X) \to K_1^{(\mathcal{A})}(X) \to K^{\mathcal{A}}(X) \to K^{(\mathcal{A}\, \hat{\otimes}\, C^{0,1})}(X) \to K^{(\mathcal{A})}(X)$$

where $K_i^{(\mathcal{A})}(X)$ denotes in general the K_i-group of the Banach algebra $\Gamma(X\ ,\ \mathcal{A})$.

3.4. Let us now assume that $\mathcal{A}$ is oriented modelled on $M_2(\mathcal{K})$. We are going to show that $\mathcal{A}$ may be written as $M_2(\mathcal{A}')$, with the obvious grading, $\mathcal{A}'$ being an ungraded bundle of algebras modelled on $\mathcal{K}$. For this purpose, we write the commutative diagram

$$\begin{array}{ccc} S^1 & \to & S^1 \\ \downarrow & & \downarrow \\ U(H) & \to & U(H) \times U(H) \\ \downarrow & & \downarrow \\ PU(H) & \to & PU^0(H \oplus H) \end{array}$$

where the first horizontal map is the identity and the others are induced by the diagonal. This shows that $H^1(X\ ;\ PU(H)) \cong H^1(X\ ;\ PU^0(H \oplus H))$, which is equivalent to saying that $\mathcal{A}$ may be written as $M_2(\mathcal{A}')$ for a certain bundle of algebras $\mathcal{A}'$.

Therefore, $\mathcal{A}\, \hat{\otimes}\, C^{0,1}$ is Morita equivalent to $\mathcal{A}' \times \mathcal{A}'$ and $K^{\mathcal{A}}(X)$ is the K-theory of the ring homomorphism

$$\mathcal{A}' \times \mathcal{A}' \to M_2(\mathcal{A}')$$

defined by $(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ (no grading). We should also note that $\mathcal{A}'$ is Morita equivalent to $\mathcal{A}$ as an ungraded bundle of algebras. Since the usual K-theory (resp. graded K-theory) is invariant under Morita equivalence (resp. graded Morita equivalence), the previous considerations lead to the following theorem :

3.5. THEOREM. *Let $\mathcal{A}$ be an oriented bundle of graded algebras modelled on $M_2(\mathcal{K})$. Then $K^{\mathcal{A}}(X)$ is isomorphic to $K^{(\mathcal{A})}(X)$ via the identification above.*

3.6. The same method may be applied in the case when $\mathcal{A}$ is an oriented bundle of graded algebras modelled on $\mathcal{K} \times \mathcal{K} = \mathcal{K}[x]/(x^2 - 1)$. The graded oriented automorphisms of $\mathcal{K} \times \mathcal{K}$ induced by $PU(H \oplus H)$ are diagonal matrices of type

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

This shows that $\mathcal{A}$ is isomorphic to $\mathcal{A}$' x $\mathcal{A}$' and $\mathcal{A} \mathbin{\hat{\otimes}} C^{0,1}$ is isomorphic to $M_2(\mathcal{A}')$. Therefore, $K^{\mathcal{A}}(X)$ is the Grothendieck group of the ring homomorphism $\mathcal{A}' \to M_2(\mathcal{A}')$, defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Hence we have the following theorem, analogous to 3.5 :

3.7. THEOREM. *Let $\mathcal{A}$ be an <u>oriented</u> bundle of graded algebras modelled on $\mathcal{K}$ x $\mathcal{K}$. Then $\mathcal{A}$ is isomorphic to $\mathcal{A}$' x $\mathcal{A}$' and the group $K^{\mathcal{A}}(X)$ is isomorphic to $K_1(\mathcal{A}')$ via the identification above.*

3.8. Remark. One may notice that if $\mathcal{A}$ is a bundle of oriented graded algebras modelled on $M_2(\mathcal{K})$, the associated bundle with fiber $M_2(\mathcal{K}^+)$ has a section ε of degree 0 and of square 1, which commutes (resp. anticommutes) with the elements of degree 0 (resp. 1) ; it is simply defined by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.9. Finally, we would like to give an equivalent definition of $K^{\mathcal{A}}(X)$ in terms of Fredholm operators as in § 2. This was done in [DK] if the class of $\mathcal{A}$ belongs to the torsion group of $H^3(X\,;\mathbf{Z})$ and later on in [AS] for the general case. We shall give a simplified treatment here, using again the *K*-theory of graded Banach algebras.

Following the general notations of § 2, we have an exact sequence of bundles of graded Banach algebras

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0$$

Since the graded *K*-groups of $\mathcal{B}$ are trivial, we see as in 2.6 that $GrK(\Gamma(X\,,\mathcal{A}))$ is isomorphic to $GrK_1(\Gamma(X\,,\mathcal{B}/\mathcal{A}))$.

In order to shorten the notations, we denote by B the graded Banach algebra $\Gamma(X\,,\mathcal{B})$, by Λ the graded Banach algebra $\Gamma(X\,,\mathcal{B}/\mathcal{A})$ and by ϕ the surjective map $B \to \Lambda$. The following lemma[10] may be proved in the same way as in [K2] p. 78 :

3.10. LEMMA. *Any element of $GrK_1(\Lambda)$ may be written as the homotopy class of a couple (E, ε) where E is a free graded B-module and ε is a grading of degree one of the associated Λ-module (see 1.4 for the definition of a grading).*

[10] It might be helpful for a better understanding to notice that the category of finitely generated free *End*(*H*)-modules is equivalent to the category of Hilbert spaces H^n for $n \in \mathbf{N}$. This "local" situation is twisted by the group *PU*(*H*).

3.11. By the well-known dictionary between modules and bundle theory, we may view ε as a grading of a suitable bundle E of free $End(H)/\mathcal{K}$-modules. By spectral theory, we may also assume that ε is self-adjoint. Finally, following [K2], we define a quasi-grading[11] of E as a family of Fredholm endomorphisms D such that

1) $D^* = D$

2) D is of degree one

The following theorem is the analogue in the graded case of Theorem 2.8 (cf [K2] p. 78/79).

3.12. THEOREM. *The (graded) twisted K-group $K^{\mathcal{A}}(X)$ is the Grothendieck group associated to the semi-group of homotopy classes of couples (E, D) where E is a free* $\mathbf{Z}/2$*-graded $\mathcal{B}$-module and D is a family of Fredholm endomorphisms of E which are self-adjoint and of degree 1.*

3.13. Remark. Let us assume that $\mathcal{A}$ is oriented modelled on $M_2(\mathcal{K})$. The description above gives a Fredholm description of $GrK_1(\mathcal{A}) = K_1(\mathcal{A})$: we just take the homotopy classes of sections of the associated bundle of self-adjoint Fredholm operators $\mathcal{Fredholm}^*(\mathcal{B})$ whose essential spectrum is divided into two non empty parts[12] in $\mathbf{R}^{+*}$ and $\mathbf{R}^{-*}$

3.14. This Fredholm description of $K^{\mathcal{A}}(X)$ enables us to define a cup-product

$$K^{\mathcal{A}}(X) \times K^{\mathcal{A}'}(X) \to K^{\mathcal{A} \hat{\otimes} \mathcal{A}'}(X)$$

where $\mathcal{A} \hat{\otimes} \mathcal{A}'$ denotes the graded tensor product of $\mathcal{A}$ and $\mathcal{A}'$. This cup-product is given by the same formula as in [DK] p. 19 and generalizes it :

$$(E, D) \cup (E', D') = (E \hat{\otimes} E', D \hat{\otimes} 1 + 1 \hat{\otimes} D')$$

(see also [AS] p. 20 for a later reference).

3.15. To conclude this section, let us consider a locally compact space X and a bundle of graded algebras $\mathcal{A}$ on X. For technical reasons, we assume the existence of a compact space Z containing X as an open subset, such that $\mathcal{A}$ extends to a bundle (also called $\mathcal{A}$) on Z. There is an obvious definition of $K^{\mathcal{A}}(X)$ as a relative term in the following exact sequence (where $T = Z - X$ and $\mathcal{A}' = \mathcal{A} \hat{\otimes} C^{0,1}$) :

$$K^{\mathcal{A}'}(Z) \to K^{\mathcal{A}'}(T) \to K^{\mathcal{A}}(X) \to K^{\mathcal{A}}(Z) \to K^{\mathcal{A}}(T)$$

By the usual excision theorem in topological K-theory, one may prove that this definition of $K^{\mathcal{A}}(X)$ is independent from the choice of Z.

The method described before and also in [K2] § 3 shows how to generalize the definition of $K^{\mathcal{A}}(X)$ in this case : one takes homotopy classes of couples (E, D) as in 3.12, with the added assumption that the family D is acyclic at infinity. In other words, there is a compact set $S \subset X$, such that D_x is an isomorphism when $x \notin S$ (see [K2] p. 89-97 for the technical details of this approach). This Fredholm description of $K^{\mathcal{A}}(X)$ will be important in the next section for the description of the Thom isomorphism.

[11] "quasi-graduation" in French.

[12] There are two independent references in the non twisted case, i.e. for the usual classifying spaces of real or complex K-theory, which are [ASi] and [K5].

4. The Thom isomorphism in twisted K-theory[13].

4.1. Let V be a finite-dimensional real vector bundle on a locally compact space X which extends over a compactification of X as was assumed in 3.15. Then the complexified Clifford bundle $C(V)$ has a well-defined class in the (graded) twisted K-theory of X. If $\mathcal{A}$ is another twist on X, we can consider the graded tensor product $\mathcal{A} \hat{\otimes} C(V)$ and the associated group $K^{\mathcal{A} \hat{\otimes} C(V)}(X)$. As was described in 3.15, it is the group[14] associated to couples (E, D) where E is a graded bundle of free $\mathcal{B}$-modules and D is a family of Fredholm endomorphisms which are of degree 1, self-adjoint and acyclic at ∞ . Let us now consider the projection $\pi : X \times V \to X$. For simplicity's sake, we shall often call E, $\mathcal{A}$, $\mathcal{B}$, ... the respective pull-backs of E, $\mathcal{A}$, $\mathcal{B}$, ... via this projection. Since $\mathcal{B}$ and $C(V)$ are subbundles of $\mathcal{B} \hat{\otimes} C(V)$, E may be provided with induced $\mathcal{B}$-module and $C(V)$-module structures.

4.2. THEOREM. *Let $d(E, D)$ be an element of $K^{\mathcal{A} \hat{\otimes} C(V)}(X)$ with the notations above. We define an element $t(d(E, D))$ in the group $K^{\mathcal{A}}(V)$ as $d(\pi^*(E), D')$ where D' is the family of Fredholm operators on $\pi^*(E)$, defined over the point v in V (with projection x on X) as*

$$D'_{(x,v)} = D_x + \rho(v)$$

where $\rho(v)$ denotes the action of the element v of the vector bundle V considered as a subbundle of $C(V)$. The homomorphism

$$t : K^{\mathcal{A} \hat{\otimes} C(V)}(X) \to K^{\mathcal{A}}(V)$$

(t for "Thom") defined by the formula above is then an isomorphism.

Proof[15]**.** We should first notice that V may be identified with the open unit ball bundle of the vector bundle V and is therefore an open subset of the closed unit ball bundle of V. Moreover, since V extends to a compactification of X, as well as $\mathcal{A}$, the required conditions in 3.15 for the definition of the twisted K-theory of X and V are fulfilled.

We shall now provide two different proofs of the theorem.

The first one, more elementary in spirit, consists in using a Mayer Vietoris argument which we can apply here since the two sides of the formula above behave as cohomology theories[16] with respect to the base X. Therefore, we may assume that $\mathcal{A}$ and V are trivial : this is a special case of the theorem stated in [K3] p. 211/212.

The second one is more subtle and may be generalized to the equivariant case. Let us first describe the Thom isomorphism for complex V. This is a slight modification of Atiyah's argument using the elliptic Dolbeault complex [A2]. More precisely, we consider the composite map

[13] See also [Ca].

[14] We should note that E is a $\mathcal{B}$ -module, not an $\mathcal{A}$-module. Nevertheless, we shall keep the notation $K^{\mathcal{A}}$.

[15] According to a suggestion of J. Rosenberg, it should be possible to give a proof with the KK-theory of Kasparov by describing an explicit inverse to the homomorphism t. However, KK-theory is out of the scope of this paper.

[16] Strictly speaking, one has to "derive" the two members of the formula, which can be done since they are Grothendieck groups of graded Banach categories.

$$\phi : K^{\mathcal{A}}(X) \to K^{\mathcal{A}\,\hat{\otimes}\,C(V)}(X) \xrightarrow{t} K^{\mathcal{A}}(V)$$

The first map θ is the cup-product with the algebraic "Thom class" which is ΛV provided with the classical Clifford graded module structure. This map is an isomorphism from well- known algebraic considerations (Morita equivalence). Therefore t is an isomorphism if and only if ϕ is an isomorphism. On the other hand, ϕ is just the cup-product with the topological Thom class T_V which belongs to the usual topological K-theory $K(V)$ of Atiyah and Hirzebruch [AH]. In order to prove that ϕ is an isomorphism, we may now use the exact sequence in 3.3 to reduce ourselves to the ungraded twisted case. In other words, it is enough to show that the cup-product with T_V induces an isomorphism

$$\psi \ : K^{(\mathcal{A})}(X) \to K^{(\mathcal{A})}(V)$$

In order to prove this last point, Atiyah defines a reverse map[17]

$$\psi' \ : K^{(\mathcal{A})}(\mathrm{V}) \to K^{(\mathcal{A})}(X)$$

He shows that $\psi' \ \psi$ = Id and, by an ingenious argument, deduces that $\psi \ \psi'$ = Id as well !

Now, as soon as Theorem 4.2 is proved for complex V, the general case follows from a trick already used in [K3] p. 241 : we consider the following three Thom homomorphisms which behave "transitively" :

$$K^{\mathcal{A}\,\hat{\otimes}\,C(V)\,\hat{\otimes}\,C(V)\,\hat{\otimes}\,C(V)}(X) \to K^{\mathcal{A}\,\hat{\otimes}\,C(V)\,\hat{\otimes}\,C(V)}(V) \to K^{\mathcal{A}\,\hat{\otimes}\,C(V)}(V \oplus V)$$

$$\to K^{\mathcal{A}}(V \oplus V \oplus V)$$

We know that the composites of two consecutive arrows are isomorphisms since $V \oplus V$ carries a complex structure. It follows that the first arrow is an isomorphism, which is essentially the stated theorem (using Morita equivalence again).

4.3. Let $\mathcal{A}$ be a graded twist. As we have seen before, it has two invariants in $H^1(X\,;\mathbf{Z}/2)$ and in $H^3(X\,;\mathbf{Z})$. The first one provides a line bundle L in such a way that the graded tensor product $\mathcal{A}_1 = \mathcal{A} \,\hat{\otimes}\, C(L)$ is oriented. From the Thom isomorphism and the considerations in 3.5/7, we deduce the following theorem which gives the relation between the ungraded and graded twisted K-groups.

4.4. THEOREM. *Let $\mathcal{A}$ be a graded twist of type $M_2(\mathcal{K})$ and $\mathcal{A}_1 = \mathcal{A} \,\hat{\otimes}\, C(L)$, where L is the orientation bundle of $\mathcal{A}$. Then $\mathcal{A}_1$ may be written as $\mathcal{A}_2$ x $\mathcal{A}_2$ where $\mathcal{A}_2$ is ungraded of type $\mathcal{K}$. Therefore, we have the following isomorphisms*

$$K^{\mathcal{A}}(X) \cong K^{\mathcal{A}\,\hat{\otimes}\,C(L)\,\hat{\otimes}\,C(L)}(X) \cong K^{\mathcal{A}_1}(L) \cong K_1^{(\mathcal{A}_2)}(L)$$

Let $\mathcal{A}$ be a graded twist of type $\mathcal{K}$ x $\mathcal{K}$. With the same notations, we have the following isomorphisms

$$K^{\mathcal{A}}(X) \cong K^{\mathcal{A}\,\hat{\otimes}\,C(L)\,\hat{\otimes}\,C(L)}(X) \cong K^{\mathcal{A}_1}(L) \cong K^{(\mathcal{A}_1)}(L)$$

[17] More precisely, one has to define an index map parametrized by a Banach bundle, which is also classical. [FM].

5. General equivariant *K*-theory.

Note : this section is inserted here for the convenience of the reader as an introduction to § 6. It is mainly a summary of results found in [S], [B], [LTX] and [AS].

5.1. Let A be a Banach algebra and G a compact Lie group acting on A via a continuous group homomorphism

$$G \to Aut(A)$$

where $Aut(A)$ is provided with the norm topology. We are interested in the category whose objects are finitely generated projective A-modules E together with a continuous left action of G on E such that we have the following identity ($g \in G$, $a \in A$, $e \in E$)

$$g.(a.e) = (g.a).(g.e)$$

with an obvious definition of the dots. We define $K_G(A)$ as the Grothendieck group of this category $\mathcal{C} = \mathcal{P}_G(A)$. By the well-known dictionary between bundles and modules, we recover the usual equivariant K-theory defined by Atiyah and Segal [S] if A is the ring of continuous maps on a compact space X (thanks to the lemma below). It may also be defined as a suitable semi-direct product of G by A (cf. [B]).
More generally, if A is a $\mathbf{Z}/2$-graded algebra (where G acts by degree 0 automorphisms), we define the graded equivariant K-theory of A, denoted by $GrK_G(A)$, as the Grothendieck group of the forgetful functor

$$\mathcal{C}^{0,1} \to \mathcal{C}$$

where $\mathcal{C}^{0,1}$ is the category of "graded objects" in $\mathcal{P}_G(A)$ which is defined as $\mathcal{P}_G(A \hat{\otimes} C^{0,1})$.

In the same spirit as in § 1, we define "derived" groups $K_G^{p,q}(A)$ as $GrK_G(A \hat{\otimes} C^{p,q})$. They satisfy the same formal properties as the usual groups $K_n(A)$ (also denoted by $K^{-n}(A)$ with $n = q - p$), for instance Bott periodicity. The following key lemma enables us to translate many general theorems of K-theory into the equivariant framework :

5.2. LEMMA. *Let E be an object of the category $\mathcal{P}_G(A)$. Then E is a direct summand of an object of type $A \otimes_{\mathbf{C}} M$ where M is a finite-dimensional G-module.*
Proof (compare with [S] p. 134). Let us consider the union Γ of all finite-dimensional invariant subspaces of the G-Banach space E. According to a version of the Peter-Weyl theorem theorem quoted in [S] (loc. cit.), this union is dense in A. We now consider a set $e_1, ..., e_n$ of generators of E as an A-module. Since Γ is dense in A and E is projective, one may choose these generators in the subspace Γ. Let $M_1, ..., M_n$ be finite-dimensional invariant subspaces of E containing $e_1, ..., e_n$ respectively and let M be the following direct sum

$$M = M_1 \oplus ... \oplus M_n$$

We define an equivariant surjection

$$\phi : A \otimes_{\mathbf{C}} M \cong (A \otimes_{\mathbf{C}} M_1) \oplus ... \oplus (A \otimes_{\mathbf{C}} M_n) \to E$$

between projective left A-modules by the formula

$$\phi(\lambda_1 \otimes m_1, ..., \lambda_n \otimes m_n) = \lambda_1 m_1 + ... + \lambda_n m_n$$

This surjection admits a section, which we can average out thanks to a Haar measure in order to make it equivariant. Therefore, E is a direct summand in $A \otimes_{\mathbf{C}} M$ as stated in the lemma.

5.3. As for usual K-theory, one may also define equivariant K-theory for non unital rings and, using Lemma 5.2 above, prove that any equivariant exact sequence of rings on which G acts

$$0 \to A' \to A \to A'' \to 0$$

induces an exact sequence of equivariant K-groups

$$K_G^{n-1}(A) \to K_G^{n-1}(A'') \to K_G^{n}(A') \to K_G^{n}(A) \to K_G^{n}(A'')$$

and an analogous exact sequence in the graded equivariant framework.

5.5. After these generalities, let us assume that G acts on a compact space X and let $\mathcal{A}$ be a bundle of algebras modelled on $\mathcal{K}$. We define the (ungraded) equivariant twisted K-group $K_G^{(\mathcal{A})}(X)$ as $K_G(A)$, where A is the Banach algebra of sections of the bundle $\mathcal{A}$. Similarly, if $\mathcal{A}$ is a bundle of graded algebras modelled on $\mathcal{K}$ x $\mathcal{K}$ or $M_2(\mathcal{K})$, we define the (graded) equivariant twisted K-group as GrK_G(A), where A is the graded Banach algebra of sections of $\mathcal{A}$.
As seen in § 3, it is also natural to consider free graded $\mathcal{B}$-modules together with a continuous action of G (compatible with the action on X) and a family of Fredholm operators D which are self-adjoint of degree one, commuting with the action of G. With the same ideas as in § 3, we can show that the Grothendieck group of this category is isomorphic to $K_G^{\mathcal{A}}(X)$. This is essentially the definition proposed in [AS].

5.6. An example of such twisted equivariant K-theory was given at the beginning of the history of twisted K-theory. In [DK] § 8, we defined a "power operation"

$$\mathrm{P} : K^{\mathcal{A}}(X) \to K_{S_n}^{\mathcal{A}^{\otimes n}}(X)$$

where S_n denotes the symmetric group on n letters acting on $\mathcal{A}^{\hat{\otimes}\, n}$ (in the spirit of Atiyah's paper on power operations [A1]). According to the general philosophy of Adams and Atiyah, one can deduce from this n^{th} power operation new "Adams operations" (for n <u>odd</u>):

$$\Psi^n : K^{\alpha}(X) \to K^{\alpha^n}(X) \otimes \Omega_n$$

where α belongs to $\mathbf{Z}/2$ x $H^1(X ; \mathbf{Z}/2)$ x $H^3(X ; \mathbf{Z})$ and Ω_n denotes the free $\mathbf{Z}$-module generated by the n^{th} roots of unity in $\mathbf{C}$ (the ring of cyclotomic integers). One can prove, following [DK], that these additive maps Ψ^n satisfy all the required properties proved by Adams. We shall come back to these operations in § 7, making the link with operations recently defined by Atiyah and Segal [AS2].

5.7. In order to fix ideas, let us consider the graded version of equivariant twisted K-theory $K_G^{\mathcal{A}}(X)$, where $\mathcal{A}$ is modelled on $\mathbf{C}$ x $\mathbf{C}$ with the obvious grading. We have again a "Thom isomorphism" :

$$t: K_G^{\mathcal{A}\hat{\otimes}C(V)}(X) \to K_G^{\mathcal{A}}(V)$$

whose proof is the same as in the non equivariant case (using elliptic operators). On the other hand, if L is the orientation line bundle of $\mathcal{A}$, the group G acts on L in a way compatible with the action on X. Therefore, $\mathcal{A}_1 = \mathcal{A} \hat{\otimes} C(L)$ is an algebra bundle with trivial orientation. and $\mathcal{A}_1 \hat{\otimes} C^{0,1}$ is naturally isomorphic to $\mathcal{A}_1$ x $\mathcal{A}_1 = \mathcal{A}_1[x]/(x^2 - 1)$. As in 4.4, this shows that the graded twisted group $K_G^{\mathcal{A}}(X)$ is naturally isomorphic to the ungraded twisted K-group $K_G^{(\mathcal{A}_1)}(L)$. The same method shows that we can reduce graded twisted K-groups to ungraded ones if $\mathcal{A}$ is a bundle of graded algebras modelled on $M_2(\mathcal{K})$.

5.8. Let us mention finally one of the main contributions of Atiyah and Segal to the subject ([AS] § 6, see also [CTX]), which we interpret in our language. One is interested in algebra bundles $\mathcal{A}$ on X modelled on $\mathcal{K}$, provided with a left G-action. The isomorphism classes of such bundles are in bijective correspondence with principal bundles P over $PU(H)$ (acting on the right) together with a left action of G.
Such an algebra bundle $\mathcal{T}$ is called "trivial" if $\mathcal{T}$ may be written as the bundle of algebras of compact operators in $End(V)$, where V is a G-Hilbert bundle. Equivalently, this means that the structure group of $\mathcal{T}$ can be lifted equivariantly to $U(H)$ (in a way compatible with the G-action).
We now say that two such algebra bundles $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if there exist two trivial algebra bundles $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{A} \otimes \mathcal{T}$ and $\mathcal{A}' \otimes \mathcal{T}'$ are isomorphic as G-bundles of algebras. The quotient is a group since the dual of $\mathcal{A}$ is its inverse via the tensor product of principal $PU(H)$-bundles. This is the "equivariant Brauer group" $Br_G(X)$.
A closely related definition (in the framework of C*-algebras and for a locally compact group G) has been given in [CKRW]. It is very likely that it coincides with this one for compact Lie groups, in the light of a very interesting filtration described in this paper, which is probably associated to a spectral sequence.

5.9. THEOREM (cf. [AS] prop. 6.3). *Let $X_G = EG \times_G X$ be the Borel space associated to X. Then the natural map*

$$Br_G(X) \to Br(X_G)$$

is an isomorphism.

5.10. The interest of this theorem lies in the fact that the equivariant K-theories $K_G(\Gamma(X, \mathcal{A}))$ and $K_G(\Gamma(X, \mathcal{A}'))$ are isomorphic if $\mathcal{A}$ and $\mathcal{A}'$ are equivalent. This follows from the well-known Morita invariance in operator K-theory. We shall study concrete applications of this principle in the next section.

6. Some computations of twisted equivariant K-groups.

6.1. Let us look at the particular case of the ungraded twisted K-groups $K_G^{(\mathcal{A})}(X)$ where G is a finite group acting on the trivial bundle of algebras $\mathcal{A} = X \times M_n(\mathbf{C})$ via a group homomorphism $G \to PU(n)$ [18]. We define $\tilde{G}$ as the pull-back diagram

$$\begin{array}{ccc} \tilde{G} & \to & SU(n) \\ \downarrow & & \downarrow \\ G & \to & PU(n) \end{array}$$

Therefore, $\tilde{G}$ is a central covering of G with fiber μ_n (whose elements are denoted by Greek letters such as λ). The following definition is already present in [K4] § 2.5 (for n = 2) :

6.2. DEFINITION. *A finite-dimensional representation ρ of $\tilde{G}$ is of "linear type" if $\rho(\lambda u) = \lambda\, \rho(u)$ for any $\lambda \in \mu_n$.*

We now consider the category $E_{\tilde{G}}^{\mathcal{A}}(X)_l$ whose objects are $\tilde{G}$ -$\mathcal{A}$-modules as before, except that we request that the $\tilde{G}$ -action be of linear type <u>and commute with the action of $\mathcal{A}$</u>. By Morita invariance, $E_{\tilde{G}}^{\mathcal{A}}(X)_l$ is equivalent to the category $E_{\tilde{G}}(X)_l$ of finite-dimensional $\tilde{G}$ -bundles on X, the action of $\tilde{G}$ on the fibers being of linear type .

6.3. THEOREM[19]**.** *The (ungraded) twisted K-theory $K_G^{(\mathcal{A})}(X)$ is canonically isomorphic to the Grothendieck group of the category $E_{\tilde{G}}(X)_l$.*

Proof. One just repeats the argument in the proof of Theorem 2.6 in [K4], where A is a Clifford algebra $C(V)$ and $\mathbf{Z}/2$ plays the role of μ_n. We simply "untwist" the action of G thanks to the formula (F) written explicitly in the proof of 2.6 (loc. cit.).

6.4. For $\mathcal{A} = X \times A$ with $A = M_n(\mathbf{C})$, the previous argument shows that $K_G^{(\mathcal{A})}(X)$ is a subgroup of the usual equivariant K-theory $K_{\tilde{G}}(X)$. From now on, we shall write $K_G^{(A)}(X)$ instead of $K_G^{(\mathcal{A})}(X)$. Similarly, in the graded case ($A = M_n(\mathbf{C}) \times M_n(\mathbf{C})$ or $M_{2n}(\mathbf{C})$), we shall write $K_G^{A}(X)$ instead of $K_G^{\mathcal{A}}(X)$. If X is a point and G is finite, $K_G^{(A)}(X)$ is just the K-theory of the semi-direct product $G \prec A$.

6.5. THEOREM. *Let G be a finite group acting on the algebra of matrices $A = M_n(\mathbf{C})$ and let $\tilde{G}$ be the central extension G by μ_n described in 6.1. Then, for X reduced to a point, the group $K_G^{(\mathcal{A})}(X) = K(G \prec A)$ is a free group of rank the number of conjugacy classes in G which split into n conjugacy classes in $\tilde{G}$.*

[18] However, we don't assume that G acts trivially on X in general.

[19] There is an obvious generalization when A is infinite-dimensional. However, for our computations, we restrict ourselves to the finite-dimensional case.

Proof. We can apply the same techniques as the ones detailed in [K4] § 2.6/12 (for $n = 2$). By the theory of characters on $\tilde{G}$, one is looking for functions f on $\tilde{G}$ (which we call of "linear type") such that

1) $f(hgh^{-1}) = f(g)$

2) $f(\lambda x) = \lambda f(x)$ if λ is an n^{th} root of the unity

The **C**-vector space of such functions is in bijective correspondence with the space of functions on the set of conjugacy classes of G which split into n conjugacy classes of $\tilde{G}$.

6.6. Like the Brauer group of a space X, one may define in a similar way the Brauer group $Br(G)$ of a finite group G by considering algebras $A = M_n(\mathbf{C})$ as above with a G-action (see [FW] for a broader perspective ; this is also a special case of the general theory of Atiyah and Segal mentioned at the end of § 5). From the diagram written in 6.1, one deduces a cohomology invariant

$$w_2(A) \in H^2(G\,;\mu_n)$$

and therefore (via the Bockstein homomorphism) a second invariant $W_3(A) \in H^2(G\,;S^1) = H^2(G\,;\mathbf{Q}/\mathbf{Z}) = H^3(G\,;\mathbf{Z})$. It is easy to show that this correspondence induces a well-defined map

$$W_3 : Br(G) \to H^3(G\,;\mathbf{Z})$$

The following theorem is a very special case of 5.9 in a more algebraic situation.

6.7. THEOREM. *Let G be a finite group. Then the previous homomorphism*

$$W_3 : Br(G) \to H^3(G\,;\mathbf{Z})$$

is bijective.

Proof. First of all, we remark that $H^3(G\,;\mathbf{Z}) \cong H^2(G\,;\mathbf{Q}/\mathbf{Z})$ is the direct limit of the groups $H^2(G\,;\mu_m)$ through the maps $H^2(G\,;\mu_m) \to H^2(G\,;\mu_p)$ when m divides p. This stabilization process corresponds on the level of algebras to the tensor product $A \mapsto A \otimes End(V)$, where V is a G-vector space of dimension p/m. Therefore, the map W_3 is injective.

The proof of the surjectivity is deeper (see also 5.9). It relies on a much more general result proved by A. Fröhlich and C.T.C. Wall [FW] about the equivariant Brauer group of an arbitrary field k : there is a split exact sequence (with their notations)

$$0 \to Br(k) \to BM(k, G) \to H^2(G\,;U(k)) \to 0$$

where $Br(k)$ is the usual Brauer group of k, $U(k)$ is the group of invertible elements in k and $BM(k, G)$ is a group built out of central simple algebras over k with a G-action. Since $Br(\mathbf{C}) = 0$ and $H^2(G\,;U(k)) = H^2(G\,;\mathbf{Q}/\mathbf{Z})$, the theorem is an immediate consequence.

6.8.1. Remark Let us consider an arbitrary central extension $\tilde{G}_1$ of G by $\mu_n \cong \mathbf{Z}/n$ associated to a cohomology class $c \in H^2(G\,;\mathbf{Z}/n)$. We are interested in the set of elements g of G such that the conjugacy class <g> splits into n conjugacy classes in $\tilde{G}_1$. This set only depends on the image of c in $H^3(G\,;\mathbf{Z})$ via the Bockstein homomorphism $H^2(G\,;\mathbf{Z}/n) \to H^3(G\,;\mathbf{Z})$ (cf. 6.6). In other words, two central extensions of G by $\mathbf{Z}$/n with the same associated image by the Bockstein homomorphism have the same set of n-split conjugacy classes.

In order to show this fact, let us consider the following diagram

$$\begin{array}{ccc} \mu_n & \to & \mu_{nm} \\ \downarrow & & \downarrow \\ \tilde{G}_1 & \to & \tilde{G} \\ \pi\downarrow & & \downarrow \\ G & = & G \end{array}$$

and an element g_1 of $\tilde{G}_1$. The conjugacy class of $g = \pi(g_1)$ splits into n conjugacy classes in $\tilde{G}_1$ if and only if there is a trace function f on $\tilde{G}_1$ with values in $\mathbf{C}$ such that $f(g_1c) = f(g_1)c$ when $c \in \mu_n$. Such a trace function extends obviously to $\tilde{G}$, which yields to the result, since the direct limit of the groups $H^2(G\,;\mathbf{Z}/nm)$ is precisely $H^2(G\,;\mathbf{Q}/\mathbf{Z}) = H^3(G\,;\mathbf{Z})$.

6.8.2. The previous remark may be generalized as follows : let $\tilde{G}_1$ and $\tilde{G}$ be two group extensions (not necessary central) of G by abelian groups C_1 and C of orders m_1 and m respectively, such that the following diagram commutes (with α injective) :

$$\begin{array}{ccc} C_1 & \xrightarrow{\alpha} & C \\ \downarrow & & \downarrow \\ \tilde{G}_1 & \to & \tilde{G} \\ \pi\downarrow & & \downarrow \\ G & = & G \end{array}$$

The previous argument shows that if an element g of G splits into m conjugacy classes in $\tilde{G}$, it splits into m_1 conjugacy classes in $\tilde{G}_1$: take trace functions f on $\tilde{G}$ with values in C such that $f(gc) = f(g)$ c (we write multiplicatively the abelian group C). The converse is true if the extension of G by C is central.

6.9. Let us now assume that X is not reduced to a point. We can use the Baum-Connes-Kuhn-Slominska character [BC][Kuh][S] which is defined on $K_\Gamma(X)$ (for any finite group Γ), with values in the following direct sum $\bigoplus_{<\gamma>} H^{even}(X^\gamma)^{C(\gamma)}$. In this formula, $<\gamma>$ runs through all the conjugacy classes of Γ, $C(\gamma)$ being the centralizer of γ (the cohomology is taken with complex coefficients). One of the main features of this "Chern character"

$$K_\Gamma(X) \to \bigoplus_{<\gamma>} H^{even}(X^\gamma)^{C(\gamma)}$$

is the isomorphism it induces between $K_\Gamma(X) \otimes_Z \mathbf{C}$ and the cohomology with complex coefficients on the right-hand side. If E is a Γ-vector bundle, the map is defined explicitly by Formula 1.13 p. 170 in [BC].

Let us now take for Γ the group $\tilde{G}$ previously considered and let us analyse the formula in this case. We shall view the right-hand side not just as a function on the set of conjugacy classes $<\gamma>$, but as a function f on the full group Γ with certain extra properties which we are going to explain.

If we replace γ by γ' such that $\gamma' = \sigma\, \gamma\, \sigma^{-1}$, we have a canonical isomorphism σ^* : $H^*(X^{\gamma'})^{C(\gamma')} \to H^*(X^{\gamma})^{C(\gamma)}$ induced by $x \mapsto \sigma x$. This map exchanges $f(\gamma)$ and $f(\gamma')$ and we have the relation $f(\gamma) = \sigma^*(f(\gamma'))$, which says that f is essentially a function on the conjugacy classes.

On the other hand, if the action of $\tilde{G}$ is of linear type, we have an extra relation, an easy consequence of the formula in [BC], which is $f(\mu\gamma) = \mu\, f(\gamma)$ when μ is an n^{th} root of unity. To summarize, we get the following theorem.

6.10. THEOREM. *Let G be a finite group and $A = M_n(\mathbf{C})$ with a G-action. Then the ungraded twisted equivariant K-theory $K_G^{(A)}(X)$ is a subgroup of the equivariant K-theory $K_{\tilde{G}}(X)$, where $\tilde{G}$ is the pull-back diagram*

$$\begin{array}{ccc} \tilde{G} & \to & SU(n) \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ G & \to & PU(n) \end{array}$$

More precisely, $K_G^{(A)}(X) \otimes_{\mathbf{Z}} \mathbf{C}$ may be identified with the $\mathbf{C}$-vector space of functions f on $\Gamma = \tilde{G}$ with $f(\gamma)$ in $H^{even}(X^g)^{C(\gamma)}$, $\pi(\gamma) = g$, such that the following two identities hold :

1) If $\gamma' = \sigma\, \gamma\, \sigma^{-1}$, one has $f(\gamma) = \sigma^*(f(\gamma'))$, according to the formula above

2) $f(\mu\, \gamma) = \mu\, f(\gamma)$ if μ is an n^{th} root of unity

In particular, if X is reduced to a point, we have $\sigma^* = Id$ and $K_G^{(A)}(X)$ is free with rank the number of conjugacy classes of G which split into n conjugacy classes in $\tilde{G}$, as we have seen in 6.5.

6.11. Remarks. This theorem is not really new. In a closely related context, one finds similar results in [AR] and [TX]. We should also notice that the same ideas have been used before in [K4] for representations of "linear type". Finally, the theorem easily extends to locally compact spaces if we consider cohomology with compact supports on the right-hand side.

6.12. THEOREM. *Let A be any finite-dimensional graded semi-simple complex algebra with a graded action of a finite group G. Then the graded K-theory $GrK_0(A') \oplus GrK_1(A')$ of the semi-direct product $A' = G \prec A$ is a non trivial free $\mathbf{Z}$-module. In particular, if V is a real finite-dimensional vector space with a G action, the group $K_G^A(V) \oplus K_G^A(V \oplus 1)$ is free non trivial thanks to the Thom isomorphism.*

Proof. The algebra $G \prec A$ is graded semi-simple over the complex numbers. Therefore, it is a direct sum of graded algebras Morita equivalent to $M_n(\mathbf{C})$ x $M_n(\mathbf{C})$ or $M_{2n}(\mathbf{C})$. In both cases, the graded K-theory is non trivial. The last part of the theorem follows from 4.2.

The following theorem is a direct consequence of the previous considerations :

6.13. THEOREM. *Let us now assume that* $A = M_{2n}(\mathbf{C})$ *is G-oriented as a graded algebra :in other words, there is an involutive element* ε *of A of degree 0 which commutes with the action of G and commutes (resp. anticommutes) with the elements of A of degree 0 (resp. 1). Then, the graded K-theory* $GrK_*(A')$, *with* $A' = G \prec A$, *is a finitely generated free module concentrated in degree 0. More precisely, one has* $GrK_0(A') = K(A')$ *and* $GrK_1(A') = 0$. *In particular, if V is an even-dimensional real vector space and if* $A \hat{\otimes} C(V)$ *is G-oriented, we have (via the Thom isomorphism)*

$$K_G^A(V) = K_G^{A\hat{\otimes}C(V)}(P) = K(G \prec (A \hat{\otimes} C(V))) \text{ and } K_G^A(V \oplus 1) = 0$$

where P is a point. If we write $A \hat{\otimes} C(V)$ *as an algebra of matrices* $M_r(\mathbf{C})$ *with a representation* ρ *of G and call* $\tilde{G}$ *the associated central extension by* μ_r, *the rank of* $K_G^A(V)$ *is the number of conjugacy classes of G which split into r conjugacy classes in* $\tilde{G}$.

In the abelian case, the following two theorems are related to results obtained by P. Hu and I. Kriz [HK], using different methods.

6.14. THEOREM. *Let us consider the algebra* $A = M_n(\mathbf{C})$ *provided with an action of an abelian group G. Then the ungraded twisted K-theory* $K_G^{(A)}(P)_* = K_*(A')$, *with* $A' = G \prec A$, *is concentrated in degree* 0 *and is a free* **Z**-*module. If we tensor this group with the rationals and if we look at it as an* $R(G) \otimes \mathbf{Q} = \mathbf{Q}[G]$-*module, it may be identified with* $R(G') \otimes \mathbf{Q}$ *for a suitable subgroup G' of G. In particular, the rank of* $K_0(A')$ *divides the order of G.*

Proof. The first part of the theorem is a consequence of the previous more general considerations. As we have shown before, the algebra A' gives rise to the following commutative diagram

$$\begin{array}{ccc} \tilde{G}_n & \rightarrow & SU(n) \\ {}^{\pi}\downarrow & & \downarrow \\ G & \rightarrow & PU(n) \end{array}$$

the fibers of the vertical maps being μ_n. The subset of elements $\tilde{g}$ in $\tilde{G}_n$ such that $\pi(\tilde{g})$ splits into n conjugacy classes is just the center $Z(\tilde{G}_n)$ of $\tilde{G}_n$ (since G is abelian). Let us put $\Gamma_n = \pi(Z(\tilde{G}_n))$. Then $K(A')$ may be written as $K_G^A(P)$ where P is a point. According to Theorem 6.10, this is the subgroup of the representation ring of $\tilde{G}_n$ generated by representations of linear type. At this stage, it is convenient to make $n = \infty$ by extension of the roots of unity, so that we have an extension of G by $\mathbf{Q}/\mathbf{Z}$

$$\mathbf{Q}/\mathbf{Z} \rightarrow \tilde{G} \rightarrow G$$

(the "linear type" finite-dimensional representations of $\tilde{G}$ are the same as the original linear

type finite-dimensional representations of $\tilde{G}_n$). We call $R(\tilde{G})_1$ the associated Grothendieck group. By the theory of characters, we see that $R(\tilde{G})_1 \otimes \mathbf{Q}$ is isomorphic to $R(Z(\tilde{G}))_1 \otimes \mathbf{Q}$, since the characters of such linear type representations of $\tilde{G}$ vanish outside $Z(\tilde{G})$. On the other hand, if we denote by G' the image of $Z(\tilde{G})$ in G, the extension of abelian groups

$$\mathbf{Q}/\mathbf{Z} \to Z(\tilde{G}) \to G'$$

splits (non canonically). This means that we can identify $R(\tilde{G})_1 \otimes \mathbf{Q}$ with the representation ring $R(G') \otimes \mathbf{Q}$ as an $R(G) \otimes \mathbf{Q}$ -module. This proves the last part of the theorem.

6.15. THEOREM. *Let us consider a graded algebra $A = M_{2n}(\mathbf{C})$ provided with a non oriented action of an abelian group G (with respect to the grading). Then the graded twisted K-theory $K_G^A(P)_* = \mathrm{Gr}K_*(A')$, with $A' = G \prec A$, is concentrated in a single degree (0 or 1) and is a free $\mathbf{Z}$-module. If we tensor this group with the complex numbers and if we look at it as an $R(G) \otimes \mathbf{C} = \mathbf{C}[G]$-module, it may be identified with $R(G') \otimes \mathbf{Q}$ for a suitable subgroup G' of G. In particular, the rank of $K_G^A(P)_*$ divides the order of G.*

Proof. Let L be the orientation bundle of A (with respect to the action of G). If we change A into $A \hat{\otimes} C(L)$ and if we apply the Thom isomorphism theorem, we have to compute $K_G^A(L)_*$ where G acts on A (resp. L) in an oriented way (resp. non oriented way).
Let us apply Theorem 6.10 in this situation : since G is abelian, the function f of the theorem must be equal to 0 on the elements of $\tilde{G}$ which are not in $Z(\tilde{G})$. Therefore, the relevant group $K_G^A(L)_*$ is reduced to $K_{G'}(L)$ (after tensoring with $\mathbf{C}$ and where G' is the image of $Z(\tilde{G})$ in G). We now consider two cases :

1) the action ρ of G' on L is oriented, in which case we only find $R(G')$ (with a shift of dimension). Therefore, the dimension of $K_G^A(L)_* \otimes \mathbf{C}$ is the order of G' which divides the order of G.
2) the action ρ of G' on L is not oriented. We then find a direct sum of copies of $\mathbf{C}$, each one corresponding to an element of G' such that $\rho(g') = -1$. The dimension of $K_G^A(L)_* \otimes \mathbf{C}$ is therefore half the order of G', hence divides $|G|/2$.

6.16. Remarks. For an oriented action of G on $M_{2n}(C)$, Theorem 3.5 enables us to reduce the analogous problem in Theorem 6.15 to ungraded twisted K-theory, which is done in 6.14.
On the other hand, as we already mentioned, the two last theorems (for G abelian) are related to results found by P. Hu and I. Kriz [HK].

In [K4] we also perform other types of computations when A is a Clifford algebra and G is <u>any</u> finite group, again using the Thom isomorphism. For instance, if G is the symmetric group S_n acting on the Clifford algebra of $\mathbf{R}^n$ via the canonical representation of S_n on $\mathbf{R}^n$, there is a nice relation with the pentagonal identity of Euler (cf. [K4] p. 532).

As a concluding remark, we would like to point out also that the theory $K_{\pm}(X)$ introduced recently by Atiyah and Hopkins [AHo] is a particular case of twisted equivariant K-theory. As

a matter of fact, it was explicitly present in [K1] § 3, 40 years ago, before the formal introduction of twisted K-theory !

According to [AHo], the definition of $K_{\pm}(X)$ (in the complex or real case) is the group

$K_{\mathbf{Z}/2}(X \times \mathbf{R}^8)$, where $\mathbf{Z}/2$ acts on X and also on $\mathbf{R}^8 = \mathbf{R} \times \mathbf{R}^7$ by $(\lambda, \mu) \mapsto (-\lambda, \mu)$. According to the Thom isomorphism in equivariant K-theory (proved in [K3]), it coincides with an explicit graded twisted K-group $K^A_{Z/2}(X)$, as defined in [K1]. Here A is the Clifford algebra $C(\mathbf{R}^2) = C^{1,1}$ of $\mathbf{R}^2$ provided with the quadratic form $x^2 - y^2$ and where $\mathbf{Z}/2$ acts via the involution $(\lambda, \mu) \mapsto (-\lambda, \mu)$ on $\mathbf{R} \times \mathbf{R}$ (this is also mentioned briefly in [AHo] p. 2, footnote 1). This identification is valid as well in the real framework, where we have 8-periodicity.

These groups $K^A_{Z/2}(X)$ were precisely those considered in [K1] § 3.3 in a broader context : A may be any Clifford algebra bundle $C(V)$ (where V is a <u>real</u> vector bundle provided with a non degenerate quadratic form) and $\mathbf{Z}/2$ may be replaced by any compact Lie group acting in a coherent way on X and V. Note that the real and complex Fredholm descriptions for the non twisted case were also considered with great detail independently in [ASi] and [K5]. These descriptions overlap with the results of our § 3 and 4. The paper [K4] gives a method to compute these equivariant K-groups, at least rationally, in a more general context, using the equivariant Thom isomorphism.

7. Operations on twisted K-groups.

Note : this section is a partial synthesis of [DK] (1970) and [AS2] (2005).

7.1. Let us start with the simple case of bundles of (ungraded) infinite C*-algebras modelled on $\mathcal{K}$, as in [AS2]. As it was shown in [A1] and [DK], we have a n^{th} power map

$$\text{P: } K^{(\mathcal{A})}(X) \to K^{(\mathcal{A}^{\otimes n})}_{S_n}(X)$$

where the symmetric group S_n acts on $\mathcal{A}^{\otimes n}$ by permutation of the factors.

7.2. LEMMA. *The group* $K^{(\mathcal{A}^{\otimes n})}_{S_n}(X)$ *is isomorphic to the group* $K^{(\mathcal{A}^{\otimes n})_0}_{S_n}(X)$ *where the symbol* $_0$ *means that* S_n *is acting* <u>*trivially*</u> *on* $\mathcal{A}^{\otimes n}$.

Proof. As we have shown many times in § 6, this "untwisting" of the action of the symmetric group on $\mathcal{A}^{\otimes n}$ is due to the following fact : the standard representation

$$S_n \to PU(H^{\otimes n})$$

can be lifted into a representation $\rho : S_n \to U(H^{\otimes n})$ in a way compatible with the diagonal action of elements of $PU(H)$, a fact which is obvious to check. If E is a module[20] over $\mathcal{A}^{\otimes n}$ and S_n, we then have to untwist the action of S_n by a new action $*$ defined by

[20] As a matter of fact, we should add a unit fibrewise to $\mathcal{A}$ in order to make sense of the module structure.

$$g{*}e = \rho(g)^{-1}(g.m)$$

This $*$ action of S_n now <u>commutes</u> with the action of $\mathcal{A}^{\otimes n}$ (see again and for instance [K4] § 2.6).

7.3. Remark. If we take a bundle of finite dimensional algebras modelled on $A = End(E)$ where $E = \mathbf{C}^r$, there is another way to check (functorially) the untwisting : we identify $A^{\otimes n}$ with $End(E^{\otimes n})$ and $(A^{\otimes n})^*$ with $Aut(E^{\otimes n})$. We have the following commutative diagram

$$\begin{tikzcd} & End(E^{\otimes n})^* = Aut(E^{\otimes n}) \arrow[d, "\pi"] \\ S_n \arrow[ur, "\theta"] \arrow[r] & End(E^{\otimes n}) = End(E)^{\otimes n} \end{tikzcd}$$

The vertical map sends the invertible element α to the automorphism $(u \mapsto \alpha u \alpha^{-1})$. If σ is a permutation, the horizontal map sends σ to the automorphism

$$u_1 \otimes \cdots \otimes u_n \mapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

while the map θ sends σ to the automorphism of $E^{\otimes n}$ defined by

$$x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

Finally, the composition $\pi\theta$, computed on a decomposable vector of $E^{\otimes n}$ gives the required result :

$$x_1 \otimes \ldots \otimes x_n \xrightarrow{\sigma} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)} \xrightarrow{u} u_1(x_{\sigma(1)}) \otimes \ldots \otimes u_n(x_{\sigma(n)})$$
$$\xrightarrow{\sigma^{-1}} u_{\sigma(1)}(x_1) \otimes \ldots \otimes u_{\sigma(n)}(x_n)$$

7.4. As it was shown in [A1], a **Z**-module map

$$R(S_n) \to \mathbf{Z}$$

defines an operation in twisted K-theory by taking the composite of the following maps

$$K^{(\mathcal{A})}(X) \to K_{S_n}^{(\mathcal{A}^{\otimes n})}(X) \cong K_{S_n}^{(\mathcal{A}^{\otimes n})_0}(X) \cong K^{(\mathcal{A})^{\otimes n}}(X) \otimes R(S_n) \to K^{(\mathcal{A})^{\otimes n}}(X)$$

This is essentially[21] what was done in [AS2] § 10, in order to define the λ^n operation of Grothendieck in this context for instance.

Let us call F (for short) the image of $(E\,,D)$ by the composite of the maps

[21] The second homomorphism was not explicitly given however.

$$K^{(\mathcal{A})}(X) \to K_{S_n}^{(\mathcal{A}^{\otimes n})}(X) \to K_{S_n}^{(\mathcal{A}^{\otimes n})_0}(X) \cdot$$

We can also define the Adams operations Ψ^n with the method described in [DK] (which we intend to generalize later on). For this, we restrict the action of S_n to the cyclic group **Z**/n identified with the group of n^{th} roots of the unity. Let us now call F_r the subgroup of F where the action of **Z**/n is given by ω^r, ω being a fixed primitive root of the unity. Then $\Psi^n(E, D)$ is defined by the following sum

$$\Psi^n(E, D) = \sum_0^{n-1} F_r \omega^r$$

It belongs formally to $K^{(\mathcal{A})^{\otimes n}}(X) \otimes \Omega_n$ where Ω_n is the ring of n-cyclotomic integers. However, if n is prime, using the action of the symmetric group S_n, it is easy to see that F_r is isomorphic to F_1 if $r \neq 1$. Therefore we end up in $K^{(\mathcal{A})^{\otimes n}}(X)$, considered as the subgroup of $K^{(\mathcal{A})^{\otimes n}}(X) \otimes \Omega_n$, as it was expected. It is proved in [A1] that this definition of Ψ^n agrees with the classical one.

There is another operation in twisted K-theory which is "complex conjugation", denoted classically by Ψ^{-1}, which maps $K^{(\mathcal{A})}(X)$ to $K^{(\mathcal{A})^{-1}}(X)$ (if we write multiplicatively the group law in $Br(X)$). It is shown in [AS2] § 10 how we can combine this operation with the previous ones in order to get "internal" operations in twisted K-theory, i.e. mapping $K^{(\mathcal{A})}(X)$ to itself.

7.5. It is more tricky to define operations in graded twisted K-theory. If Λ is a Z/2 graded algebra, it is no longer true in general that a graded involution of Λ is induced by an inner automorphism with an element of degree 0 and of order 2. A typical example is the Clifford algebra

$$(\mathbf{C} \oplus \mathbf{C})^{\hat{\otimes} 2} = (C^{0,1})^{\hat{\otimes} 2}$$

which may be identified with the graded algebra $M_2(\mathbf{C})$. If we put

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we see that there is no inner automorphism by an element of order 2 and degree 0 permuting e_1 and e_2.

On the other hand, we know by the general theory that the standard representation

$$\rho : S_n \to Aut(\mathcal{A})^{\hat{\otimes} n}$$

where $\mathcal{A}$ is a bundle of graded algebras modelled on $\mathcal{K} \times \mathcal{K}$ or $M_2(\mathcal{K})$ can be lifted by inner automorphisms[22] with elements of degree 0 or 1. Therefore, if we restrict the representation ρ to the alternating group A_n, we get indeed a representation from A_n to the 0-degree part of

$(\mathcal{A}^{\hat{\otimes} n})^*$ such that the following diagram commutes

$$\begin{array}{ccc} & & (\mathcal{A}^{\hat{\otimes} n})^* \\ & \nearrow & \downarrow \\ A_n & \rightarrow & Aut(\mathcal{A})^{\hat{\otimes} n} \end{array}$$

7.6. In order to define the operations Ψ^n in graded twisted K-theory, we now follow the scheme developed in [DK] and choose the description of twisted K-theory given by a family of Fredholm operators (cf. 3.14). The power map

$$K^{\mathcal{A}}(X) \to K_{S_n}^{\mathcal{A}^{\hat{\otimes} n}}(X)$$

still makes sense : we just take the n^{th} (graded) tensor product of (E, D) by itself with the notations of 3.12/14. The graded representation of S_n in $\mathcal{B}^{\hat{\otimes} n}$ does not lift in general to a representation of S_n in the 0-degree part of $(\mathcal{B}^{\hat{\otimes} n})^*$. However, as we have seen before, it is the case if we restrict this representation to the alternating group A_n. Therefore, any **Z**-module map $R(A_n) \to \mathbf{Z}$ gives rise to an operation in graded twisted K-theory which is the following composite :

$$K^{\mathcal{A}}(X) \to K_{A_n}^{\mathcal{A}^{\hat{\otimes} n}}(X) \to K^{\mathcal{A}^{\hat{\otimes} n}}(X) \otimes R(A_n) \to K^{\mathcal{A}^{\hat{\otimes} n}}(X)$$

Among these operations, we should mention the Adams operations Ψ^n as defined in [DK]. For this we restrict even more the action of A_n to the cyclic group $\mathbf{Z}/n$ with n <u>odd</u> (such that $\mathbf{Z}/n$ is a subgroup of A_n).

The Adams operation Ψ^n is then given by the same formula as in 7.4

$$\Psi^n : K^{\mathcal{A}}(X) \to K^{\mathcal{A}^{\otimes n}}(X) \otimes \Omega_n$$

where Ω_n is the ring of n-cyclotomic integers. We can copy the proof in [DK] to show that this operation is additive and multiplicative up to canonical isomorphisms (see Theorem 30, p. 23 in [DK]).

7.7. Although, we have been forced to introduce the alternating group in order to define operations in graded twisted K-theory (see 7.9 below), there are important cases where we can stay in the classical situation described at the beginning of the section. In particular, if $\mathcal{A}$ is a bundle of <u>oriented</u> graded algebras modelled on $M_2(\mathcal{K})$, we have seen in 3.5 that the two groups $K^{\mathcal{A}}(X)$ and $K^{(\mathcal{A})}(X)$ coincide. As a matter of fact, in this case, we have a commutative diagram

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[22] Again, we must add a unit in order to make sense of $(\mathcal{A}^{\hat{\otimes} n})^*$. Another possibility which we will consider in 7.6 is to work with the bundle $\mathcal{B}$ rather than $\mathcal{A}$ (see 2.6).

$$\begin{tikzcd} & \mathrm{U}(\mathrm{H} \oplus \mathrm{H}) \arrow[d] \\ S_n \arrow[r] \arrow[ur] & \mathrm{PU}(\mathrm{H} \oplus \mathrm{H}) \end{tikzcd}$$

with S_n instead of A_n since the image of S_n by the horizontal map is inside $\mathrm{PU}^0(\mathrm{H} \oplus \mathrm{H})$, with the notation of 3.4.

7.8. As a concluding remark, we should notice that the image of Ψ^n as defined in 7.6 is not arbitrary. If k and n are coprimes, the multiplication by k on the group $\mathbf{Z}/n$ defines an element of the symmetric group S_n. The signature of this permutation is called the Legendre symbol $\binom{k}{n}$. We denote by F_r (as in 7.4) the element associated to the eigenvalue $e^{2i\pi r}$. Then we see that F_r and F_{rk} are isomorphic if the Legendre symbol $\binom{k}{n}$ is equal to 1 (consider again the multiplication by k mod. n). If n is prime for instance, $\Psi^n(E)$ may therefore be written in the following way :

$$\Psi^n(E) = F_0 + \sum_{\binom{k}{n}=1} U\omega^k + \sum_{\binom{k}{n}=-1} V\omega^k$$

where U (resp. V) is any F_k with Legendre's symbol equal to 1 (resp. -1).

We should also notice, following [AS2], that there is no problem to define the Adams operation Ψ^{-1} in graded twisted K-theory and combine it with compositions of the Ψ^n 's in order to define "internal" operations.

7.9. The simplest non-trivial example of such an operation is

$$\Psi^n : \mathbf{Z} \cong K^1(S^1) \to K^n(S^1) \otimes \Omega_n$$

where n is a product of different odd primes. Since the operation Ψ^n on $K^2(S^2)$ is the multiplication by n, we deduce that $\theta = \sqrt{(-1)^{(n-1)/2} n}$ belongs to Ω_n (a well-known result) and that Ψ^n on $K^1(S^1)$ is essentially the inclusion of $\mathbf{Z}$ in Ω_n defined by $1 \mapsto \theta$.

REFERENCES

[AR] A. ADEM and Y. RUAN. Twisted Orbifold *K*-theory. Communications in Mathematical Physics 237, pp. 533--556 (2003).

[A1] M. F. ATIYAH. *K*-theory. Notes by D.W. Anderson. Second edition. Advanced Book Classics. Addison-Wesley Publishing Company (1989).

[A2] M. F. ATIYAH. Bott periodicity and the index of elliptic operators. The Quarterly Journal of Math. 1968, pp. 113-140 (1968).

[AH] M. F. ATIYAH and F. HIRZEBRUCH. Riemann-Roch theorems for differentiable manifolds. Bull. Amer. Math Society, p. 276-281 (1959).

[AHo] M. F. ATIYAH and M. HOPKINS. A variant of K-theory : $K_{\pm}$. Topology, geometry and quantum field theory. London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, pp. 5-17 (2004).

[AS] M.F. ATIYAH and G. SEGAL. Twisted *K*-theory. Ukr. Mat. Visn. 1, pp. 287-330 (19) (2004) MR 2172633, ArXiv math.KT/0407054 (2005).

[AS2] M.F. ATIYAH and G. SEGAL. Twisted *K*-theory and cohomology. ArXiv math. KT/0510674 (2005).

[ASi] M.F. ATIYAH and I.M. SINGER. Index theory for skew adjoint Fredholm operators. Publ. Math. IHES Paris 37, pp. 5-26 (1969). [Compare with [K5]]

[BC] P. BAUM and A. CONNES. Chern character for discrete groups in " A fête of Topology", pp. 163-232. Academic Press, Boston, MA (1988).

[B] B. BLACKADAR. *K*-theory for operator algebras. Second edition. Cambridge University Press (1998).

[BCMMS] P. BOUWKNEGT, A.L. CAREY, V. MATHAI, M.K. MURRAY and D. STEVENSON. Twisted *K*-theory and *K*-theory of Bundle Gerbes. Comm. Math. Phys. 228, pp. 17-45 (2002).

[Ca] A.L. CAREY and Bai-Ling WANG. Thom isomorphism and push-forward maps in twisted *K*-theory. math. KT/0507414 (2006).

[CKRW] D. CROCKER, A. KUMJIAN, I. RAEBURN, D.P. WILLIAMS. An equivariant Brauer group and action of groups on C*-algebras. J. Funct. Anal. 146, pp. 151-184 (1997).

[DD]. J. DIXMIER et A. DOUADY. Champs continus d'espaces hilbertiens et de C*-algèbres. Bull. Soc. Math. France 91, pp. 227-284 (1963).

[DK] P. DONOVAN and M. KAROUBI. Graded Brauer groups and *K*-theory with local coefficients. Publ. Math. IHES 38, p. 5-25 (1970). French summary in : Groupe de Brauer et coefficients locaux en *K*-théorie. Comptes Rendus Acad. Sci. Paris, t. 269, pp. 387-389 (1969).

[FM] A.T. FOMENKO and A.S. MISCENKO. The index of elliptic operators over C*-algebras. Izv. Akad. Nauk. SSSR, Ser. Mat. 43, pp. 831-859 (1979).

[FHT] D.S. FREED, M. HOPKINS and C. TELEMAN. Twisted *K*-theory and loop group representations. ArXiv math/0206237, 0312155 and 0511232.

[FW] A. FROHLICH and C.T.C. WALL. Equivariant Brauer groups. Contemp. Math. 272, pp. 57-71 (2000).

[G] A. GROTHENDIECK. Le groupe de Brauer. Séminaire Bourbaki 290 (1965).

[J] K. JANICH. Vektorraumbündel und der Raum der Fredholm-operatoren. Math. Ann. 161, pp. 129-142 (1965).

[K1] M. KAROUBI. Algèbres de Clifford et *K*-théorie. Ann. Sci. Ecole Norm. Sup. (4), pp. 161-270 (1968).

[K2] M. KAROUBI. Algèbres de Clifford et opérateurs de Fredholm. Springer Lecture Notes in Maths N° 136, p. 66-106 (1970). Summary in Comptes Rendus Acad. Sci. Paris, t. 267, pp. 305 (1968).

[K3] M. KAROUBI. Sur la *K*-théorie équivariante. Springer Lecture Notes in Math. N° 136, pp. 187-253 (1970).

[K4] M. KAROUBI. Equivariant *K*-theory of real vector spaces and real projective spaces. Topology and its applications 122, pp. 531-546 (2002).

[K5] M. KAROUBI. Espaces classifiants en *K*-théorie. Trans. of the Amer. Math. Soc., Vol 147, pp. 75-115 (1970). [compare with [ASi]].

[KH] I. KRIZ and P. HU. The $RO(G)$-graded coefficients of $(\mathbf{Z}/2)^n$-equivariant *K*-theory. ArXiv math. KT/0609067 (2006).

[Kuh] N. KUHN. Character rings in algebraic topology, Advances in Homotopy Theory, Proceedings of Cortona 1988, LMS Lecture Note Series 139 pp. 111-126(1989).

[Ku] N. KUIPER. The homotopy type of the unitary group in Hilbert space. Topology 3, pp. 19-30 (1965).

[LTX] C. LAURENT, J.-L. TU and P. XU. Twisted K-theory of differential stacks. Ann. Scient. Ec. Norm. Sup. 37, pp. 841-910 (2004).

[MS] V. MATHAI and D. STEVENSON. On a generalized Connes-Hochschild-Kostant-Rosenberg theorem. Advances in Math. 200, pp. 303-335 (2006).

[Mi] E.A. MICHAEL. Convex structures and continuous selections. Canad. J. Math., vol. 11, pp. 556-575 (1959).

[R] J. ROSENBERG. Continuous-trace algebras from the bundle theoretic point of view. J. Austral. Math. Soc. A 47, pp. 368-381 (1989).

[S] G. SEGAL. Equivariant *K*-theory. Publ. Math. de l'IHES, tome 34, pp. 129-151 (1968).

[S] J. SLOMINSKA. On the equivariant Chern homomorphism. Bulletin de l'Académie Polonaise des Sciences, vol. XXIV, N° 10, pp. 909-913 (1976).

[TX] J.-L. TU and Ping XU. The Chern character for twisted *K*-theory of orbifolds. ArXiv math. KT/0505267 (2005).

[Wi] E. WITTEN. D-branes and K-theory. J. High Energy Phys. 12, paper 19, hep-th/0007175 (1998).

[W] R. WOOD. Banach algebras and Bott periodicity. Topology, vol. 4, pp. 371-389 (1966).