Motivic interpretation of Milnor $K$-groups attached to Jacobian varieties

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Abstract

In the paper [Som90] p.105, Somekawa conjectures that his Milnor $K$-group $K(k, G_1, \ldots, G_r)$ attached to semi-abelian varieties $G_1, \ldots, G_r$ over a field $k$ is isomorphic to $\operatorname{Ext}_{\mathcal{M}_k}(Z, G_1[-1] \otimes \ldots \otimes G_r[-1])$ where $\mathcal{M}_k$ is a certain category of motives over $k$. The purpose of this note is to give remarks on this conjecture, when we take $\mathcal{M}_k$ as Voevodsky's category of motives $\operatorname{DM}^e(k)$.

Key words: motivic cohomology, 1-motives, Milnor $K$-groups, Weil reciprocity law

Contents

0 Introduction 2

1 Milnor $K$-groups attached to semi-abelian varieties 3
  1.1 Extension of valuations and tame symbols 3
  1.2 Definition of the Milnor $K$-groups attached to semi-abelian varieties 4

2 Triangulated categories of motives 5
  2.1 Triangulated category of effective geometric motives 5
  2.2 Triangulated category of effective motivic complexes 6
  2.3 Motives with compact support 9
  2.4 Tate object 10
  2.5 The triangulated category of geometric motives 11

3 Various morphisms between motives 12
  3.1 Transpose for finite equidimensional morphisms 12
  3.2 Pull back for flat equidimensional morphisms 12
  3.3 Motives with closed support 13
  3.4 Thom isomorphism 14
  3.5 Normal cone deformation 16
  3.6 Gysin triangles 16

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0 Introduction

To unify the Moore exact sequence and the Bloch exact sequence, K. Kato defined the generalized Milnor $K$-groups attached to finite family of semi-abelian varieties over a base field $k$ in [Som90]. (See also [Akh00], [Kah92].) That is, for semi-abelian varieties $G_1, \ldots, G_r$ over $k$, he associated the group $K(k, G_1, \ldots, G_r)$. (For precise definition, see 1.4) This group is a generalization of the Milnor $K$-group as the following example shows.

Example 0.1. In the notation above, if $G_1 = G_2 = \ldots = G_r = \mathbb{G}_m$, the following equality holds.

$$K(k, \mathbb{G}_m, \ldots, \mathbb{G}_m) = K^M_r(k)$$

On the other hand this group is also a generalization of the Bloch group $V$.

Example 0.2. Let $C$ be a projective smooth curve over $k$ such that $C(k) \neq \phi$. We have the following equality

$$K(k, \text{Jac} C, \mathbb{G}_m) = V(C),$$

where $V(C)$ is defined by S. Bloch (c.f. [Blo81]) as the following way

$$V(C) = \frac{\text{Ker} \left( \bigoplus_{x \in C^1} k(x)^* \xrightarrow{N} k(x)^* \right)}{\text{Im} \left( K_2(k(C)) \xrightarrow{\partial_2} \bigoplus_{x \in C^1} k(x)^* \right)}.$$

As is explained in [Som90], there is the generalized Bloch-Moore exact sequence.

0.3. Let $k$ be a number field and $A$ a semi-abelian variety over $k$. We write $G = \text{Gal}(\bar{k}/k)$ and $G_v = \text{Gal}(\bar{k}_v/k_v)$ for a place $v$. Let $T(A)$ be the Tate module of $A$. $S$ is a finite set of places including all Archimedean and places where $A$ has bad reduction. Then $T(A)_G$ is a finite group by owing to [KL81]. Let $m$ be a nonzero integer divisible by the order of $T(A)_G$. Somekawa proves the following generalized Moore-Bloch exact sequence (c.f. [Som90] Theorem 4.1):

$$K(k, A, \mathbb{G}_m) \to \bigoplus_{v \notin S} T(A)_{G_v} \oplus \bigoplus_{v \in S} K(k_v, A_v, \mathbb{G}_m)/m \to T(A)_G \to 0$$
In the case of $A = \mathbb{G}_m$, the above exact sequence is proved by Moore (c.f. [Moo69])

$$K_2(k) \to \bigoplus_{v:\text{not complex}} \mu(k_v) \to \mu(k) \to 0.$$ 

In the case of $A = \text{Jac} C$ in the notation in 0.2, the above exact sequence is proved by S. Bloch, K. Kato and S. Saito (c.f. [Blo81], [KS83])

$$V(C) \to \bigoplus_{v \in S} T(\text{Jac} C)_{G_v} \oplus \bigoplus_{v \in S} V(C \times_k k_v)/m \to T(\text{Jac} C)_G \to 0.$$ 

In [Som90], Somekawa conjectures that the Somekawa $K$-groups should be motivic cohomology groups attached to semi-abelian varieties. More precisely

**Conjecture 0.4. (Somekawa conjecture)**

Let $G_1, \ldots, G_r$ be semi-abelian varieties over $k$, then $K(k,G_1,\ldots,G_r)$ is isomorphic to $\text{Ext}^*_{\mathcal{M}_k}(\mathbb{Z}, G_1[-1] \otimes \cdots \otimes G_r[-1])$, where $\mathcal{M}_k$ is a certain category of motives over $k$ and $G_i[-1]$ means $1$-motif (c.f [Del74]).

In this paper we will examine this conjecture, if we take $\mathcal{M}_k$ as Voevodsky’s category of motives $\text{DM}^{eff}(k)$.

**Main Theorem 0.5. (Somekawa conjecture for Jacobian varieties)**

Let $(C_1, a_1), \ldots, (C_n, a_n)$ be pointed projective smooth curves over perfect field $k$ which admits resolution of singularities. Then

$$K(k, \text{Jac} C_1, \ldots, \text{Jac} C_n) \cong \text{Hom}_{\text{DM}^{eff}(k)}(\mathcal{M}_{\text{sm}}(\text{Spec } k), \mathbb{Z}(\bigotimes_{i=1}^n (C_i, a_i))[n]).$$ 

0.6. In this paper, let $k$ be a perfect field which admits resolution of singularity.

1 Milnor $K$-groups attached to semi-abelian varieties

1.1 Extension of valuations and tame symbols

1.1. Suppose $k$ is a field and $G$ is a semi-abelian variety defined over $k$, that is, there is an exact sequence of group schemes (viewed as sheaves in the flat topology) over $k$:

$$0 \to T \to G \to A \to 0$$

where $T$ is a torus and $A$ is an abelian variety.

1.2. In the notation above, let $K/k$ be an algebraic function field and $v$ a place of $K/k$. Let $L/K_v$ be a finite unramified Galois extension such that $T \times_k F \cong \mathbb{G}_m^n$ for the residue field $F$ of $L$ and some $n$; let $w$ be the unique extension of $v$ of $L$. We obtain the following commutative diagram of exact sequences defining a
map \( r_w = (r^1_w, \ldots, r^n_w) \);

\[
\begin{array}{cccc}
0 & \longrightarrow & T(O_w) & \longrightarrow & G(O_w) & \longrightarrow & A(O_w) & \longrightarrow & 0 \\
0 & \longrightarrow & T(L) & \longrightarrow & G(L) & \longrightarrow & A(L) & \longrightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{c}
\text{ord}_w \\
\text{id}
\end{array}
\]

\[
\begin{array}{c}
\mathbb{Z}^n \\
\mathbb{Z}^n
\end{array}
\]

1.3. In the notation above, we are going to construct a map

\[
\partial_v : \mathcal{K}_v \otimes \mathcal{K}_v^\times \rightarrow G(k(v)).
\]

Fix \( g \in \mathcal{K}_v \) and \( h \in \mathcal{K}_v^\times \). For each \( i = 1, \ldots, n \), we define \( h_i \in T(L) \) to be the \( n \)-th tuple having \( h \) in the \( i \)-th coordinate and \( 1 \) elsewhere. Then set

\[
\varepsilon(g, h) = ((-1)^{\text{ord}_w(h)} r^1_w(g), \ldots, (-1)^{\text{ord}_w(h)} r^n_w(g)) \in T(O_w) \subset G(O_w)
\]

and

\[
\partial_v(g, h) = \varepsilon(g, h) g^{\text{ord}_w(h)} \prod_{i=1}^n h_i^{-r^i_w(g)} \in G(O_w).
\]

We define the “extended tame symbol” \( \partial_v(g, h) \) to be the image of \( \partial_v(g, h) \) under the canonical map \( G(O_w) \rightarrow G(F) \); Then \( \partial_v(g, h) \) is invariant under the action of \( \text{Gal}(F/k(v)) \), so that it belongs to \( G(k(v)) \). This definition of \( \partial_v \) is independent of the choice of \( L \) and of the isomorphic from the torus to \( \mathbb{G}_m \otimes \mathbb{Z}^n \).

1.2 Definition of the Milnor \( K \)-groups attached to semi-abelian varieties

1.4. Let \( k \) be a field and \( G_1, \ldots, G_r \) a finite (possibly empty) family of semi-abelian varieties defined over \( k \). We define Milnor \( K \)-groups attached to semi-abelian varieties \( K(k, G_1, \ldots, G_r) \) as follows. If \( r = 0 \), we write \( K(k, \phi) \) for our groups and set \( K(k, \phi) = \mathbb{Z} \).

For \( r \geq 1 \), we define

\[
K(k, G_1, \ldots, G_r) = F/R
\]

where

\[
F = \bigoplus_{E/k \text{ finite}} G_1(E) \otimes \cdots \otimes G_r(E)
\]
and $R \subset F$ is the subgroup generated by the relation $\textbf{R1-R2}$ below.

**R1** For any finite extensions $k \hookrightarrow E_1 \overset{\phi}{\rightarrow} E_2$, let $g_{i_0} \in G_{i_0}(E_2)$ and $g_i \in G_i(E_1)$ for $i \neq i_0$, the relation

$$(\phi^*(g_1) \otimes \ldots \otimes g_{i_0} \otimes \ldots \otimes \phi^*(g_r))_{E_2} - (g_1 \otimes \ldots \otimes N_{E_2/E_1}(g_{i_0}) \otimes \ldots \otimes g_r)_{E_1}$$

(Here $N_{E_2/E_1}$ denotes the norm map on the group scheme $G_{i_0}$)

**R2** For every algebraic function field $K/k$ and all choices $g_i \in G_i(K), h \in K^\times$ such that for each place $v$ of $K/k$, there exists $i(v)$ such that $g_i \in G_i(O_v)$ for all $i \neq i(v)$, the relation

$$\sum_{v: 	ext{place of } K/k} (g_1(v) \otimes \ldots \otimes \partial_v(g_{i(v)}), h) \otimes \ldots \otimes g_r(v))_{k(v)/k}$$

Here $g_i(v) \in G_i(k(v))$ ($i \neq i(v)$) denotes the reduction of $g_i \in O_v$ modulo $m_v$ and $\partial_v(g_{i(v)}, h)$ is the extended tame symbol as defined in 1.3.

The class in $F/R$ of an element $a_1 \otimes \ldots \otimes a_r \in G_1(E) \otimes \ldots \otimes G_r(E)$ will be denoted $\{a_1, \ldots, a_r\}_{E/k}$.

**Remark 1.5.** By the relation $\textbf{R1}$, if $\phi$ is a $k$-isomorphism $E_1 \overset{\sim}{\rightarrow} E_2$, then $\{g_1, \ldots, g_r\}_{E_1/k} = \{\phi^*(g_1), \ldots, \phi^*(g_r)\}_{E_2/k}$. This shows that symbols form a set.

## 2 Triangulated categories of motives

In this section, we will briefly review the definition of the triangulated categories of motives. (c.f. [TriCa]).

### 2.1 Triangulated category of effective geometric motives

First we will review the construction of the category of geometric motives.

1. Let $\Sm/k$ be the category of schemes which are separated smooth, and of finite type over $k$.
2. Recall the definition of the category $\SmCor(k)$: its objects are those of $\Sm/k$. The set of morphism from $Y$ to $X$ is given by the group $\Cor(Y, X)$ of finite correspondences from $Y$ to $X$, defined as the free abelian group on the symbol $(Z)$, where $Z$ runs through the integral closed subschemes of $Y \times_k X$ which are finite over $Y$ and surjective over a connected component of $Y$. We will denote the object of $\SmCor(k)$ which corresponds to a smooth scheme $X$ by $[X]$.

2.2. The category $\SmCor(k)$ is an additive category. Consider the homotopy category $\mathcal{H}^b(\SmCor(k))$ of bounded complexes over $\SmCor(k)$. Let $\mathcal{T}$ be the class of complexes of the following two forms:

1. For any smooth scheme $X$ over $k$ the complex

$$[X \times \mathbb{A}_k^1] \xrightarrow{[\phi_1]} [X]$$

5
belongs to \( \mathcal{T} \).

2. For any smooth scheme \( X \) over \( k \) and an open covering \( X = U \cup V \) of \( X \) the complex

\[
[U \cap V] \xrightarrow{[iv]} [U] \oplus [V] \xrightarrow{[iv]} [X]
\]

belongs to \( \mathcal{T} \). (Here \( j_U, j_V, i_U, i_V \) are the obvious open embeddings.)

Denote by \( \tilde{T} \) the minimal thick subcategory of \( \mathcal{H}^b(\text{SmCor}(k)) \) which contains \( \mathcal{T} \).

The triangulated category \( \text{DM}_{\text{gm}}^\text{eff}(k) \) of effective geometric motives over \( k \) is the pseudo-Abelian envelope of the localization of \( \mathcal{H}^b(\text{SmCor}(k)) \) with respect to the thick subcategory \( \tilde{T} \). We denote the obvious functor \( Sm/k \to \text{DM}_{\text{gm}}^\text{eff}(k) \) by \( M_{\text{gm}} \).

2.3. For a pair of smooth schemes \( X, Y \) over \( k \), we set

\[
[X] \otimes [Y] := [X \times Y].
\]

For any smooth schemes \( X_1, Y_1, X_2, Y_2 \) the external product of cycles defines a homomorphism:

\[
\text{Cor}(X_1, Y_1) \otimes \text{Cor}(X_2, Y_2) \to \text{Cor}(X_1 \times X_2, Y_1 \times Y_2)
\]

which gives us a definition of tensor product of morphisms in \( \text{SmCor}(k) \). This structure defines in the usual way a tensor category structure on \( \mathcal{H}^b(\text{SmCor}(k)) \) which can be descended to the category \( \text{DM}_{\text{gm}}^\text{eff}(k) \) by the universal property of localization.

Note that the unit object our tensor structure is \( M_{\text{gm}}(\text{Spec } k) \). We will denote it by \( Z \).

**Example 2.4.** Let \( x, y : \text{Spec } k \to \mathbb{P}^1_k \) be two \( k \)-rational points. Then \( M_{\text{gm}}(x) = M_{\text{gm}}(y) : M_{\text{gm}}(\text{Spec } k) \to M_{\text{gm}}(\mathbb{P}^1_k) \).

**Proof.** We take an affine open set \( \mathbb{A}^1_k \xrightarrow{j} \mathbb{P}^1_k \) which contains \( x \) and \( y \). That is, there are \( \tilde{x}, \tilde{y} : \text{Spec } k \to \mathbb{A}^1_k \) such that \( x = j \circ \tilde{x} \) and \( y = j \circ \tilde{y} \). Then we have \( M_{\text{gm}}(x) = M_{\text{gm}}(j) \circ M_{\text{gm}}(\tilde{x}) = M_{\text{gm}}(j) \circ M_{\text{gm}}(\tilde{x}) = M_{\text{gm}}(y) \), where \( p : \mathbb{A}^1_k \to \text{Spec } k \) is the structure morphism.

### 2.2 Triangulated category of effective motivic complexes

To study the fundamental property of \( \text{DM}_{\text{gm}}^\text{eff}(k) \), Voevodsky uses sheaf theoretic method in [TriCa]. More precisely, he constructs another category \( \text{DM}_{\text{gm}}^\text{eff}(k) \) using a sheaf category and he proves \( \text{DM}_{\text{gm}}^\text{eff}(k) \) admits a natural full embedding as a tensor category and a triangulated category to the category \( \text{DM}_{\text{eff}}^\text{eff}(k) \). We will review the construction of \( \text{DM}_{\text{eff}}^\text{eff}(k) \).

2.5. 1. A presheaf with transfers on \( \text{Sm } k \) is an additive contravariant functor from the category \( \text{SmCor}(k) \) to the category of abelian groups. We denote by \( \text{PST}(k) \) the category of presheaf with transfers on \( \text{Sm } k \).
2. A presheaf with transfers on $Sm/k$ is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups on $Sm/k$ is a sheaf in the Nisnevich topology. We denote by $\text{Shv}_{\text{Nis}}(\text{SmCor}(k))$ the category of Nisnevich sheaves with transfers.

**Example 2.6.** For any smooth scheme $X$ over $k$, a presheaf $Z_{\text{tr}}(X) := \text{Cor}(?, X)$ is a Nisnevich sheaf with transfers on $Sm/k$, (c.f. [TriCa] Lemma 3.1.2).

For a $k$-rational point $x : \text{Spec} \, k \to X$, we put

$$Z_{\text{tr}}(X, x) := \text{Coker}(Z_{\text{tr}}(\text{Spec} \, k) \xrightarrow{Z_{\text{tr}}(x)} Z_{\text{tr}}(X)).$$

2.7. 1. A presheaf with transfers $F$ is called homotopy invariant if for any smooth scheme $X$ over $k$ the projection $X \times A^1_k \to X$ induces the isomorphism $F(X) \to F(X \times A^1_k)$.

2. A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transfers.

2.8. $\text{Shv}_{\text{Nis}}(\text{SmCor}(k))$ is an abelian category. (c.f. [TriCa] Theorem 3.1.4) Inside the derived category $D^{-}(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$ of complexes bounded from above, one defines the full subcategory $\text{DM}_{\text{eff}}(k)$ of effective motivic complexes over $k$ as the one consisting of objects whose cohomology sheaves are homotopy invariant. This subcategory is triangulated. (Need the assumption of perfection of $k$.) (c.f. [TriCa] Proposition 3.1.13).

2.9. 1. Let $F$ be a presheaf with transfers. There is a canonical surjection of presheaves

$$\bigoplus_{(X, x \in F(X))} Z_{\text{tr}}(X) \to F.$$ 

Iterating this construction we get a canonical left resolution $\mathcal{L}(F)$ of $F$ which consists of direct sums of presheaves of the form $Z_{\text{tr}}(X)$ for smooth schemes $X$ over $k$.

2. We set for two smooth schemes $X, Y$:

$$Z_{\text{tr}}(X) \otimes Z_{\text{tr}}(Y) := Z_{\text{tr}}(X \times Y)$$

and for two presheaves with transfers $F, G$:

$$F \otimes G := \mathbb{H}_0(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

3. This construction provides us with a tensor structure on the derived category $D^{-}(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$.

To define the tensor structure on $\text{DM}_{\text{eff}}(k)$, we will need an alternative description of $\text{DM}_{\text{eff}}(k)$.

2.10. Let $\Delta^\bullet$ be the standard cosimplicial object in $Sm/k$. For any presheaf with transfers $F$ on $Sm/k$ let $C_\ast(F)$ be the complex of presheaves on $Sm/k$ of the form $C_\ast(F) = \text{Hom}(\Delta^\bullet, F)$ with differentials given by alternated sums of morphisms which correspond to the boundary morphisms of $\Delta^\bullet$. This complex
is called the singular simplicial complex of $F$.
The following properties are fundamental.
1. If $F$ is a presheaf with transfers (resp. a Nisnevich sheaf with transfers) then
$C_*(F)$ is a complex of presheaves with transfers (resp. Nisnevich sheaves with transfers).
2. For any presheaf with transfers $F$ over $k$, the cohomology presheaf $h_i(F)$
of the complex $C_*(F)$ and its Nisnevich sheafification $h_i^{Nis}(F)$ are homotopy
invariant. (Need the assumption of perfection of $k$) (c.f. [TriCa] Lemma 3.2.1),
3. In view of 1. and 2., $C_*(?)$ is a functor from the category of Nisnevich sheaves
with transfers on $Sm/k$ to $DM_{eff}(k)$.

**Proposition 2.11.** (c.f. [TriCa] Proposition 3.2.3)
The functor $C_*(?)$ can be extended to a functor

$$RC : D^-(\text{Shv}_{Nis}(\text{SmCor}(k))) \to DM_{eff}(k)$$

which is left adjoint to the natural embedding. The functor $RC$ identifies $DM_{eff}(k)$
with localization of $D^-(\text{Shv}_{Nis}(\text{SmCor}(k)))$ with respect to the localizing subcat-egory
$\mathcal{A}$ generated by complexes of the form

$$Z_{tr}(X \times \mathbb{A}_{k}^1) \xrightarrow{Z_{tr}(pr_1)} Z_{tr}(X)$$

for smooth schemes $X$ over $k$.

**2.12.** 1. In the notation above, $\mathcal{A}$ is an $\otimes$-ideal, that is, for any object $T$ of
$D^-(\text{Shv}_{Nis}(\text{SmCor}(k)))$ and an object $S$ of $\mathcal{A}$ the object $T \otimes S$ belongs to $\mathcal{A}$.
(c.f. [TriCa] Lemma 3.2.4).
2. We define tensor structure on $DM_{eff}(k)$ as the descent of the tensor structure
on $D^-(\text{Shv}_{Nis}(\text{SmCor}(k)))$ with respect to the projector $RC$. Note that such a
descent exists by the universal property of localization and 1.

**Theorem 2.13.** (c.f. [TriCa] Theorem 3.2.6)
There is a commutative diagram of functors of the form such that the following
conditions hold:

$$
\begin{array}{ccc}
\mathcal{H}^b(\text{SmCor}(k)) & \xrightarrow{L} & D^-(\text{Shv}_{Nis}(\text{SmCor}(k))) \\
\downarrow & & \downarrow \circ \kern-2.7ex R_C \\
DM_{nm}^{eff}(k) & \xrightarrow{i} & DM_{eff}(k)
\end{array}
$$

1. The functor $i$ is a full embedding with a dense image.
2. For any smooth scheme $X$ over $k$ the object $RC(L(X))$ is canonically iso-
morphic to the $C_*(Z_{tr}(X))$.
3. All functors preserve tensor and triangulated structures.
Example 2.14. Let \( x : \text{Spec} \, k \to X \) be a \( k \)-rational point of a smooth scheme \( X \). Then we have an identification

\[
C_*(Z_{14}(X, x)) \cong \text{Cone}(\mathcal{M}_{\text{gm}}(\text{Spec} \, k) \xrightarrow{M_{\text{gm}}(x)} \mathcal{M}_{\text{gm}}(X))
\]

defined by

\[
\begin{bmatrix}
Z_{14}(\text{Spec} \, k) \\
Z_{14}(x) \\
Z_{14}(X)
\end{bmatrix} \to \begin{bmatrix}
0 \\
Z_{14}(X, x)
\end{bmatrix}.
\]

2.3 Motives with compact support

In this subsection, we will briefly review the notation and fundamental result of [FV00], [ReCy] and [TriCa].

2.15. For any scheme of finite type \( X \) over \( k \) and any \( r \geq 0 \) we denote by \( \mathcal{Z}_{\text{equi}}(X, r) \) the presheaf on the category of smooth schemes over \( k \) which takes a smooth scheme \( Y \) to free abelian groups generated by closed integral subschemes \( Z \) of \( X \times Y \) which are equidimensional of relative dimension \( r \) over \( Y \).

This presheaf has the following property.
1. \( \mathcal{Z}_{\text{equi}}(X, r) \) is a sheaf in the Nisnevich topology.
2. It has a canonical structure of a presheaf with transfers.
3. The presheaf with transfers \( \mathcal{Z}_{\text{equi}}(X, r) \) is covariantly functorial with respect to proper morphisms of \( X \) by means of the usual proper push-forward of cycles.
4. It is contravariantly functorial with an appropriate dimension shift with respect to flat equidimensional morphisms.
5. There is a pairing

\[
\times : \mathcal{Z}_{\text{equi}}(X, r) \otimes \mathcal{Z}_{\text{equi}}(X', r') \to \mathcal{Z}_{\text{equi}}(X \times X', r + r')
\]

of presheaves.

Let \( U \) be a smooth scheme over \( k \). For any pair of integral closed subschemes \( Z \subset X \times U, Z' \subset X' \times U \) equidimensional over \( U \). Sending \( Z, Z' \) to the cycle associated to the subscheme \( Z \times_U Z' \subset X \times X' \times U \) determines a pairing.

6. (Need the assumption of 0.6.) The flat pull-back morphism induces a quasi-isomorphism

\[
C_*(\mathcal{Z}_{\text{equi}}(X, 0)) \to C_*(\mathcal{Z}_{\text{equi}}(X \times \mathbb{A}^n, n))
\]

2.16. For any scheme of finite type over \( k \) the object \( C_*(\mathcal{Z}_{\text{equi}}(X, 0)) \) belongs to \( \text{DM}_{\text{eff}}^+(k) \). Moreover it belongs to \( \text{DM}_{\text{gm}}(k) \). (Using the assumption 0.6) (c.f. [TriCa], Corollary 4.1.6). We will denote \( M_{\text{gm}}(X) := C_*(\mathcal{Z}_{\text{equi}}(X, 0)) \) and call it a motivic complex of \( X \) with compact support.

Since \( Z_{14}(X) \) is a subsheaf of \( \mathcal{Z}_{\text{equi}}(X, 0) \), the inclusion morphism induces the natural morphism \( M_{\text{gm}}(X) \to M_{\text{gm}}^c(X) \).

The following properties are fundamental. (c.f. [TriCa], Proposition 4.1.5, Proposition 4.1.7)
1. If $X$ is proper then the canonical morphism $\text{M}_{\text{gm}}(X) \to M_{\text{gm}}^c(X)$ is the isomorphism.
2. (Need the assumption 0.6.) Let $Z$ be a closed subscheme of $X$. Then there is a canonical distinguished triangle of the form

$$M_{\text{gm}}^c(Z) \to M_{\text{gm}}^c(X) \to M_{\text{gm}}^c(X - Z) \to M_{\text{gm}}^c(Z)[1].$$

3. For $X, Y$ of finite type over $k$, the pairing $\mathcal{Z}_{\text{equi}}(X, 0) \otimes \mathcal{Z}_{\text{equi}}(Y, 0) \to \mathcal{Z}_{\text{equi}}(X \times_k Y, 0)$ induces an isomorphism

$$M_{\text{gm}}^c(X) \otimes M_{\text{gm}}^c(Y) \xrightarrow{\sim} M_{\text{gm}}^c(X \times_k Y).$$

### 2.4 Tate object

2.17. For any smooth scheme $X$ over $k$, the morphism $X \to \text{Spec } k$ gives us a morphism in $\text{DM}_{\text{gm}}^\text{eff}(k)$ of the form $\text{M}_{\text{gm}}(X) \to \mathcal{Z}$. There is a canonical split distinguished triangle

$$\widehat{\text{M}_{\text{gm}}(X)} \to \text{M}_{\text{gm}}(X) \to \mathcal{Z} \to \widehat{\text{M}_{\text{gm}}(X)}[1]$$

where $\widehat{\text{M}_{\text{gm}}(X)}$ is the reduced motif of $X$ represented in $\mathcal{H}^0(\text{SmCor}(k))$ by the complex $[X] \to [\text{Spec } k]$.

**Example 2.18.** In the notation above, for any $k$-rational point $x: \text{Spec } k \to X$, we have the canonical identification $\text{M}_{\text{gm}}(X, x) \xrightarrow{\sim} \widehat{\text{M}_{\text{gm}}(X)}$ as the following way.

$$\begin{bmatrix}
\text{[Spec } k] \\
x \\
\text{[X]} \\
0
\end{bmatrix} \quad \xrightarrow{x^* - \text{M}_{\text{gm}}(x \circ p)} \quad \begin{bmatrix}
0 \\
\text{[X]} \\
p
\end{bmatrix}$$

where $p: X \to \text{Spec } k$ is the structure morphism.

$x: \text{Spec } k \to X$ defines splitting $\text{M}_{\text{gm}}(X) \xrightarrow{\sim} \text{M}_{\text{gm}}(X, x) \oplus \mathcal{Z}$.

2.19. We define the Tate object $\mathcal{Z}(1)$ of $\text{DM}_{\text{gm}}^\text{eff}(k)$ as $\widehat{\text{M}_{\text{gm}}(\mathcal{P}^1_k)[-2]}$. We further define $\mathcal{Z}(n)$ to be the $n$-th tensor power of $\mathcal{Z}(1)$.

For any object $A$ of $\text{DM}_{\text{gm}}^\text{eff}(k)$ we put

$$A(n) = A \otimes \mathcal{Z}(n)$$

$$A\{n\} = A \otimes \mathcal{Z}(n)[n]$$

$$A((n)) = A \otimes \mathcal{Z}(n)[2n].$$

By Example 2.18, for any $x: \text{Spec } k \to \mathcal{P}^1_k$, we have the canonical isomorphism $\text{M}_{\text{gm}}(\mathcal{P}^1_k, x) \xrightarrow{\sim} \mathcal{Z}((1))$. 

10
2.20. Let $x: \text{Spec} \ k \to \mathbb{P}^1_k$ be a $k$-rational point. Comparing the following split distinguished triangles

\[
\begin{array}{cccc}
M_{\text{gm}}(\text{Spec} \ k) & \xrightarrow{\text{id}} & M_{\text{gm}}(\mathbb{P}^1_k) & \xrightarrow{\text{id}} M_{\text{gm}}(\mathbb{A}^1_k) & \xrightarrow{\text{id}} M_{\text{gm}}(\text{Spec} \ k)[1] \\
M_{\text{gm}}(\text{Spec} \ k) & \xrightarrow{M_{\text{gm}}(x)} & M_{\text{gm}}(\mathbb{P}^1_k) & \xrightarrow{\mathbb{Z}(1)} & M_{\text{gm}}(\text{Spec} \ k)[1],
\end{array}
\]

where $M_{\text{gm}}(\mathbb{P}^1_k) \to \mathbb{Z}(1)$ is defined by

\[M_{\text{gm}}(\mathbb{P}^1_k) \rightarrow M_{\text{gm}}(\mathbb{P}^1_k, x) \xrightarrow{\text{can}} \mathbb{Z}(1),\]

we know that there is a natural isomorphism $\mathbb{Z}(1) \xrightarrow{\sim} M_{\text{gm}}^0(\mathbb{A}^1_k)$. It does not depend on the choice of a $k$-rational point by Example 2.4. Similarly using Mayer-Vietoris sequence for canonical covering of $\mathbb{P}^1_k$, we know also that there is a natural isomorphism $\mathbb{Z} \{1\} \xrightarrow{\sim} \tilde{M}_{\text{gm}}(\mathbb{A}^1_k - \{0\}).$

2.5 The triangulated category of geometric motives

In this subsection, we will define the triangulated category $\text{DM}_{\text{gm}}(k)$ of geometric motives over $k$.

2.21. 1. We define the category $\text{DM}_{\text{gm}}(k)$: its objects are pairs of the form $(A, n)$ where $A$ is an object of $\text{DM}_{\text{gm}}^0(k)$ and $n \in \mathbb{Z}$ and morphisms are defined by the following formula

\[\text{Hom}_{\text{DM}_{\text{gm}}(k)}((A, n), (B, m)) := \lim_{k \geq n-m} \text{Hom}_{\text{DM}_{\text{gm}}(k)}(A(k+n), B(k+m)).\]

2. The category $\text{DM}_{\text{gm}}(k)$ with the obvious shift functor and class of distinguished triangles is a triangulated category.

3. The permutation involution on $\mathbb{Z}(1) \otimes \mathbb{Z}(1)$ is identity in $\text{DM}_{\text{gm}}^0(k)$. (c.f. [TriCa] Corollary 2.1.5)

4. Using the fact of 3. and general theory, $\text{DM}_{\text{gm}}(k)$ has a natural tensor structure.

Theorem 2.22. (c.f. [Voe02] The cancellation theorem)

(\text{Need the assumption of perfectness of } k.) \text{ For objects } A, B \text{ in } \text{DM}_{\text{gm}}^0(k) \text{ the natural map}

\[? \otimes \text{id}_{\mathbb{Z}(1)} : \text{Hom}_{\text{DM}_{\text{gm}}(k)}(A, B) \rightarrow \text{Hom}_{\text{DM}_{\text{gm}}(k)}(A(1), B(1))\]

is an isomorphism. Thus the canonical functor

\[\text{DM}_{\text{gm}}^0(k) \rightarrow \text{DM}_{\text{gm}}(k)\]

is a full embedding.
3 Various morphisms between motives

In this section, we will briefly review [TriCa] and [MotGe]

3.1 Transpose for finite equidimensional morphisms

3.1. Let \( X, Y \) be smooth schemes over \( k \) and \( f : X \to Y \) a finite equidimensional morphism. Then we have the transpose of \( f, \mathcal{t} f : Y \to X \) in \( \text{SmCor}(k) \). That is \( \mathcal{s}(\Gamma f) \in \text{Hom}_{\text{SmCor}(k)}(Y, X) \), where \( s : X \times Y \to Y \times X \) is the switch morphism.

Example 3.2. Let \( L/K/k \) be finite field extensions. Then we have a canonical morphism \( i : \text{Spec} \ L \to \text{Spec} \ K \). Hence we get \( \mathcal{M}_{\text{gm}}(i) : \mathcal{M}_{\text{gm}}(\text{Spec} \ K) \to \mathcal{M}_{\text{gm}}(\text{Spec} \ L) \) in \( \text{DM}_{\text{gm}}(k) \). We shall write this map as \( N_{L/K} \) because there is the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(\mathcal{M}_{\text{gm}}(\text{Spec} \ L), \mathbb{Z}\{n\}) & \longrightarrow & K_n^M(L) \\
\text{Hom}(N_{L/K}, \mathbb{Z}\{n\}) & \longrightarrow & N_{L/K} \\
\text{Hom}(\mathcal{M}_{\text{gm}}(\text{Spec} \ K), \mathbb{Z}\{n\}) & \longrightarrow & K_n^M(K).
\end{array}
\]

(c.f. [BKcon] Lemma 3.4.4. See also Theorem 5.16)

3.2 Pull back for flat equidimensional morphisms

3.3. Let \( X, Y \) be smooth schemes and \( f : X \to Y \) a flat equidimensional morphism of relative dimension \( n \). Then one can define a morphism \( f^* : \mathcal{M}_{\text{gm}}^c(Y)((n)) \to \mathcal{M}_{\text{gm}}^c(X) \) as follows. (This is slightly different from the definition in [TriCa] Corollary 4.2.4).

\[
\mathcal{M}_{\text{gm}}^c(Y)((n)) = C_*(\mathcal{Z}_{\text{equil}}(Y \times \mathbb{A}^n, 0)) \\
C_*(f \times \text{id}_{\mathbb{A}^n})^* C_*(\mathcal{Z}_{\text{equil}}(X \times \mathbb{A}^n, n)) \leftarrow q_{\text{is}} C_*(\mathcal{Z}_{\text{equil}}(X, 0)) = \mathcal{M}_{\text{gm}}^c(X)
\]

where every quasi-isomorphisms are induced from flat pull backs.

3.4. The following properties are easily proved.

1. In the notation above, if \( f \) is an open immersion, this morphism coincides with the canonical morphism \( \mathcal{M}_{\text{gm}}^c(Y) \to \mathcal{M}_{\text{gm}}^c(X) \).
2. In the notation above, if \( X \) and \( Y \) are proper over \( \text{Spec} \ k \) and \( f \) is flat finite equidimensional, then \( \mathcal{M}_{\text{gm}}^c(f) = f^* \). (c.f. [MotGe] Lemma 1.1.2)
3. Let \( X, Y \) and \( Z \) be smooth schemes and \( f : X \to Y, \ g : Y \to Z \) flat equidimensional morphisms of relative dimension \( n \) and \( m \) respectively. Then we have \( f^* \circ g^*((m)) = (g \circ f)^* \).

Lemma 3.5.

Let \( p : \mathbb{A}^1_k \to \text{Spec} \ k \) be the structure morphism. Then \( p^* : \mathcal{M}_{\text{gm}}^c(\text{Spec} \ k)((1)) = \mathcal{M}_{\text{gm}}^c(\mathbb{A}^1_k) \to \mathcal{M}_{\text{gm}}^c(\mathbb{A}^1_k) \) coincides with \( \text{id}_{\mathcal{M}_{\text{gm}}^c(\mathbb{A}^1_k)} \).
Proof. By definition (See 3.3),
\[ p^* = C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) \xrightarrow{(pr_1^*)} C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) \xrightarrow{(pr_2^*)^{-1}} C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) \]
where \( pr_1, pr_2 \) are two projections \( pr_1, pr_2 : A^2_k \to A^1_k \). We assert that two projections \( pr_1, pr_2 \) induce the same morphism \( pr_1^* = pr_2^* : C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) \to C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) \) in \( \text{DM}^{eff}(k) \). Since
\[ pr_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ pr_2, \]
it suffices to prove that the action of \( GL_2(k) \) on \( A^2_k \) induces trivial action on \( C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) \) in \( \text{DM}^{eff}(k) \) by flat pull back. On the other hand \( GL_2(k) \) is generated by the elements of conjugate of \( \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \). For \( A = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \in GL_2(k) \), considering the following diagram
\[ \begin{array}{ccc} C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) & \xrightarrow{A^*} & C_*(\mathbb{Z}_{\text{equi}}(A^2_k, 1)) \\ pr_1^* \downarrow & & \downarrow pr_1^* \\ C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)) & \xrightarrow{\text{id}} & C_*(\mathbb{Z}_{\text{equi}}(A^1_k, 0)), \end{array} \]
we get the result. \( \square \)

3.3 Motives with closed support

3.6. We call \((X, Z)\) a closed pair if \(X\) is a smooth scheme over \(k\) and \(Z\) is a closed subscheme. If \(Z\) is smooth over \(k\), we call \((X, Z)\) a smooth pair. We call a pair of morphisms of scheme \((f, g) : (Y, T) \to (X, Z)\) a morphism of closed pair if a commutative square \( \begin{array}{ccc} T & \xrightarrow{f} & X \\ \downarrow g & & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array} \) is a Cartesian square as underlying topological spaces. Such a morphisms called Cartesian (resp. excissive) if the diagram above is a Cartesian square (resp. \(f\) is étale and \(g_{\text{red}}\) is an isomorphism.)

3.7. Let \(X\) be a smooth scheme and \(U\) its open subset. Then we define
\[ M_{gm}(X/U) := C^*(\text{Coker}(\mathbb{Z}_{\text{et}}(U) \to \mathbb{Z}_{\text{et}}(X))). \]
For any closed pair \((X, Z)\), we define relative motif associated \((X, Z)\) by \(M_Z(X) := M_{gm}(X/X - Z)\). By definition there is a canonical distinguished triangle of the form
\[ M_{gm}(X - Z) \xrightarrow{M_{gm}(j)} M_{gm}(X) \xrightarrow{i} M_Z(X) \to M_{gm}(X - Z)[1]. \]
where \(i : Z \hookrightarrow X\) is a closed immersion and \(j : X - Z \hookrightarrow X\) is an open immersion.
3.8. For any morphisms of closed pair \((f, g) : (Y, T) \to (X, Z)\), we associate a morphism \((f, g)_* : M_T(Y) \to M_Z(X)\) which makes the following diagram commute

\[
\begin{array}{c}
0 \rightarrow M_{gm}(Y - T) \rightarrow M_{gm}(Y) \rightarrow M_T(Y) \rightarrow 0 \\
\downarrow M_{gm}(h) \quad \downarrow M_{gm}(f) \quad \downarrow (f, g)_* \\
0 \rightarrow M_{gm}(X - Z) \rightarrow M_{gm}(X) \rightarrow M_Z(X) \rightarrow 0
\end{array}
\]

where \(h : Y - T \to X - Z\) is a induced morphism from \(f\).

3.9. In the notation above, if \(f\) is finite equidimensional, we associate a morphism \((f, g)^* : M_T(Y) \to M_Z(X)\) which makes the following diagram commute

\[
\begin{array}{c}
0 \rightarrow M_{gm}(Y - T) \rightarrow M_{gm}(Y) \rightarrow M_T(Y) \rightarrow 0 \\
\downarrow M_{gm}(h) \quad \downarrow M_{gm}(f) \quad \downarrow (f, g)^* \\
0 \rightarrow M_{gm}(X - Z) \rightarrow M_{gm}(X) \rightarrow M_Z(X) \rightarrow 0.
\end{array}
\]

**Proposition 3.10.** (c.f. [IntMo] Proposition 2.3)

(\textbf{Red}) Reduction: If \((X, Z)\) is a closed pair, the canonical morphism \((X, Z_{red}) \to (X, Z)\) induces identity map \(M_{Z_{red}}(X) \to M_Z(X)\).

(\textbf{Add}) Additivity: Let \(X\) be a smooth scheme, and \(Z, W\) disjoint closed subschemes of \(X\). Then induced morphism \(M_{Z \amalg W}(X) \to M_Z(X) \oplus M_W(X)\) is an isomorphism.

(\textbf{Exc}) Excision: Any excisive morphism \((Y, T) \to (X, Z)\) induces an isomorphism \(M_T(Y) \to M_Z(X)\).

(\textbf{MV}) Mayer-Vietoris: Let \(X\) be a smooth scheme over \(k\), \(U\) and \(V\) two open subsets of \(X\) such that \(X = U \cup V\), and \(Z\) a closed subscheme of \(X\). Then we have a distinguished triangle of the form

\[
M_{Z \cap U \cap V}(U \cap V) \rightarrow M_{Z \cap U}(U) \oplus M_{Z \cap V}(V) \rightarrow M_Z(X) \xrightarrow{\pm}.
\]

(\textbf{Hyp}) Homotopy invariance: A Cartesian morphism \(\pi : (A_k^X, A_k^Z) \to (X, Z)\) induced from the canonical projection induces an isomorphism

\[
M_{gm}(\pi) : M_{A_k^X} \rightarrow M_{A_k^Z} \rightarrow M_Z(X).
\]

### 3.4 Thom isomorphism

3.11. Let \(X\) be a scheme and \(E/X\) a vector bundle. We consider \(X\) as a closed subscheme of \(E\) by zero section. We define the Thom motif of \(E/X\) by \(M_{gm}(ThE) := M_X(E)\).

3.12. In the notation above, if rank of \(E\) is \(n\), there is the Thom isomorphism \(\theta(E) : M_{gm}(ThE) \xrightarrow{\sim} M_{gm}(X)((n))\). We will briefly review the construction of this isomorphism.
3.13. Let $X$ be a smooth scheme and $\Delta_X : X \to X \times_k X$ the diagonal immersion. Let $\mathcal{M}, \mathcal{N}$ be objects of $\text{DM}_\text{eff}(k)$ and

$$\alpha : M_{\text{gm}}(X) \to \mathcal{M}$$

$$\beta : M_{\text{gm}}(X) \to \mathcal{N}$$

are morphisms in $\text{DM}_\text{eff}(k)$.

We define external cup product of $\alpha$ and $\beta$ over $X$ is composition of the following morphisms

$$M_{\text{gm}}(X) \xrightarrow{M_{\text{gm}}(\Delta_X)} M_{\text{gm}}(X) \otimes M_{\text{gm}}(X) \xrightarrow{\alpha \otimes \beta} \mathcal{M} \otimes \mathcal{N}.$$ We denote this morphism by $\alpha \boxtimes_X \beta$ or $\alpha \boxtimes \beta$.

3.14. In the notation above, if $\mathcal{M} = \mathbb{Z}((m))$ and $\mathcal{N} = \mathbb{Z}((n))$, then we have the canonical isomorphism $\varepsilon : \mathbb{Z}((m)) \otimes \mathbb{Z}((n)) \xrightarrow{\sim} \mathbb{Z}((m + n))$.

We put $\alpha \cup \beta := \varepsilon \circ \alpha \boxtimes \beta$ and call it internal cup product of $\alpha$ and $\beta$.

3.15. There is a natural isomorphism as a presheaf with transfers (c.f. [MotGe] Corollary 2.2.7)

$$c_1 : \text{Pic} (?) \to \text{Hom}_{\text{DM}_{\text{gm}}(k)} (M_{\text{gm}}(?), \mathbb{Z}((1))).$$

For any smooth scheme $X$ and $\mathcal{L} \in \text{Pic}(X)$, we will call $c_1(\mathcal{L})$ a motivic Chern class of $\mathcal{L}$.

3.16. Let $X$ be a smooth scheme, $E$ a vector bundle over $X$ of rank $n$, $\lambda_E$ the canonical invertible sheaf on $\mathbb{P}(E)$ and $p : \mathbb{P}(E) \to X$ the canonical projection. We define the $r$-th motivic Lefschetz projector of $E$ by

$$l_r(E) := c_1(\lambda_E)^{-1} \boxtimes M_{\text{gm}}(p) : M_{\text{gm}}(\mathbb{P}(E)) \to M_{\text{gm}}(X)((r))$$

We put the motivic Lefschetz operator as

$$l(E) := \sum_{r=0}^{n-1} l_r(E).$$

**Proposition 3.17.** (c.f. [TriCa] Proposition 3.5.1)

In the notation above, the morphism

$$l(E) : M_{\text{gm}}(\mathbb{P}(E)) \to \bigoplus_{r=0}^{n-1} M_{\text{gm}}(X)((r))$$

is an isomorphism.

3.18. Let $X$ be a smooth scheme and $E/X$ a vector bundle of rank $n$. Put $\tilde{E} := E \times_X \mathbb{A}_X^n$. Then we have the canonical isomorphisms

$$M_{\text{gm}}(\text{Th}E) \xrightarrow{l} M_X(\mathbb{P}(\tilde{E})) \xrightarrow{\sim} M_{\text{gm}}(\mathbb{P}(\tilde{E})/\mathbb{P}(E)).$$
The first morphism is induced from an open immersion \( E \to \mathbb{P}(\hat{E}) \) which is an isomorphism by (Exc). The second morphism is induced from the projection \( \mathbb{P}(\hat{E}) \to X \to \mathbb{P}(E) \) which is an isomorphism by (MV) and (Htp). From these isomorphisms, we get the following distinguished triangle
\[
M_{gm}(\mathbb{P}(E)) \to M_{gm}(\mathbb{P}(\hat{E})) \xrightarrow{\pi} M_{gm}(ThE) \xrightarrow{\theta(E)} .
\]
Using this distinguished triangle and Proposition 3.17, we get the following isomorphism
\[
M_{gm}(X)((n)) \to \bigoplus_{r=0}^{n} M_{gm}(X)((r)) \xrightarrow{\iota[E]^{-1}} M_{gm}(\mathbb{P}(\hat{E})) \xrightarrow{\pi} M_{gm}(ThE).
\]
We call the inverse of this isomorphism the Thom isomorphism and denote it by \( \theta(E) \).

### 3.5 Normal cone deformation

3.19. Let \((X, Z)\) be a smooth pair of pure codimension \(c\) over \(k\) such that dimension of \(X\) is \(n\). We will write \(B_{Z}X\) by the blow up of \(X\) in \(Z\). Put \(D_{Z}X := B_{0 \times Z}(\mathbb{A}^1 \times X) - B_{Z}X\). There are canonical isomorphisms
\[
M_{Z}(X) \to M_{\mathbb{A}^2}(D_{Z}X) \leftrightarrow M_{gm}(ThN_{Z}X).
\]
Hence we get the isomorphism \(M_{Z}(X) \to M_{gm}(ThN_{Z}X)\).

### 3.6 Gysin triangles

3.20. Let \((X, Z)\) be a smooth pair of pure codimension \(c\) over \(k\) and \(i : Z \to X\) a closed immersion. In [IntMo], Déglise constructs the following functorial Gysin triangle in \(DM_{gm}(k)\)
\[
M_{gm}(X - Z) \to M_{gm}(X) \xrightarrow{\eta} M_{gm}(Z)((c)) \xrightarrow{\partial_{X,Z}[1]} M_{gm}(X - Z)[1].
\]
This is constructed from the following distinguished triangle
\[
M_{gm}(X - Z) \to M_{gm}(X) \to M_{Z}(X) \to M_{gm}(X - Z)[1]
\]
and the following isomorphisms
\[
M_{Z}(X) \xrightarrow{\sim} M_{gm}(Th(N_{Z}(X))) \xrightarrow{\theta(N_{Z}(X))} M_{gm}(Z)((c))
\]
where the first isomorphism is induced from the normal cone deformation.

3.21. Let \((f, g) : (Y, T) \to (X, Y)\) be a morphism of closed pairs. We assume \(Z\) (resp. \(T\)) is connected and smooth over \(k\) of codimension \(n\) in \(X\) (resp. \(m\) in \(Y\)).

16
Then we define Gysin morphism associated to \((f, g)\), denote \((f, g)_!\) by the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
M_T(Y) & \xrightarrow{(1)} & M_{gm}(Th(N_Y^r Y)) \xrightarrow{\theta(N_Y^r Y)} M_{gm}(T)((m)) \\
(f, g)^! & \downarrow & \downarrow \\
M_Z(X) & \xrightarrow{(1)} & M_{gm}(Th(N_Z^r X)) \xrightarrow{\theta(N_Z^r X)} M_{gm}(Z)((n))
\end{array}
\end{array}
\]

where morphisms (1) are isomorphisms induced from the morphisms of normal cone deformations.

3.22. In the notation above, consider \(i : Z \to X\), \(j : X - Z \to X\), \(k : T \to Y\) and \(l : Y - T \to T\) the canonical immersions. The following diagram is commutative:

\[
\begin{array}{c}
\begin{array}{ccc}
M_{gm}(Y - T) & \xrightarrow{M_{gm}(l)} & M_{gm}(Y) \\
M_{gm}(h) & \downarrow & \downarrow \\
M_{gm}(X - Z) & \xrightarrow{M_{gm}(j)} & M_{gm}(X) \\
M_{gm}(X - Z) & \xrightarrow{M_{gm}(k)} & M_{gm}(X - Z)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{k^*} & M_{gm}(T)((m)) & \xrightarrow{\partial_Y, r[1]} M_{gm}(Y - T)[1] \\
\xrightarrow{j^*} & M_{gm}(Z)((n)) & \xrightarrow{\partial_X, r[1]} M_{gm}(X - Z)[1]
\end{array}
\end{array}
\]

**Lemma 3.23.** Let \(x : \text{Spec } k \to \mathbb{A}_k^1 \xrightarrow{j} \mathbb{P}^1\) be a \(k\)-rational point. Then the following diagram is commutative.

\[
\begin{array}{c}
\begin{array}{ccc}
M_{gm}(\mathbb{P}^1) & \xrightarrow{x^*} & M_{gm}(\text{Spec } k)((1)) \\
id & \downarrow & \downarrow \\
M_{gm}(\mathbb{P}^1) & \xrightarrow{j^*} & M_{gm}(\mathbb{A}_k^1)
\end{array}
\end{array}
\]

where the vertical isomorphism \(M_{gm}(\text{Spec } k)((1)) \xrightarrow{\sim} M_{gm}^c(\mathbb{A}_k^1)\) is defined in 2.20.

**Proof.** It is just a matter of considering two split distinguished triangles below

\[
\begin{array}{c}
\begin{array}{ccc}
M_{gm}(\mathbb{A}_k^1) & \rightarrow & M_{gm}(\mathbb{P}^1_k) \\
M_{gm}(\mathbb{A}_k^1) & \rightarrow & M_{gm}(\text{Spec } k)((1)) \\
M_{gm}(\mathbb{A}_k^1) & \rightarrow & M_{gm}(\text{Spec } k)[1]
\end{array}
\end{array}
\]

where the commutativity of 1 follows from Example 2.4.

3.24. In the notation above, if \(f\) is finite equidimensional, then we define \((f, g)^!\) by the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
M_T(Y) & \xrightarrow{(1)} & M_{gm}(Th(N_Y^r Y)) \xrightarrow{\theta(N_Y^r Y)} M_{gm}(T)((m)) \\
(f, g)^* & \downarrow & \downarrow \\
M_Z(X) & \xrightarrow{(1)} & M_{gm}(Th(N_Z^r X)) \xrightarrow{\theta(N_Z^r X)} M_{gm}(Z)((n))
\end{array}
\end{array}
\]
where the morphisms (1) are the isomorphisms induced from the morphisms of normal cone deformations.

The following diagram is commutative:

\[
\begin{array}{c}
M_{gm}(Y - T) \xrightarrow{M_{gm}(f)} M_{gm}(Y) \xrightarrow{k^*} M_{gm}(T)(m) \xrightarrow{\partial_{Y,m}} M_{gm}(Y - T)[1] \\
M_{gm}(h) \downarrow \quad \downarrow (f, g) \downarrow \quad \downarrow M_{gm}(h)[1] \\
M_{gm}(X - Z) \xrightarrow{M_{gm}(j)} M_{gm}(X) \xrightarrow{r} M_{gm}(Z)(n) \xrightarrow{\partial_{X,Z}} M_{gm}(X - Z)[1]
\end{array}
\]

**Proposition 3.25.** ([MotGe] Proposition 2.5.2)
In the notation above, if \((f, g)\) is Cartesian, then

\[(f, g)^! = M_{gm}(g)(m)).\]

Next we cite the some proposition in [MotGe]. This is needed to prove that the motivic reciprocity law implies the Weil reciprocity law for Milnor \(K\)-groups.

3.26. By 2.20, we have the following distinguished triangle

\[Z \rightarrow M_{gm}(\mathbb{G}_m) \xrightarrow{f} \mathbb{Z}\{1\} \xrightarrow{\partial} \]

where \(Z \rightarrow M_{gm}(\mathbb{G}_m)\) is induced from the unit morphism Spec \(k \rightarrow \mathbb{G}_m\).

**Proposition 3.27.** (c.f. [MotGe] Proposition 2.6.6)
Let \((X, Z)\) be a smooth closed pair of codimension 1. We denote \(i : Z \rightarrow X\) a closed immersion and \(j : X - Z \rightarrow X\) a canonical open immersion.

Suppose there is a regular function \(\pi : X \rightarrow \mathbb{A}^1\) which parameterizes \(Z\). Hence we have a morphism \(\pi|_{X - Z} : X - Z \rightarrow \mathbb{G}_m\). Then the following diagram is commutative

\[
\begin{array}{c}
M_{gm}(Z)\{1\} \xrightarrow{\partial_{X,Z}} M_{gm}(X - Z) \\
M_{gm}(i)\{1\} \downarrow \quad \downarrow M_{gm}(i) \circ \partial_{X,Z} \downarrow M_{gm}(X - Z)\{1\}
\end{array}
\]

4 Motivic cohomology groups attached to pointed smooth curves

4.1 Definition

4.1. For pointed smooth curves \((C_1, x_1), \ldots, (C_r, x_r)\) over field \(k\), we define a motivic complex \(Z((C_1, x_1) \land \ldots \land (C_r, x_r))\), or \(Z(C_1 \land \ldots \land C_r)\) in \(DM^eff(k)\) as follows

\[Z(C_1 \land \ldots \land C_r) = C^*(Z_{tr}(C_1, x_1) \otimes \ldots \otimes Z_{tr}(C_r, x_r))[-r]\]
4.2. The restriction \( Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \) of \( Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \) to the Zariski site of \( X \in Sm/k \) is a complex of sheaves in Zariski topology and the motivic cohomology groups \( H^n_M(X, \hat{\Lambda}_{i=1}^r (C_i, x_i)) \), or \( H^n_U(X, \hat{\Lambda}_{i=1}^r C_r) \) are defined to be the hyper cohomology of the motivic complexes \( Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \) with respect to Zariski topology:

\[
H^n_M(X, \hat{\Lambda}_{i=1}^r (C_i, x_i)) = \mathbb{H}_{Zar}^n(X, Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \mid X).
\]

4.3. As in [TriCa] we have

\[
\mathbb{H}_{Zar}^n(X, Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \mid X) = \text{Hom}_{DM^*(\mathbb{S})} (\mathbb{M}_{gm}(X), Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \mid n])
\]

Moreover if \( k \) is a perfect field as in [CohTh] Proposition 3.1.11, we have

\[
\mathbb{H}_{Zar}^n(X, Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \mid X) = \mathbb{H}_{Nis}^n(X, Z(\hat{\Lambda}_{i=1}^r (C_i, x_i)) \mid X).
\]

4.2 Fundamental properties

4.4. (Product structure) Let \((C_1, a_1), \ldots, (C_r, a_r), (D_1, b_1), \ldots, (D_t, b_t)\) be pointed smooth curves over field \( k \). As usual motivic complexes, we have canonical morphism

\[
Z(\hat{\Lambda}_{i=1}^r (C_i, a_i)) \otimes Z(\hat{\Lambda}_{j=1}^t (D_j, b_j)) \to Z(\hat{\Lambda}_{i=1}^r (C_i, a_i) \wedge \hat{\Lambda}_{j=1}^t (D_j, b_j)).
\]

Hence we get for any \( X \in Sm/k \) the pairing

\[
H^n_M(X, \hat{\Lambda}_{i=1}^r (C_i, a_i)) \otimes H^n_M(X, \hat{\Lambda}_{j=1}^t (D_j, b_j)) \to H^{n+t}_M(X, \hat{\Lambda}_{i=1}^r (C_i, a_i) \wedge \hat{\Lambda}_{j=1}^t (D_j, b_j)).
\]

4.5. Let \((C_1, a_1), \ldots, (C_r, a_r)\) be pointed smooth curves over field \( k \). For any field extension \( L/k \), we abbreviate \( H^n_M(\text{Spec} \ L, \hat{\Lambda}_{i=1}^r (C_i \times_k L, a_i \times_k \text{id}_L)) \) as \( H^n_M(L, \hat{\Lambda}_{i=1}^r C_i) \).

By definition, we have

\[
H^n_M(L, \hat{\Lambda}_{i=1}^r C_i) = H_{i-p}(C_* (\bigotimes_{i=1}^r Z_{i=1}^r (C_i \times_k L, a_i \times_k \text{id}_L))) (\text{Spec} \ L).
\]

4.6. (Norm map) In 4.5, if we assume \( L/k \) is a finite field extension, then using the description of 4.5 and the proper push-forward of cycles induces a map

\[
\text{N}_{L/k} : H^n_M(L, \hat{\Lambda}_{i=1}^r C_i) \to H^n_M(k, \hat{\Lambda}_{i=1}^r C_i).
\]

From the corresponding properties of proper push-forward, the following properties are immediately verified.

For finite field extension \( k \subset L \subset M \) and \( x \in H^n_M(M, \hat{\Lambda}_{i=1}^r C_i) \) and \( y \in H^n_M(L, \hat{\Lambda}_{i=1}^r C_i) \)
then we have
(1) \( N_{M/L}(y_M, x) = y, N_{M/L}(x, y_M) = N_{M/L}(x, y) \)
(2) \( N_{M/k}(x) = N_{M/L}(N_{L/k}(x)) \)
(3) If \( M/k \) is a normal extension, we have
\[
N_{L/k}(x)^{M} = \frac{[L : k]}{\text{insep}} \sum_{j : M \to L} j^*(x).
\]

**Example 4.7.** Let \((C, x)\) be a pointed projective smooth curve over \(k\). Then we have
\[
H^1_M(k, (C, x)) = \text{Ker} \left( \text{CH}_0(C) \to \mathbb{Z} \right).
\]

**Example 4.8.** Let \(X\) be a smooth curve over \(k\). A good compactification of \(X\) is a pair \((\tilde{X}, X_\infty)\) such that there is an open embedding \(X \hookrightarrow \tilde{X}\), \(\tilde{X}\) is proper non-singular curve over \(k\) and \(X_\infty = \tilde{X} - X\) has an affine open neighborhood in \(\tilde{X}\).

Let \((C, x)\) be a pointed smooth affine curve with a good compactification \((\tilde{X}, X_\infty)\), then
\[
H^1_M(k, (C, x)) = \text{Ker} \left( \text{Pic}(\tilde{X}, X_\infty) \to \mathbb{Z} \right)
\]

where \(\text{Pic}(\tilde{X}, X_\infty)\) is the relative Picard group. The elements of \(\text{Pic}(\tilde{X}, X_\infty)\) are the isomorphism classes \((\mathcal{L}, t)\) of line bundle \(\mathcal{L}\) on \(\tilde{X}\) with a trivialization \(t\) on \(X_\infty\).

\[4.9.\] In 4.7, for finite field extension \(L/k\), using the property 4.6 (3), we know through isomorphisms in 4.7, norm maps in 4.6 and classical one are compatible.

## 5 Calculation of motivic cohomology groups attached to pointed smooth curves

### 5.1 Pro-motives

In this subsection, we will briefly review the result of [MotGe].

**5.1.** Let \(\mathcal{A}\) be a tensor triangulated category. We consider Pro-\(\mathcal{A}\) the pro-category of \(\mathcal{A}\). Then the following facts are fundamental.
1. Pro-\(\mathcal{A}\) is additive.
2. The shift functor of \(\mathcal{A}\) induces an auto-functor of Pro-\(\mathcal{A}\).
3. There is a unique tensor structure over Pro-\(\mathcal{A}\) such that \(\otimes\) commutes projective limits.

**5.2.** In the notation above, we call any triangle in Pro-\(\mathcal{A}\) isomorphic to formal projective limit of distinguished triangles of \(\mathcal{A}\) a pro-distinguished triangle. Let \(H : A^{op} \to Ab\) be a cohomological functor. Then the functor
\[
\mathbf{H} : (\text{Pro-}A)^{op} \ni (X_i)_{i \in I} \mapsto \lim_{i \in I^{op}} H(X_i) \in Ab
\]

sends pro-distinguished triangles to long exact sequences.

20
5.3. Let $\mathcal{O}$ be a $k$-algebra. We say $\mathcal{O}$ is local smooth over $k$ iff there is an formally smooth of finite type $k$-algebra $A$, and a prime ideal $x$ of $A$ and an isomorphism $\mathcal{O} \xrightarrow{\sim} A$. Since $k$ is perfect, $\mathcal{O}$ is local smooth iff it is regular and essentially of finite type.

5.4. Let $\mathcal{O}$ be a local smooth $k$-algebra. A model of $\mathcal{O}/k$ is a pair $(X, x)$ consist of a smooth scheme $X$ and a morphism $x : \text{Spec} \mathcal{O} \to X$ such that, if we write the image of closed point of $\text{Spec} \mathcal{O}$ to $x$, induced morphism $x^\sharp : X_{x, x} \to \mathcal{O}$ is an isomorphism.

We put

$$M^\text{sm}(\mathcal{O}/k) := \{A \subset \mathcal{O}; (\text{Spec} A, \text{Spec} \mathcal{O} \xrightarrow{\text{hiss}} \text{Spec} A) \text{ is a model of } \mathcal{O}/k\}.$$

$M^\text{sm}(\mathcal{O}/k)$ is not empty and filtrant for inclusion. (c.f. [MotGe] Lemma 3.1.5).

5.5. 1. Let $\mathcal{O}$ be a local smooth $k$-algebra. We consider a pro-object of $\text{Sm} / k$

$$(\mathcal{O}) := \{\text{Spec } A\}_{A \in M^\text{sm}(\mathcal{O}/k)}.$$

2. Let $X$ be a smooth scheme and $x \in X$, we define localization of $X$ in $x$ as a pro-object of $\text{Sm} / k$.

$$X_x := \{U\}_{x \in U \subset X}$$

where $U$ runs through the open neighborhood of $x$.

5.6. (c.f. [MotGe] Lemma 3.1.8) Let $\mathcal{O}$ be a local smooth $k$-algebra, and $(X, x)$ a model of $\mathcal{O}$. Then $x$ induces a canonical isomorphism

$$(\mathcal{O}) \to X_x.$$

5.7. Let $\mathcal{O}$ be a local smooth $k$-algebra and $n, m \in \mathbb{Z}$. We consider a pro-object of $\text{Pro-DM}_{\text{gm}}(k)$

$$M_{\text{gm}}(\text{Spec } \mathcal{O})(n)[m] := \{M_{\text{gm}}(\text{Spec } A)(n)[m]\}_{A \in M^\text{sm}(\mathcal{O}/k)}.$$

Next we define a residue morphism associated to a discrete valuation.

5.8. Let $E/k$ be a field extension of finite type, $v$ a valuation of $E/k$, $\mathcal{O}_v$ a valuation ring of $v$ and $(X, t)$ a $k$-model of $\mathcal{O}_v$. We say a special point of $(X, t)$ for image of closed point of $\text{Spec } \mathcal{O}_v$ for $t$ and denote by $s$.

We say that $(X, t)$ is a strict $k$-model of $\mathcal{O}_v$ iff closure $\{s\}$ in $X$ is a smooth scheme.

Any discrete valuation ring $\mathcal{O}_v$ essentially of finite type over $k$ admits a strict $k$-model. (c.f. [MotGe] Lemma 4.5.3).

5.9. Let $E/k$ be a field extension of finite type, $v$ a valuation of $E/k$, $\mathcal{O}_v$ a valuation ring of $v$ and $(X, t)$ a strict $k$-model of $\mathcal{O}_v$. Put $Z := \{s\}$. Since $(X, Z)$ is a smooth closed pair of codimension 1. We have a distinguished triangle of the form

$$M_{\text{gm}}(Z)[1] \xrightarrow{\alpha} M_{\text{gm}}(X - Z) \xrightarrow{\beta} M_{\text{gm}}(X) \xrightarrow{\gamma}.$$

21
Since this triangle is natural for inclusions of open sets in $X$. (c.f. 3.22) Considering a cofiltrant system of open neighborhoods of $s$ in $X$, we get a pro-distinguished triangle

$$M_{gm}(Z_s) \{1\} \rightarrow_{\partial_{X,s}^{\partial,s}} M_{gm}(X_s - Z_s) \rightarrow_{\partial} M_{gm}(X_s) \hookrightarrow.$$

Since $(X, s)$ is a $k$-model of $O_v$, a morphism $s : \text{Spec } O_v \rightarrow X$ induces an isomorphism of pro-object $(O_v) \rightarrow X_s$; Hence we get a pro-distinguished triangle isomorphic to the form

$$M_{gm}(\text{Spec } k(v)) \{1\} \rightarrow_{\partial_{X,s}^{\partial,s}} M_{gm}(\text{Spec } E) \rightarrow_{\partial} M_{gm}(\text{Spec } O_v) \hookrightarrow$$

where $E$ (resp. $k(v)$) is a fraction field (resp. residue field) of $v$, $i : O_v \rightarrow E$ is a canonical inclusion.

**Lemma 5.10.** (c.f. [MotGe] Lemma 4.5.6.)
Let $E/k$ be a field extension of finite type, $v$ a discrete valuation of $E/k$, $O_v$ a valuation ring of $v$ and $k(v)$ is this residue field.
Then adopting the notation above, if $(X, s)$ and $(Y, t)$ are two strict $k$-models of $O_v$, put $Z := \{s\}$ closure in $X$ and $T := \{t\}$ closure in $Y$. Then we have

$$\partial_{X,s, Z} = \partial_{Y,t, T}.$$

5.11. Let $E/k$ be a field extension of finite type, $v$ a discrete valuation of $E/k$. We define a residue morphism associated to $v$, denoted by $\partial_v$, defined by $\partial_v := \partial_{X, s, Z}$ where $(X, s)$ is a strict $k$-model of valuation ring of $v$. By Lemma 5.10 this does not depend on a choice of a strict $k$-model of valuation ring of $v$.

So we have the following pro-distinguished triangle of the form

$$M_{gm}(\text{Spec } k(v)) \{1\} \rightarrow_{\partial_v} M_{gm}(\text{Spec } E) \rightarrow_{\partial} M_{gm}(\text{Spec } O_v) \hookrightarrow.$$

Having defined residue morphisms, we explain the connection of Milnor $K$-groups and Hom sets in $\text{Pro-DM}_{gm}(k)$.

5.12. Using the distinguished triangle in 3.26 and the definition of tensor structure in $\text{DM}^{\text{eff}}(k)$, for any $n \in \mathbb{N}$, there is a distinguished triangle of the form

$$\bigoplus_{i=1}^{n} M_{gm}(\mathbb{G}_m^{n-1}) \rightarrow M_{gm}(\mathbb{G}_m^n) \rightarrow \mathbb{Z}\{n\} \rightarrow.$$  

where the first morphism is induced from sum of $n$ closed immersions

$$\iota_i := \text{id} \times \text{id} \times \ldots \times \text{id} \times \text{id} \times \ldots \times \text{id} \times \text{id}.$$

5.13. Let $E/k$ be a field extension of finite type, $f : M_{gm}(\text{Spec } E) \rightarrow M$ and $g : M_{gm}(\text{Spec } E) \rightarrow N$ are morphisms in $\text{Pro-DM}_{gm}(k)$. Then we can extend the definition of external cup product $f \boxtimes g : M_{gm}(\text{Spec } E) \rightarrow M \otimes N$.  

22
If $\mathcal{M} = \mathbb{Z}\{p\}$ and $\mathcal{N} = \mathbb{Z}\{q\}$, we have a canonical isomorphism $\mathbb{Z}\{p\} \otimes \mathbb{Z}\{q\} \cong \mathbb{Z}\{p+q\}$. Then we have the following identity
\[ \alpha \boxtimes \beta = -\beta \boxtimes \alpha. \]

(c.f. [MotGe] Remarque 4.4.2.)

5.14. Let $E/k$ be a field extension of finite type. Then we have a morphism
\[
\begin{array}{ccc}
(E^\times)^n & \xrightarrow{\sim} & \text{Hom}(\text{Spec } E, \mathbb{G}_m^n) \\
M_{\text{gm}} & \xrightarrow{\text{Hom}(M_{\text{gm}}(\text{Spec } E), M_{\text{gm}}(\mathbb{G}_m^n)))} & \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(M_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\}).
\end{array}
\]

This map induces a morphism
\[ \alpha : K^M_n(E) \to \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(M_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\}). \]

5.15. In the notation above, for any $x \in (E^\times)^{\otimes n}$ and $y \in (E^\times)^{\otimes m}$, we have
\[ \alpha(x \otimes y) = \alpha(x) \boxtimes \alpha(y). \]

**Theorem 5.16.** (c.f. [MotGe] Theorem 4.4.4)

(need the assumption of perfection of $k$.) In the notation above,
\[ \alpha : K^M_n(E) \to \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(M_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{\ast\}) \]
is an algebra isomorphism.

### 5.2 Motivic reciprocity law

The classical theorems “Weil reciprocity law” and “residue formula” are unified using Milnor $K$-groups. More precisely, the following statement is known. (c.f. [Sus82])

5.17. (Reciprocity law for Milnor $K$-groups)

Let $K$ be an algebraic function field over a field $k$. Then the following composition are the zero maps for all non-negative integers $n$.
\[
K^M_{n+1}(K) \overset{\partial_n}{\longrightarrow} \bigoplus_v K^M_n(k(v)) \overset{\Sigma N_{k(v)/k}}{\longrightarrow} K^M_n(k)
\]

In this subsection, we will prove more fundamental style of the following reciprocity law.

**Theorem 5.18.** (Motivic reciprocity law)
The following composition
\[
\text{M}_{\text{gm}}(\text{Spec } k)\{1\} \overset{\Sigma N_{k(1)/k}}{\longrightarrow} \prod_v \text{M}_{\text{gm}}(\text{Spec } k(v))\{1\} \overset{\prod_q}{\longrightarrow} \text{M}_{\text{gm}}(\text{Spec } K)
\]
is the zero map in Pro-DM$_{\text{gm}}(k)$. 

23
5.19. Let \( K/k \) be a field extension of transcendental degree one. Let \( C/k \) be a projective nonsingular curve such that \( K(C) = K \). As in the previous subsection, we can construct the following pro-distinguished triangle in \( \text{Pro-DM}_{\text{gm}}(k) \).

\[
\text{M}_{\text{gm}}(\text{Spec } K) \rightarrow \text{M}_{\text{gm}}(C) \\
\rightarrow \prod_{x \in C: \text{closed points}} \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) \xrightarrow{\prod[1]} \text{M}_{\text{gm}}(\text{Spec } K)[1]
\]

This is constructed as follows: For any closed set \( Z \subset C \), there is the Gysin triangle

\[
\text{M}_{\text{gm}}(C - Z) \rightarrow \text{M}_{\text{gm}}(C) \rightarrow \bigoplus_{x \in Z} \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) \xrightarrow{\text{ev}} \text{M}_{\text{gm}}(C - Z)[1]
\]

and we consider \( \text{M}_{\text{gm}}(\text{Spec } K) = \{ \text{M}_{\text{gm}}(C-Z) \}_{Z \subset C: \text{closed subsets}} \in \text{Pro-DM}_{\text{gm}}(k) \)

**Lemma 5.20.**

In the notation above, for any closed point \( x \in C \), the diagram of structure morphisms

\[
\begin{array}{ccc}
\text{Spec } k(x) & \xrightarrow{i} & C \\
\downarrow{p} & \text{Spec } k\\
\end{array}
\]

induces the following commutative diagram:

\[
\begin{array}{ccc}
\text{M}_{\text{gm}}(\text{Spec } k(x))((1)) & \xrightarrow{i^*} & \text{M}_{\text{gm}}(C) \\
\downarrow{p^*} & \text{M}_{\text{gm}}(\text{Spec } k)((1))
\end{array}
\]

\[
\xrightarrow{N_{k(x)/k((1))}}
\]

**Proof.** First choose a finite equidimensional morphism \( C \xrightarrow{\pi} \mathbb{P}^1 \) which is unramified at every points over \( \pi(x) \). (This can be done by using Bertini theorem.) Next using 3.4 and Proposition 3.25, we may assume \( C = \mathbb{P}^1 \). Replacing \( \mathbb{P}^1 \) by \( \mathbb{P}^1_{k(x)} \) and using Proposition 3.25 again, we may assume \( k(x) = k \). In this case, \( i^* \circ p^* = \text{id} \) by Lemma 3.5 and 3.23. \( \square \)

5.21. Hence we get the following diagram:

\[
\begin{array}{ccc}
\text{M}_{\text{gm}}(\text{Spec } k)((1)) & \xrightarrow{p^*} & \text{M}_{\text{gm}}(C) \\
\downarrow{\Sigma N_{k(x)/k((1))}} & \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) & \text{M}_{\text{gm}}(C - Z)[1].
\end{array}
\]

Taking a limit with respect to \( Z \), we get the motivic reciprocity law.
Next we prove that the motivic reciprocity law implies the Weil reciprocity law for Milnor $K$-groups.

**Lemma 5.22.**
In the notation above, let $v$ be a valuation of $K/k$, $\mathcal{O}_v$ a valuation ring of $v$, $\pi$ a uniformizer element of $\mathcal{O}_v$. Then the following diagram is commutative.

$$
\begin{array}{c}
M_{gm}(\text{Spec } k(v))\{1\} \xrightarrow{\partial_v} M_{gm}(\text{Spec } K) \\
\downarrow \hspace{2cm} \downarrow \\
M_{gm}(\text{Spec } \mathcal{O}_v)\{1\} & \xleftarrow{\alpha(\pi) \circ d_{M_{gm}(\text{Spec } K)}} M_{gm}(\text{Spec } K)\{1\}
\end{array}
$$

**Proof.** Take a strict $k$-model of $\text{Spec } \mathcal{O}_v$. Denote it by $(X, s)$. If we take $X$ sufficiently small, $\pi \in \mathcal{O}_v \twoheadrightarrow \mathcal{O}_{X, s}$ determines a regular function $X \to A^1_k$ which parameterizes $s$. Using Proposition 3.27 and considering a cofiltrant system of open neighborhoods of $s$ in $X$, we get the following commutative diagram of pro-motives

$$
\begin{array}{c}
M_{gm}(Z_s)\{1\} \xrightarrow{\partial_{X, s}} M_{gm}(X_s - Z_s) \\
\downarrow \hspace{2cm} \downarrow \\
M_{gm}(X_s)\{1\} & \xleftarrow{M_{gm}(\pi) \circ d_{M_{gm}(X_s - Z_s)}} M_{gm}(X_s - Z_s)\{1\}
\end{array}
$$

Hence we get the result. $\square$

**Example 5.23.** In the notation above, for any discrete valuation $v$ of $K/k$, there is a commutative diagram

$$
\begin{array}{c}
\text{Hom}(M_{gm}(\text{Spec } K), \mathbb{Z}\{n+1\}) \xleftarrow{(-1)^n \partial_v} K^n_{m+1}(K) \\
\text{Hom}(\partial_v, \mathbb{Z}\{n+1\}) \downarrow \hspace{2cm} \downarrow \hspace{2cm} \downarrow \\
\text{Hom}(M_{gm}(\text{Spec } k(v))\{1\}, \mathbb{Z}\{n+1\}) & \xleftarrow{K^n_{m}(k(v))} \text{Hom}(\partial_v, \mathbb{Z}\{n+1\})
\end{array}
$$

**Proof.** For $u_1, \ldots, u_{n+1} \in \mathcal{O}_v^\times$ and a uniformizer $\pi$, it is enough to check the following two conditions.

1. $\text{Hom}(\partial_v, \mathbb{Z}\{n+1\})(\alpha\{u_1, \ldots, u_{n+1}\}) = 0$
2. $\text{Hom}(\partial_v, \mathbb{Z}\{n+1\})(\alpha\{u_1, \ldots, u_{n+1}, \pi\}) = (-1)^n\alpha(\{u_1, \ldots, u_{n+1}\})$

To prove 1: Notice that there is a pro-distinguished triangle as follows (c.f. 5.11)

$$
M_{gm}(\text{Spec } k(v))\{1\} \xrightarrow{\partial_v} M_{gm}(\text{Spec } K) \to M_{gm}(\text{Spec } \mathcal{O}_v) \oplus 1
$$

To prove 2: Anti-commutativity of $\boxtimes$ (c.f. 5.13) and Lemma 5.22. $\square$
Example 5.24. Let $K/k$ be a field extension of transcendental degree 1, $(C, x)$ be a pointed smooth curve and $v$ a place of $K/k$. There is a tame symbol $\partial_v : \text{Jac}(K_v) \otimes K_v^\times \to \text{Jac}(k(v))$. Then the following diagram is commutative.

$$
\begin{array}{ccc}
\text{Jac}(K_v) \otimes K_v^\times & \longrightarrow & \text{Hom}_{\text{DM}^{tr}(\mathbb{A})}(\text{M}(\text{Spec } K_v), \mathbb{Z}(C, x)[1]) \\
& & \otimes \\
\partial_v & & \text{Hom}_{\text{DM}^{tr}(\mathbb{A})}(\text{M}(\text{Spec } K_v), \mathbb{Z}(\mathbb{G}_m, 1)[1]) \\
\downarrow & & \downarrow \\
\text{Jac}(k(v)) & \longrightarrow & \text{Hom}_{\text{DM}^{tr}(\mathbb{A})}(\text{M}(\text{Spec } k(v))[1], \mathbb{Z}(C, x)(1)[3])
\end{array}
$$

This is proved in the same way as Example 5.23.

Corollary 5.25. 
The motivic reciprocity law implies the Weil reciprocity law for Milnor $K$-groups.

Proof. Take Hom(?, $\mathbb{Z}\{n+1\}$) and use Theorem 2.22, Theorem 5.16 and notice Example 3.2 and Example 5.23. □

5.3 Main result

5.26. In this section, let $k$ be a perfect field which admits resolution of singularities and $(C_1, a_1), \ldots, (C_n, a_n)$ pointed projective smooth curves over $k$.

5.27. Let $p : Z \to \mathbb{A}^d_1$ be a finite surjective morphism and suppose that $Z$ is integral. Let $f_i \in \text{Hom}(Z, C_i)$ and

$$p^{-1}(j) = \prod n_i^j z_i^j \quad (j = 0, 1)$$

where $n_i^j$ are the multiplicities of points $z_i^j = \text{Spec } L_i^j$. Define:

$$\phi_j = \Sigma n_i^j \{f_1, \ldots, f_n\}_{L_i^j/k}$$

then we have

$$\phi_0 = \phi_1$$

in $K(k, \text{Jac } C_1, \ldots, \text{Jac } C_n)$.

The proof is similar to [MVW02], p.45 Corollary 5.5.

5.28. As $\bigotimes_{i=1}^n \mathbb{Z}_{tr}(C_i, a_i)(\text{Spec } k))$ is a quotient of the free abelian groups generated by the closed points of $C_1 \times \cdots \times C_n$ modulo the subgroup generated by all points of the form $(x_1, \ldots, a_i, \ldots, x_n)$ where the $a_i$’s can be any position. If $x$ is a closed point of $C_1 \times \cdots \times C_n$ with residue field $L$ then $x$ is defined by a
canonical sequence \((x_1, \ldots, x_n) \in \text{Jac } C_1(L) \times \ldots \times \text{Jac } C_n(L)\).

Since

\[
H^n_{\lambda^M}(k, \bigwedge^n C_i) = \mathrm{Coker}\left( \bigotimes_{i=1}^n \mathbb{Z}_\tau(C_i, a_i)(\Lambda^n_k) \to \bigotimes_{i=1}^n \mathbb{Z}_\tau(C_i, a_i)(\text{Spec } k) \right)
\]

using 5.27, we have a natural map \(H^n_{\lambda^M}(k, \bigwedge^n C_i) \to K(k, \text{Jac } C_1, \ldots, \text{Jac } C_n)\).

5.29. Using 4.7, for every finite field extension \(L/k\) we have an isomorphism

\[
\bigotimes_{i=1}^n \text{Jac } C_i(L) \cong \bigotimes_{i=1}^n H^n_{\lambda^M}(L, C_i).
\]

Combining a natural pairing in 4.4

\[
\bigotimes_{i=1}^n H^n_{\lambda^M}(L, C_i) \to H^n_{\lambda^M}(L, \bigwedge^n C_i)
\]

and a norm map (c.f. 4.6)

\[
N_{L/k} : H^n_{\lambda^M}(L, \bigwedge^n C_i) \to H^n_{\lambda^M}(k, \bigwedge^n C_i)
\]

we get a canonical map

\[
\bigoplus_{L/k: \text{finite extension}} \bigotimes_{i=1}^n \text{Jac } C_i(L) \to H^n_{\lambda^M}(k, \bigwedge^n C_i).
\]

If we use 4.6 (1), Theorem 4.9, Theorem 5.18 and Example 5.24, this map should factor through the map \(K(k, \text{Jac } C_1, \ldots, \text{Jac } C_n) \to H^n_{\lambda^M}(k, \bigwedge^n C_i)\).

5.30. Obviously the morphisms above are inverse to each other. Hence we get the following result.

**Theorem 5.31. (Somekawa conjecture for Jacobian varieties)**

Let \((C_1, a_1), \ldots, (C_n, a_n)\) be pointed projective smooth curves over perfect field \(k\) which admits resolution of singularities. Then

\[
K(k, \text{Jac } C_1, \ldots, \text{Jac } C_n) \cong \text{Hom}_{\text{DM}_{\text{eff}}^m(k)}(\text{M}(\text{Spec } k), \mathbb{Z} \bigwedge^n C_i[n]).
\]

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27


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