

Cocycle categories

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Introduction

Suppose that G is a sheaf of groups on a space X and that $U_\alpha \subset X$ is an open covering. Then a cocycle for the covering is traditionally defined to be a family of elements $g_{\alpha\beta} \in G(U_\alpha \cap U_\beta)$ such that $g_{\alpha\alpha} = e$ and $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ when all elements are restricted to the group $G(U_\alpha \cap U_\beta \cap U_\gamma)$.

A more compact way of saying this is to assert that such a cocycle is a map of simplicial sheaves $C(U) \rightarrow BG$ on the space X , where $C(U)$ is the Čech resolution associated to the covering family $\{U_\alpha\}$. The canonical map $C(U) \rightarrow *$ is a local weak equivalence of simplicial sheaves, and is a fibration in each section since $C(U)$ is actually the nerve of a groupoid — the map $C(U) \rightarrow *$ is therefore a hypercover, which is most properly defined to be a map of simplicial sheaves (or presheaves) which is a Kan fibration and a weak equivalence in each stalk. Every cocycle in the traditional sense therefore determines a picture of simplicial sheaf morphisms

$$* \leftarrow C(U) \rightarrow BG.$$

where the canonical map $C(U) \rightarrow *$ is a hypercover.

More generally, it has been known for a long time [3], [6] that the locally fibrant simplicial sheaves (ie. simplicial sheaves which are Kan complexes in each stalk) have a partial homotopy theory which is good enough to give a calculus of fractions approach to formally inverting local weak equivalences, and thereby giving a construction of the homotopy category for simplicial sheaves. Specifically, one defines morphisms $[X, Y]$ in the homotopy category by setting

$$[X, Y] = \varinjlim_{[\pi]: V \rightarrow X} \pi(V, Y), \quad (1)$$

where the filtered colimit is indexed over simplicial homotopy classes of hypercovers $\pi : V \rightarrow X$, and $\pi(V, Y)$ denotes simplicial homotopy classes of maps $V \rightarrow Y$. Thus, morphisms in the homotopy category are represented by pictures

$$X \xleftarrow{\pi} V \rightarrow Y, \quad (2)$$

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where π is a hypercover. The relation (1) is usually called the Generalized Verdier Hypercovering Theorem, and it is the historical starting point for the homotopy theory of simplicial sheaves.

Much has transpired in the intervening years. We now know that there is a plethora of Quillen model structures for simplicial sheaves and simplicial presheaves on all small Grothendieck sites [11], all of which determine the same homotopy category. In the first of these structures [17], [7], the monomorphisms are the cofibrations and the weak equivalences are the local weak equivalences (ie. stalkwise weak equivalences in the presence of enough stalks), and then the homotopy categories for simplicial sheaves and presheaves (which are equivalent) are constructed by methods introduced by Quillen. The morphisms $[X, Y]$ in the homotopy category of simplicial sheaves coincide with those specified by (1) if X and Y are locally fibrant. These basic model structures also inherit many of the good properties from simplicial sets, including right properness, which means that weak equivalences are stable under pullback along fibrations. Weak equivalences of simplicial presheaves and sheaves are also closed under finite products, essentially because the same statement holds for simplicial sets.

One still needs a cocycle or hypercover theory, because it is involved in the proofs of many of the standard homotopy classification theorems for sheaf cohomology, such as the identification of sheaf cohomology with morphisms $[\ast, K(A, n)]$ in the homotopy category. It turns out, however, that the traditional requirement that the map π in (2) should be something like a fibration in formal manipulations based on the generalized Verdier hypercover theorem is usually quite awkward in practice. This has been especially apparent in all attempts to convert n -types to more algebraic objects.

The basic point of this paper is that there is a better approach to defining and manipulating cocycles, which essentially starts with removing the fibration condition on weak equivalence π in (2).

A cocycle from X to Y is defined in this paper to be a pair of maps

$$X \xleftarrow{f} Z \xrightarrow{g} Y \quad (3)$$

where f is a weak equivalence. A morphism of cocycles is the obvious thing, namely a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow & \searrow g & \\ X & & & & Y \\ & f' \swarrow & \downarrow & \searrow g' & \\ & & Z' & & \end{array}$$

The cocycle category is denoted by $H(X, Y)$. Then the basic point is this: there is an obvious function

$$\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$$

defined by sending the cocycle (3) to the morphism gf^{-1} in the homotopy category. Then Theorem 2 of this paper implies that this function ϕ is a bijection.

Cocycle categories and the function ϕ are defined in great generality. The fact that ϕ is a bijection (Theorem 2) holds in the rather common setting of a right proper closed model structure in which weak equivalences are preserved by finite products. The formal definition of cocycle categories and their basic properties appear in the first section of this paper. The overall theory is rather easy to demonstrate.

The subsequent sections of this paper are taken up with a tour of the applications of this theory. Some of these applications are well known, and are given here with simple new proofs. This general approach to cocycles is also implicated in several recent results in non-abelian cohomology theory, which are displayed here.

Section 2 gives a quick (and hypercover-free) demonstration of the homotopy classification theorem for G -torsors for a sheaf of groups G — this is Theorem 5. It is also shown that the ideas behind the proof of Theorem 5 admit great generalization: examples include the definition and homotopy classification of torsors for a presheaf of categories enriched in simplicial sets (Theorem 6), the explicit construction of the stack completion of a sheaf of groupoids given in Section 2.3, and the calculation of the morphism set $[\ast, \operatorname{holim}_G X]$ for a diagram X of simplicial presheaves defined on a presheaf of groupoids G in Theorem 7.

Section 3 gives a new demonstration of homotopy classification theorem for abelian sheaf cohomology (Corollary 11). Section 4 gives a new description of the homotopy classification theorem for group extensions in terms of cocycles taking values in 2-groupoids (Theorem 12), and in Section 5 we discuss but do not prove a classification theorem for gerbes up to local equivalence as path components of a suitable cocycle category. This last result is Theorem 13 — it is proved in [15]. The cocycles appearing in the proofs of Theorems 5, 12 and 13 are canonically defined.

1 Cocycles

Suppose that \mathcal{M} is a right proper closed model category, and suppose further that the weak equivalences of \mathcal{M} are closed under finite products.

The assertion that \mathcal{M} is a closed model category means that \mathcal{M} has all finite limits and colimits (**CM1**), and that the class of morphisms of \mathcal{M} contain three subclasses, namely weak equivalences, fibrations and cofibrations which satisfy some properties. These properties include the two of three condition for weak equivalences (**CM2**: if any two of the maps f , g or $g \cdot f$ are weak equivalences, then so is the third), the requirement that all three classes of maps are closed under retraction (**CM3**), and the factorization axiom (**CM5**) which asserts that any map f has factorizations $f = qj = pi$ where q is a trivial fibration and j is a cofibration, and p is a fibration and i is a trivial cofibration. Note, for example, that a trivial fibration is a fibration and a weak equivalence — “trivial” things are always weak equivalences. Finally, \mathcal{M} should satisfy the

lifting axiom (CM4) which says that in any solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow \text{dotted} & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where p is a fibration and i is a cofibration, the dotted arrow exists making the diagram commute if either i or p is trivial.

A model category \mathcal{M} is said to be right proper if weak equivalences are closed under pullback along fibrations.

Not every model category is right proper, but right proper model structures are fairly common: examples include topological spaces, simplicial sets, spectra, simplicial presheaves, presheaves of spectra, and certain “good” localizations such as the motivic and motivic stable model categories. In all of these examples as well, weak equivalences are closed under finite products, meaning that if $f : X \rightarrow Y$ is a weak equivalence and Z is any other object, then the map $f \times 1 : X \times Z \rightarrow Y \times Z$ is a weak equivalence.

Suppose that X and Y are objects of \mathcal{M} . Let $H(X, Y)$ be the category whose objects are all pairs of maps (f, g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

such that f is a weak equivalence. A morphism $\alpha : (f, g) \rightarrow (f', g')$ of $H(X, Y)$ is a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow \alpha & \searrow g & \\ X & & & & Y \\ & f' \swarrow & \downarrow & \searrow g' & \\ & & Z' & & \end{array}$$

$H(X, Y)$ is the *category of cocycles* from X to Y .

Example 1. Suppose that $V \rightarrow *$ is a sheaf epimorphism (possibly arising from a covering) and that G is a sheaf of groups. Traditional cocycles for the underlying site with coefficients in G can be interpreted as simplicial sheaf maps

$$* \leftarrow C(V) \rightarrow BG,$$

where $C(V)$ is the Čech resolution for the cover, and the canonical map $C(V) \rightarrow *$ is a local weak equivalence of simplicial sheaves. All cocycles of this type therefore represent objects of the cocycle category $H(*, BG)$ in simplicial sheaves. This is a motivating example for the present definition of a cocycle.

More generally, it has been known for some time [6] that morphisms $X \rightarrow Y$ in the homotopy category of simplicial sheaves can be represented by pairs of morphisms

$$X \xleftarrow{\pi} U \rightarrow Y$$

where π is a hypercover, or locally trivial fibration. This is provided that X and Y are locally fibrant. Such pictures are members of the cocycle category $H(X, Y)$ in simplicial sheaves.

Generally, write $\pi_0 H(X, Y)$ for the class of path components of $H(X, Y)$, and write $[X, Y]$ for the set of morphisms from X to Y in the homotopy category $\text{Ho}(\mathcal{M})$ of \mathcal{M} . Recall that $\text{Ho}(\mathcal{M})$ is constructed from \mathcal{M} by formally inverting the weak equivalences. Then one sees that there is a function

$$\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$$

which is defined by $(f, g) \mapsto g \cdot f^{-1}$.

Theorem 2. *Suppose that \mathcal{M} is a right proper closed model category for which the class of weak equivalences is closed under finite products. Then the function $\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$ is a bijection for all X and Y .*

Suppose that the maps $f, g : X \rightarrow Y$ are left homotopic. Then there is a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow 1 & \downarrow d_0 & \searrow f & \\ X & \xleftarrow{s} & X \otimes I & \xrightarrow{h} & Y \\ & \nwarrow 1 & \uparrow d_1 & \nearrow g & \\ & & X & & \end{array}$$

for some choice of cylinder object $X \otimes I$, in which s is a weak equivalence and h is the homotopy. Then there are morphisms

$$(1_X, f) \rightarrow (s, h) \leftarrow (1_X, g)$$

in $H(X, Y)$, so that $(1_X, f)$ and $(1_X, g)$ are in the same path component of $H(X, Y)$. It follows that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X, Y) \rightarrow \pi_0 H(X, Y)$$

where $\pi(X, Y)$ denotes left homotopy classes of maps from X to Y .

Theorem 2 is a formal consequence of the following two results:

Lemma 3. *Suppose that $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$ are weak equivalences. Then the induced function*

$$(\alpha, \beta)_* : \pi_0 H(X, Y) \rightarrow \pi_0 H(X', Y')$$

is a bijection.

Lemma 4. *Suppose that X is cofibrant and Y is fibrant. Then the function*

$$\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$$

is a bijection.

Proof of Lemma 3. Identify the object $(f, g) \in H(X', Y')$ with a map $(f, g) : Z \rightarrow X' \times Y'$ of \mathcal{M} such that f is a weak equivalence.

There is a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & W \\ & \searrow (f, g) & \downarrow (p_{X'}, p_{Y'}) \\ & & X' \times Y' \end{array}$$

such that j is a trivial cofibration and $(p_{X'}, p_{Y'})$ is a fibration. Observe that $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{array}{ccc} W_* & \xrightarrow{(\alpha \times \beta)_*} & W \\ (p_X^*, p_Y^*) \downarrow & & \downarrow (p_{X'}, p_{Y'}) \\ X \times Y & \xrightarrow{\alpha \times \beta} & X' \times Y' \end{array}$$

Then (p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a weak equivalence. The map p_X^* is also a weak equivalence.

The assignment $(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \rightarrow \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. □

Proof of Lemma 4. The canonical function $\pi(X, Y) \rightarrow [X, Y]$ is a bijection since X is cofibrant and Y is fibrant, and there is a commutative diagram

$$\begin{array}{ccc} \pi(X, Y) & \xrightarrow{\psi} & \pi_0 H(X, Y) \\ & \searrow \cong & \downarrow \phi \\ & & [X, Y] \end{array}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k . Form the diagram

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow j & \searrow g & \\ X & & & & Y \\ & p \swarrow & & \searrow \theta & \\ & & V & & \end{array}$$

where j is a trivial cofibration and p is a fibration; the map θ exists because Y is fibrant. The object X is cofibrant, so the trivial fibration p has a section σ ,

and there is a commutative diagram

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow 1 & \downarrow \sigma & \searrow \theta\sigma & \\
 X & & & & Y \\
 & \nwarrow p & \downarrow \theta & \nearrow & \\
 & & V & &
 \end{array}$$

The composite $\theta\sigma$ is the required map k . □

2 Torsors

2.1 Torsors for sheaves of groups

Suppose that G is a sheaf of groups on a small Grothendieck site \mathcal{C} .

A G -torsor is usually defined to be a sheaf X with a free G -action such that the map $X/G \rightarrow *$ is an isomorphism in the sheaf category.

The G -action on X is free if and only if the canonical simplicial sheaf map $EG \times_G X \rightarrow X/G$ is a local weak equivalence. One sees this by noting that the fundamental groups of the Borel construction $EG \times_G X$ are isotropy subgroups for the G -action. Further, $EG \times_G X$ is the nerve of a groupoid so there are no higher homotopy groups.

It follows that a sheaf X with G -action is a G -torsor if and only if the canonical simplicial sheaf map $EG \times_G X \rightarrow *$ is a local weak equivalence. Write $G - \mathbf{Tors}$ for the groupoid of G -torsors and G -equivariant maps.

Suppose given a cocycle

$$* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$$

in the category of simplicial sheaves. Form pullback

$$\begin{array}{ccc}
 \mathrm{pb}(Y) & \longrightarrow & Y \\
 \downarrow & & \downarrow \alpha \\
 EG & \xrightarrow{\pi} & BG
 \end{array} \tag{4}$$

Then the simplicial sheaf $\mathrm{pb}(Y)$ inherits a G -action from the G -action on EG , and the induced map $EG \times_G \mathrm{pb}(Y) \rightarrow Y$ is a weak equivalence. The square (4) is locally homotopy cartesian, and it follows that the map $\mathrm{pb}(Y) \rightarrow \tilde{\pi}_0 \mathrm{pb}(Y)$ is a G -equivariant weak equivalence. Here, $\tilde{\pi}_0 \mathrm{pb}(Y)$ is the sheaf of path components of the simplicial sheaf $\mathrm{pb}(Y)$, in this case identified with a constant simplicial sheaf.

It follows that the maps

$$EG \times_G \tilde{\pi}_0 \mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \rightarrow Y \simeq *$$

are weak equivalences, so that $\tilde{\pi}_0 \mathrm{pb}(Y)$ is a G -torsor. A functor

$$H(*, BG) \rightarrow G - \mathbf{Tors} \tag{5}$$

is therefore defined by sending the cocycle $* \xleftarrow{\simeq} Y \rightarrow BG$ to the torsor $\tilde{\pi}_0 \text{pb}(Y)$.

A functor

$$G - \mathbf{Tors} \rightarrow H(*, BG) \quad (6)$$

is defined by sending a G -torsor X to the cocycle $* \xleftarrow{\simeq} EG \times_G X \rightarrow BG$.

Theorem 5. *Suppose that \mathcal{C} is a small Grothendieck site, and that G is a sheaf of groups on \mathcal{C} . Then the functors (5) and (6) induce bijections*

$$[* , BG] \cong \pi_0 H(*, BG) \cong \pi_0(G - \mathbf{Tors}) = H^1(\mathcal{C}, G).$$

Proof. The functors (5) and (6) induce functions $\pi_0 H(*, BG) \rightarrow \pi_0(G - \mathbf{Tors})$ and $\pi_0(G - \mathbf{Tors}) \rightarrow \pi_0 H(*, BG)$ which are inverse to each other. To see this, there are two statements to verify, only one of which is non-trivial.

To show that the composite

$$\pi_0(G - \mathbf{Tors}) \rightarrow \pi_0 H(*, G) \rightarrow \pi_0(G - \mathbf{Tors})$$

is the identity, one uses the fact that the diagram

$$\begin{array}{ccc} X & \longrightarrow & EG \times_G X \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG \end{array}$$

is locally homotopy cartesian for each G -sheaf X . This is a consequence of Quillen's Theorem B [4, IV.5.2]. \square

The identification of the non-abelian cohomology invariant $H^1(\mathcal{C}, G)$ with morphisms $[\ast, BG]$ in the homotopy category of simplicial sheaves of Theorem 5 is, at this writing, almost twenty years old [8]. Unlike the original proof, the demonstration given here contains no references to hypercovers or pro objects.

2.2 Diagrams and torsors

There is a local model structure for simplicial presheaves on a small site \mathcal{C} which is Quillen equivalent to the local model structure for simplicial sheaves [7] — this has also been known for some time. Just recently [13], it has been shown that both the torsor concept and the homotopy classification result Theorem 5 admit substantial generalizations, to the context of diagrams of simplicial presheaves defined on presheaves of categories enriched in simplicial sets.

To explain a little, when I say that A is a presheaf of categories enriched in simplicial sets I mean the standard thing, that A consists of a presheaf $\text{Ob}(A)$ and a simplicial presheaf $\text{Mor}(A)$, together with source and target maps $s, t : \text{Mor}(A) \rightarrow \text{Ob}(A)$, a map $e : \text{Ob}(A) \rightarrow \text{Mor}(A)$ which is a section for both s and t , and an associative law of composition

$$\text{Mor}(A) \times_{t,s} \text{Mor}(A) \rightarrow \text{Mor}(A)$$

for which the map e is a two-sided identity. Here, the simplicial presheaf $\text{Mor}(A) \times_{t,s} \text{Mor}(A)$ is defined by the pullback

$$\begin{array}{ccc} \text{Mor}(A) \times_{t,s} \text{Mor}(A) & \longrightarrow & \text{Mor}(A) \\ \downarrow & & \downarrow s \\ \text{Mor}(A) & \xrightarrow{t} & \text{Ob}(A) \end{array}$$

and describes composable pairs of morphisms in A . To say it a different way, A is a category object in simplicial presheaves with simplicially discrete objects.

An A -diagram X (expressed internally [1, p.325], [20, p.240]) consists of a simplicial presheaf map $\pi : X \rightarrow \text{Ob}(A)$, together with an action

$$\begin{array}{ccc} \text{Mor}(A) \times_{s,\pi} X & \xrightarrow{m} & X \\ \downarrow & & \downarrow \pi \\ \text{Mor}(A) & \xrightarrow{t} & \text{Ob}(A) \end{array} \quad (7)$$

of the morphism object $\text{Mor}(A)$ which is associative and respects identities.

The proof of Theorem 5 uses Quillen's Theorem B, which can be interpreted as saying that if $Y : I \rightarrow \mathbf{sSet}$ is an ordinary diagram of simplicial sets defined on a small category I such that every morphism $i \rightarrow j$ induces a weak equivalence $Y_i \rightarrow Y_j$, then the pullback diagram

$$\begin{array}{ccc} Y & \longrightarrow & \text{holim}_I Y \\ \pi \downarrow & & \downarrow \\ \text{Ob}(I) & \longrightarrow & BI \end{array}$$

is homotopy cartesian. Here, we write $Y = \bigsqcup_{i \in \text{Ob}(I)} Y_i$. This result automatically holds when the index category I is a groupoid, but for more general index categories we have to be more careful.

Say that $Y : I \rightarrow \mathbf{sSet}$ is a *diagram of equivalences* if all induced maps $Y_i \rightarrow Y_j$ are weak equivalences of simplicial sets, and observe that this is equivalent to the requirement that the corresponding action

$$\begin{array}{ccc} \text{Mor}(I) \times_{s,\pi} Y & \xrightarrow{m} & Y \\ \downarrow & & \downarrow \pi \\ \text{Mor}(I) & \xrightarrow{t} & \text{Ob}(I) \end{array}$$

is homotopy cartesian. This is the formulation that works in general: given a presheaf of categories A enriched in simplicial sets, I say that an A -diagram X is a diagram of equivalences if the action diagram (7) is homotopy cartesian for the local model structure on simplicial presheaves.

The Borel construction $EG \times_G Z$ for a sheaf (or presheaf) Z having an action by a sheaf of groups G is the homotopy colimit for the action, thought of as a diagram defined on G . In general, say that an A -diagram X is an A -torsor if

- 1) X is a diagram of equivalences, and
- 2) the canonical map $\underline{\mathrm{holim}}_A X \rightarrow *$ is a local weak equivalence.

A map $X \rightarrow Y$ of A -torsors is just a natural transformation, meaning a simplicial presheaf map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & \mathrm{Ob}(A) & \end{array}$$

over $\mathrm{Ob}(A)$ which respects actions in the obvious sense. Write $A - \mathbf{Tors}$ for the corresponding category of A -torsors. This category of torsors is not a groupoid in general, but one can show that every map of A -torsors is a local weak equivalence.

Now here's the theorem:

Theorem 6. *Suppose that A is a presheaf of categories enriched on simplicial sets on a small Grothendieck site \mathcal{C} . Then the homotopy colimit functor induces a bijection*

$$\pi_0(A - \mathbf{Tors}) \cong \pi_0 H(*, BA) \cong [* , BA].$$

This result is proved in [13], by a method which generalizes the proof of Theorem 5. This same collection of ideas is also strongly implicated in the homotopy invariance results for stack cohomology which appear in [12].

The definition of A -torsor and the homotopy classification result Theorem 6 have analogues in localized model categories of simplicial presheaves, provided that those model structures are right proper (so that Theorem 2 applies). This is proved in [13]. The motivic model category of Morel and Voevodsky [21] is an example of a localized model structure for which this result holds.

2.3 Stack completion

The overall technique displayed in this section specializes to give an explicit model for the stack associated to a sheaf of groupoids H — see [14]. In general, the stack associated to H has global sections with objects given by the “discrete” H -torsors. A discrete H -torsor is a H -torsor X as above, with the extra requirement that X is simplicially constant on a sheaf of vertices; alternatively, one could say that X is a H -functor taking values in sheaves such that the map $\underline{\mathrm{holim}}_H X \rightarrow *$ is a weak equivalence. This construction of the stack associated to H is a direct generalization of the classical observation that the groupoid of G -torsors forms the stack associated to a sheaf of groups G .

2.4 Homotopy colimits

Suppose that G is a presheaf of groupoids.

Write $s\text{Pre}(\mathcal{C})^G$ for the category of G -diagrams in simplicial presheaves, defined for $G = A$ as above. Following [12], this category of G -diagrams has an injective model structure for which the weak equivalences (respectively cofibrations) are those maps of G -diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & \text{Ob}(G) & \end{array}$$

for which the simplicial presheaf map $f : X \rightarrow Y$ is a local weak equivalence (respectively cofibration).

Write $s\text{Pre}(\mathcal{C})/BG$ for the category of simplicial presheaf maps $X \rightarrow BG$, with morphisms

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow & \swarrow \\ & BG & \end{array}$$

In a standard way, this category inherits a model structure from simplicial presheaves, for which a map as above is a weak equivalence (respectively cofibration, fibration) if and only if the simplicial presheaf map $g : X \rightarrow Y$ is a local weak equivalence (respectively cofibration, global fibration) of simplicial presheaves.

The homotopy colimit construction defines a functor

$$\underline{\text{holim}}_G : s\text{Pre}(\mathcal{C})^G \rightarrow s\text{Pre}(\mathcal{C})/BG.$$

Since G is a presheaf of groupoids, this functor has a left adjoint

$$\text{pb} : s\text{Pre}(\mathcal{C})/BG \rightarrow s\text{Pre}(\mathcal{C})^G,$$

which is defined at $X \rightarrow BG$ in sections for $U \in \mathcal{C}$ by pulling back the map $X(U) \rightarrow BG(U)$ over the canonical maps $BG(U)/x \rightarrow BG(U)$. The functor pb preserves cofibrations and weak equivalences (this by Quillen's Theorem B), but more is true: the functors pb and $\underline{\text{holim}}_G$ define a Quillen equivalence

$$\text{pb} : s\text{Pre}(\mathcal{C})/BG \rightleftarrows s\text{Pre}(\mathcal{C})^G : \underline{\text{holim}}_G$$

relating the model structures for these categories. This result is Lemma 18 of [12].

Suppose now that X is a fixed G -diagram in simplicial presheaves, and write $G - \mathbf{Tors}/X$ for the category with objects consisting of G -diagram morphisms

$A \rightarrow X$ with A a G -torsor. The morphisms of $G - \mathbf{Tors}/X$ are commutative diagrams

$$\begin{array}{ccc} A & & \\ \theta \downarrow & \searrow & \\ B & \nearrow & X \end{array}$$

of morphisms of G -diagrams. One can show that the map θ must be a weak equivalence of G -diagrams in all such pictures. The homotopy colimit functor defines a functor

$$\underline{\mathrm{holim}}_G : G - \mathbf{Tors}/X \rightarrow H(*, \underline{\mathrm{holim}}_G X)$$

in an essentially obvious way: the G -diagram map $A \rightarrow X$ is taken to the cocycle

$$* \xleftarrow{\simeq} \underline{\mathrm{holim}}_G A \rightarrow \underline{\mathrm{holim}}_A X.$$

On the other hand, given a cocycle $* \xleftarrow{\simeq} Y \rightarrow \underline{\mathrm{holim}}_G X$ the adjoint $\mathrm{pb}(Y) \rightarrow X$ is an object of $G - \mathbf{Tors}/X$, so the adjunction defines a functor

$$\mathrm{pb} : H(*, \underline{\mathrm{holim}}_G X) \rightarrow G - \mathbf{Tors}/X.$$

These functors are adjoint and therefore define inverse functions in path components, so we have the following:

Theorem 7. *Suppose that G is a presheaf of groupoids and that X is a G -diagram in simplicial presheaves. Then there are natural bijections*

$$\pi_0(G - \mathbf{Tors}/X) \cong \pi_0 H(*, \underline{\mathrm{holim}}_G X) \cong [* , \underline{\mathrm{holim}}_G X].$$

The identification of $\pi_0 H(*, \underline{\mathrm{holim}}_G X)$ with morphisms $[* , \underline{\mathrm{holim}}_G X]$ in the homotopy category of simplicial presheaves in the statement of Theorem 7 is a consequence of Theorem 2.

Theorem 7 is a generalization of Theorem 16 of [10], which deals with the case where G is a sheaf of groups and X is a sheaf (ie. constant simplicial sheaf) with G -action.

Example 8. Let $et|_S$ be the étale site for a scheme S , and suppose that G is a discrete group acting by permutations on a torus $\mathbb{G}_m^{\times n}$ defined over S (more concretely, think of the action of the Weyl group on a maximal torus in an algebraic group). The classifying object $B(G \ltimes \mathbb{G}_m^{\times n})$ is weakly equivalent to the homotopy colimit $EG \times_G B(\mathrm{St} \mathbb{G}_m)^{\times n}$, where $\mathrm{St} \mathbb{G}_m$ is the stack completion for the multiplicative group \mathbb{G}_m . The presheaf of groupoids $\mathrm{St} \mathbb{G}_m$ is equivalent to the groupoid of line bundles in each section. It follows from Theorem 7 that every torsor for the group $G \ltimes \mathbb{G}_m^{\times n}$ can be identified with a choice of family of line bundles $L_U = (L_1, L_2, \dots, L_n)$, one for each element $U \rightarrow S$ of an étale covering family for S , together with group elements $g_{U,V} \in G$ inducing permutation isomorphisms $L_U|_{U \cap V} \cong L_V|_{U \cap V}$, where the group elements $g_{U,V} \in G$ form a cocycle $C(U) \rightarrow BG$ defined on the Čech resolution associated to the covering family $U \rightarrow S$.

3 Abelian sheaf cohomology

Suppose that A is a sheaf of abelian groups, and let $A \rightarrow J$ be an injective resolution of A , thought of as a \mathbb{Z} -graded chain complex and concentrated in negative degrees. Identify A with a chain complex concentrated in degree 0, and consider the shifted chain map $A[-n] \rightarrow J[-n]$. Observe that $A[-n]$ is the chain complex consisting of A concentrated in degree n . Recall that $K(A, n) = \Gamma A[-n]$ defines the Eilenberg-Mac Lane sheaf associated to A , where $\Gamma : \mathbf{Ch}_+ \rightarrow s\mathbf{Ab}$ is the functor appearing in the Dold-Kan correspondence. Let $K(J, n) = \Gamma T(J[-n])$ where $T(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees. In particular, $T(J[-n])_0$ is the kernel of the boundary map $J_{-n} \rightarrow J_{-n-1}$.

As usual, write $\mathbb{Z}X$ for the free simplicial abelian group on a simplicial set X , and write NA for the normalized chain complex of a simplicial abelian group A . Write $\pi_{ch}(C, D)$ for chain classes of maps between presheaves of \mathbb{Z} -graded chain complexes C and D .

Lemma 9. *Every local weak equivalence of simplicial presheaves $f : X \rightarrow Y$ induces an isomorphism*

$$\pi_{ch}(N\mathbb{Z}Y, J[-n]) \xrightarrow{\cong} \pi_{ch}(N\mathbb{Z}X, J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Proof. Starting with the third quadrant bicomplex $\text{hom}(\mathbb{Z}X_p, J^q)$ one constructs a spectral sequence

$$E_2^{p,q} = \text{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\mathbb{Z}X, J[-p-q])$$

(see [6]). The map f induces a homology sheaf isomorphism $N\mathbb{Z}X \rightarrow N\mathbb{Z}Y$, and then a comparison of spectral sequences gives the desired result. \square

Recall [2] that the category of \mathcal{C}^{op} -diagrams in simplicial sets has a projective model structure for which the fibrations (respectively weak equivalences) are the maps $f : X \rightarrow Y$ which are defined sectionwise (aka. pointwise) in the sense that each map $f : X(U) \rightarrow Y(U)$, $U \in \text{Ob}(\mathcal{C})$ is a fibration (respectively weak equivalence) of simplicial sets.

If two chain maps $f, g : N\mathbb{Z}X \rightarrow J[-n]$ are chain homotopic, then the corresponding maps $f_*, g_* : X \rightarrow K(J, n)$ are right homotopic in the projective model structure for \mathcal{C}^{op} -diagrams. Choose a sectionwise trivial fibration $\pi : W \rightarrow X$ such that W is projective cofibrant. Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W , again in the projective

structure. This means that there is a diagram

$$\begin{array}{ccccc}
& & W & \xrightarrow{\pi} & X \\
& \swarrow 1 & \downarrow i_0 & & \searrow f_* \\
W & \xleftarrow{s} & W \otimes I & \xrightarrow{h} & K(J, n) \\
& \nwarrow 1 & \uparrow i_1 & & \nearrow g_* \\
& & W & \xrightarrow{\pi} & X
\end{array}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. It follows that

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J, n))$, where $H(X, K(J, n))$ is a cocycle category for the local model structure on simplicial presheaves. It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\mathbb{Z}X, J[-n]) \rightarrow \pi_0 H(X, K(J, n)).$$

This map is natural in simplicial presheaves X .

Lemma 10. *The map ϕ is an isomorphism.*

Proof. Suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J, n)$ is an object of $H(X, K(J, n))$. The map f is a local weak equivalence, so by Lemma 9 there is a unique chain homotopy class $[v] : N\mathbb{Z}X \rightarrow J[-n]$ such that $[v_*f] = [g]$. This chain homotopy class $[v]$ is independent of the choice of representative for the component of (f, g) . We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J, n)) \rightarrow \pi_{ch}(N\mathbb{Z}X, J[-n]).$$

The composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. \square

Corollary 11. *Suppose that A is a sheaf of abelian groups on \mathcal{C} , and let $A \rightarrow J$ be an injective resolution of A in the category of abelian sheaves.*

1) *Let X be a simplicial presheaf on \mathcal{C} . Then there is a natural isomorphism*

$$\pi_{ch}(N\mathbb{Z}X, J[-n]) \cong [X, K(A, n)].$$

2) *There is a natural isomorphism*

$$H^n(\mathcal{C}, A) \cong [*, K(A, n)].$$

relating sheaf cohomology to morphisms in the homotopy category of simplicial presheaves (or sheaves).

Proof. This result is a consequence of Theorem 2 and Lemma 10. Observe that the map $K(A, n) \rightarrow K(J, n)$ is a local weak equivalence.

The second statement is a consequence of the first, and arises from the case where X is the terminal simplicial presheaf $*$. \square

4 Group extensions and 2-groupoids

In this section we shall see that group extensions can be classified by path components of cocycles in 2-groupoids, by a very simple argument.

This is subject to knowing a few things about 2-groupoids and their homotopy theory. Recall that a 2-groupoid H is a groupoid enriched in groupoids. Equivalently, H is a groupoid enriched in simplicial sets such that all simplicial sets of morphisms $H(x, y)$ are nerves of groupoids. The object H is, in particular, a simplicial groupoid, and therefore has a bisimplicial nerve BH with associated diagonal simplicial set dBH .

One says that a map $G \rightarrow H$ of 2-groupoids is a weak equivalence if it induces a weak equivalence of simplicial sets $dBG \rightarrow dBH$. There is a natural weak equivalence of simplicial sets

$$dBH \simeq \overline{WH}$$

relating dBH to the space of universal cocycles \overline{WH} [4, V.7], [16], so that $G \rightarrow H$ is a weak equivalence of 2-groupoids if and only if the induced map $\overline{WG} \rightarrow \overline{WH}$ is a weak equivalence of simplicial sets. One can also show that a map $G \rightarrow H$ is a weak equivalence if and only if it induces an isomorphism $\pi_0 G_0 \cong \pi_0 H_0$ of path components, and a weak equivalence of groupoids $G(x, y) \rightarrow H(f(x), f(y))$ for all objects x, y of G . Here, G_0 is the groupoid of 0-cells and 1-cells of G , or equivalently the groupoid in simplicial degree 0 for the corresponding groupoid enriched in simplicial sets.

The weak equivalences of 2-groupoids are part of a general picture: there is a model structure on groupoids enriched in simplicial sets, due to Dwyer and Kan (see [4, V.7.6, V.7.8]), for which a map $f : G \rightarrow H$ is a weak equivalence (respectively fibration) if and only if the induced map $f_* : \overline{WG} \rightarrow \overline{WH}$ is a weak equivalence (respectively fibration) of simplicial sets. This model structure is Quillen equivalent to the standard model structure for simplicial sets [4, V.7.11]. The model structure for groupoids enriched in simplicial sets restricts to a model structure for 2-groupoids, having the same definitions of fibration and weak equivalence [19], and it is easy to see that both model structures are right proper.

Here are some simple examples of 2-groupoids:

- 1) If K is a group, then there is a 2-groupoid $\mathbf{Aut}(K)$ with a single 0-cell, with 1-cells given by the automorphisms of K , and with 2-cells given by homotopies (aka. conjugacies) of automorphisms of K .
- 2) Suppose that $p : G \rightarrow H$ is a surjective group homomorphism. Then there is a 2-groupoid \tilde{p} with a single 0-cell, 1-cells given by the morphisms of G , and there is a 2-cell $g \rightarrow h$ if and only if $p(g) = p(h)$.

There are canonical morphisms of 2-groupoids

$$H \xleftarrow{\pi} \tilde{p} \xrightarrow{F} \mathbf{Aut}(K), \tag{8}$$

The map π is p on 1-cells, and takes a 2-cell $g \rightarrow h$ to the identity on $p(g) = p(h)$, whereas the map F takes a 1-cell g to conjugation by g , and takes the 2-cell $g \rightarrow h$ to conjugation by $hg^{-1} \in K$. The map $\pi : \tilde{p} \rightarrow H$ is also a weak equivalence of 2-groupoids, since the groupoid of 1-cells and 2-cells of \tilde{p} is the “Čech groupoid” associated associated to the underlying surjective function $G \rightarrow H$. In general, if $f : X \rightarrow Y$ is a surjective function, then the associated Čech groupoid has objects given by the elements of X , and a unique morphism $x \rightarrow y$ if and only if $f(x) = f(y)$.

We have therefore produced a cocycle (8) in 2-groupoids from a short exact sequence

$$e \rightarrow K \xrightarrow{i} G \xrightarrow{p} H \rightarrow e$$

Write $\mathbf{Ext}(H, K)$ for the usual groupoid of all such exact sequences. The cocycle construction is natural, and defines a functor

$$\phi : \mathbf{Ext}(H, K) \rightarrow H_*(H, \mathbf{Aut}(K))$$

taking values in the cocycle category $H_*(H, \mathbf{Aut}(K))$ in pointed 2-groupoids. All objects in the cocycle (8) have unique 0-cells, so the maps making up the cocycle are pointed in an obvious way.

Theorem 12. *The functor ϕ induces isomorphisms*

$$\pi_0 \mathbf{Ext}(H, K) \cong \pi_0 H_*(H, \mathbf{Aut}(K)) \cong [BH, dB\mathbf{Aut}(K)]_*,$$

where $[,]_*$ denotes morphisms in the pointed homotopy category.

Proof. If

$$H \xleftarrow{\pi} A \xrightarrow{F} \mathbf{Aut}(K)$$

is a pointed cocycle, then the base point $x \in A_0$ determines a 2-groupoid equivalence $A(x, x) \rightarrow H$. The cocycle F can therefore be canonically replaced by its restriction to $A(x, x)$ at the base point x , and the 2-groupoid $A(x, x)$ can be identified with a 2-groupoid p_* arising from a surjective group homomorphism $p : L \rightarrow H$ with 2-cells consisting of pairs (g, h) such that $p(g) = p(h)$.

Suppose given a cocycle

$$H \xleftarrow{\pi} p_* \xrightarrow{F} \mathbf{Aut}(K)$$

where $\pi : p_* \rightarrow H$ is determined a surjective group homomorphism $p : L \rightarrow H$ as above. There is a group $E_F(p)$ which is the set of equivalence classes of pairs (k, x) , $x \in L, k \in K$ such that $(k, x) \sim (k', x')$ if and only if $p(x) = p(x')$ and $k' = F(x, x')k$. Recall that $F(x, x')$ is a homotopy of the automorphisms $F(x), F(x')$ of K , and is therefore defined by conjugation by an element $F(x, x') \in K$. The product is defined by

$$[(k, x)] \cdot [(l, y)] = [(kF(x)(l), xy)]$$

and there is a short exact sequence

$$e \rightarrow K \rightarrow E_F(p) \rightarrow H \rightarrow e$$

where $k \mapsto [(k, e)]$ and $[(k, x)] \mapsto p(x)$.

We have, with these constructions, described a functor

$$\psi : H_*(H, \mathbf{Aut}(K)) \rightarrow \mathbf{Ext}(H, K).$$

One shows that the associated function ψ_* on path components is the inverse of the function

$$\phi_* : \pi_0 \mathbf{Ext}(H, K) \rightarrow \pi_0 H_*(H, \mathbf{Aut}(K)).$$

The homotopy category of groupoids enriched in simplicial sets is equivalent to the homotopy category of simplicial sets, and this equivalence is induced by the universal cocycles functor \overline{W} . It follows that there is a bijection

$$[H, \mathbf{Aut}(K)]_* \cong [BH, dB\mathbf{Aut}(K)]_*,$$

where the morphisms on the left are in the pointed homotopy category of groupoids enriched in simplicial sets. The set $[H, \mathbf{Aut}(K)]_*$ can be also identified with morphisms in the pointed homotopy category of 2-groupoids. One sees this most effectively by observing that for every cocycle

$$H \xleftarrow{\simeq} B \rightarrow \mathbf{Aut}(K)$$

in groupoids enriched in simplicial sets, the object B is weakly equivalent to its fundamental groupoid, and so the cocycle can be canonically replaced by a cocycle in 2-groupoids. \square

5 Classification of gerbes

A gerbe is a stack G which is locally path connected in the sense that the sheaf of path components $\tilde{\pi}_0(G)$ is isomorphic to the terminal sheaf. Stacks are really just homotopy types of presheaves (or sheaves) of groupoids [18], [10], [5], so one may as well say that a gerbe is a locally connected presheaf of groupoids.

A morphism of gerbes is a morphism $G \rightarrow H$ of presheaves of groupoids which is a weak equivalence in the sense that the induced map $BG \rightarrow BH$ is a local weak equivalence of classifying simplicial sheaves. Write **gerbe** for the category of gerbes.

Write **Grp** for the “presheaf” of 2-groupoids whose objects are sheaves of groups, 1-cells are isomorphisms of sheaves of groups, and whose 2-cells are the homotopies of isomorphisms of sheaves of groups. The object **Grp** is not really a presheaf of 2-groupoids because it’s too big in the sense that it does not take values in small 2-groupoids.

Write $H(*, \mathbf{Grp})$ for the category of cocycles

$$* \xleftarrow{\simeq} A \rightarrow \mathbf{Grp}$$

where A is a presheaf of 2-groupoids. One can sensibly discuss such a category, even though the object \mathbf{Grp} is too big to be a presheaf. The category $H(*, \mathbf{Grp})$ is not small, and its path components do not form a set. Similarly, the path components of the category of gerbes do not form a set. It is, nevertheless, convenient to display the relationship between these objects in the following statement:

Theorem 13. *There is a bijection*

$$\pi_0 H(*, \mathbf{Grp}) \cong \pi_0(\mathbf{gerbe}).$$

The proof of Theorem 13 is a bit technical, and appears in [15]. It is relatively easy to say, however, how to get a cocycle from a gerbe G . Write \tilde{G} for the 2-groupoid whose 0-cells and 1-cells are the objects and morphisms of G , respectively, and say that there is a unique 2-cell $\alpha \rightarrow \beta$ between any two arrows $\alpha, \beta : x \rightarrow y$. Then the canonical map $\tilde{G} \rightarrow *$ is an equivalence. There is a map $F(G) : \tilde{G} \rightarrow \mathbf{Grp}$ which associates to $x \in G(U)$ the sheaf $G(x, x)$ of automorphisms of x on \mathcal{C}/U , associates to $\alpha : x \rightarrow y$ the isomorphism $G(x, x) \rightarrow G(y, y)$ defined by conjugation by α , and associates to a 2-cell $\alpha \rightarrow \beta$ the homotopy defined by conjugation by $\beta\alpha^{-1}$. This cocycle construction effectively defines the function

$$\pi_0(\mathbf{gerbe}) \rightarrow \pi_0 H(*, \mathbf{Grp}).$$

A generalized Grothendieck construction is used to define its inverse — the construction of a group from a cocycle in the proof of Theorem 12 is a special case.

One can go further: the gerbes with band $L \in H^1(\mathcal{C}, \mathbf{Out})$ are classified by cocycles in the homotopy fibre over L of a morphism of fibrant presheaves of 2-groupoids approximating the map

$$\mathbf{Grp} \rightarrow \mathbf{Out}.$$

Here, \mathbf{Out} is the groupoid of outer automorphisms, or automorphisms modulo the homotopy relation. One can also use the same techniques to classify gerbes locally equivalent to a fixed gerbe G . These results are also proved in [15].

Remark 14. Suppose that E is a sheaf. An E -gerbe is a morphism of presheaves of groupoids $G \rightarrow E$ such that the induced map $\tilde{\pi}_0 G \rightarrow E$ is an isomorphism of sheaves. Write \mathbf{gerbe}/E for the category of E -gerbes. An E -gerbe is canonically a gerbe in the category of presheaves on the site \mathcal{C}/E fibred over E [12]. It follows that Theorem 13 specializes to a homotopy classification statement

$$\pi_0(\mathbf{gerbe}/E) \cong \pi_0 H(E, \mathbf{Grp})$$

for E -gerbes.

References

- [1] F. Borceux, *Handbook of Categorical Algebra I: Basic Category Theory*, Encyclopedia of Mathematics and its Applications **50**, Cambridge Univ. Press, Cambridge (1994).
- [2] A.K. Bousfield and D.M. Kan, *Homotopy limits completions and localizations*, Springer Lecture Notes in Math. **304** (2nd corrected printing), Springer-Verlag, Berlin-Heidelberg-New York (1987).
- [3] K.S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. AMS **186** (1973), 419–458.
- [4] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Math. **174**, Birkhäuser, Basel-Boston-Berlin (1999).
- [5] S. Hollander, *A homotopy theory for stacks*, Preprint (2001).
- [6] J.F. Jardine, *Simplicial objects in a Grothendieck topos*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory I, Contemporary Math. **55,I** (1986), 193–239.
- [7] J.F. Jardine, *Simplicial presheaves*, J. Pure Appl. Algebra **47** (1987), 35–87.
- [8] J.F. Jardine, *Universal Hasse-Witt classes*, “Algebraic K-theory and Algebraic Number Theory”, Contemp. Math. **83** (1989) 83–100.
- [9] J.F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552.
- [10] J.F. Jardine, *Stacks and the homotopy theory of simplicial sheaves*, Homology Homotopy Appl. **3**(2) (2001), 361–384.
- [11] J.F. Jardine, *Intermediate model structures for simplicial presheaves*, Preprint (2003).
- [12] J.F. Jardine, *Fibred sites and stack cohomology*, Preprint (2004).
- [13] J.F. Jardine, *Diagrams and torsors*, Preprint (2005).
- [14] J.F. Jardine, *Torsors and stacks*, Preprint (2005).
- [15] J.F. Jardine, *The homotopy classification of gerbes*, Preprint (2006).
- [16] J.F. Jardine and Z. Luo, *Higher principal bundles*, Preprint (2004), to appear in Math. Proc. Cambridge Philos. Soc.
- [17] A. Joyal, *Letter to A. Grothendieck* (1984).
- [18] A. Joyal and M. Tierney, *Strong stacks and classifying spaces*, Category Theory (Como, 1990), Springer Lecture Notes in Math. **1488** (1991), 213–236.
- [19] Z. Luo, *Closed model categories for presheaves of simplicial groupoids and presheaves of 2-groupoids*, Preprint (2003).
- [20] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer-Verlag, Berlin-Heidelberg-New York (1992).
- [21] F. Morel and V. Voevodsky, \mathbb{A}^1 homotopy theory of schemes, Publ. Math. IHES **90** (1999), 45–143.