Motivic decomposition of a compactification of a Merkurjev-Suslin variety

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Abstract

We provide a motivic decomposition of a twisted form of a smooth hyperplane section of $\text{Gr}(3,6)$. This variety is a norm variety corresponding to a symbol in $K^M_3/3$.

1 Introduction

In the present paper we study certain twisted forms of a smooth hyperplane section of $\text{Gr}(3,6)$. These twisted forms are smooth $\text{SL}_1(A)$-equivariant compactifications of a Merkurjev-Suslin variety corresponding to a central simple algebra $A$ of degree 3. On the other hand, these twisted forms are norm varieties corresponding to symbols in $K^M_3/3$ given by the Serre-Rost invariant $g_3$. In the present paper we provide a complete decomposition of the Chow motives of these varieties.

The history of this question goes back to Rost and Voevodsky. Namely, Rost obtained the celebrated decomposition of a norm quadric (see [Ro98]) and later Voevodsky found some direct summand, called a generalized Rost motive, in the Chow motive of any norm variety (see [Vo03]). Note that the $F_4$-varieties from [NSZ05] can be considered as a mod-3 analog of a Pfister quadric. In its turn, our variety can be considered as a mod-3 analog of a norm quadric.

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The paper is organized as follows. In section 2 we provide a background to the category of Chow motives. In section 3 we define a smooth compactification of a Merkurjev-Suslin variety MS(A, c) with A a central simple algebra of degree 3, describe its geometrical properties, and decompose its Chow motive. The main ingredients of our proofs are results of Białynicki-Birula [BB73], Lefschetz hyperplane theorem, and Segre embedding.

2 Notation

2.1. The matrix notation of the present paper follows [Inv].

We use Galois descent language, i.e., identify a (quasi-projective) variety \( X \) over a field \( k \) with the variety \( X_s = X \times_{\text{Spec} \kappa} \text{Spec} \kappa \) over the separable closure \( \kappa \) equipped with an action of the absolute Galois group \( \Gamma = \text{Gal}(k_s/k) \). The set of \( k \)-rational points of \( X \) is precisely the set of \( k_s \)-rational points of \( X_s \) stable under the action of \( \Gamma \).

We consider the Chow group \( CH^i(X) \) of classes of algebraic cycles of codimension \( i \) on an algebraic variety \( X \) modulo rational equivalence (see [Ful]).

A generating function for a variety \( X \) is, by definition, the polynomial \( \sum a_i t^i \in \mathbb{Z}[t] \) with \( a_i = \text{rk} CH^i(X) \).

The structure of the Chow ring of a Grassmann variety is of our particular interest. The study of this ring is a subject of the classical Schubert calculus. We do a lot of computations using formulae from [Ful, 14.7].

Next we introduce the category of Chow motives over a field \( k \) following [Ma68] and [CM04]. We remind the notion of a rational cycle and state the Rost Nilpotence Theorem following [CGM05].

2.2. Let \( k \) be a field and \( \text{Var}_k \) be a category of smooth projective varieties over \( k \). Let \( S \) denote any commutative ring. For any variety \( X \) we set \( \text{Ch}(X) := CH(X) \otimes_{\mathbb{Z}} S \). First, we define the category of correspondences with \( S \)-coefficients (over \( k \)) denoted by \( \text{Cor}_k(S) \). Its objects are smooth projective varieties over \( k \). For morphisms, called correspondences, we set \( \text{Mor}(X, Y) := CH^{\dim X}((X \times Y) \otimes_{\mathbb{Z}} S) \). For any two correspondences \( \alpha \in \text{Ch}(X \times Y) \) and \( \beta \in \text{Ch}(Y \times Z) \) we define their composition \( \beta \circ \alpha \in \text{Ch}(X \times Z) \) as

\[
\beta \circ \alpha = \text{pr}_{13\ast} (\text{pr}_{12\ast} (\alpha) \cdot \text{pr}_{23\ast} (\beta)),
\]  

(1)
where \( pr_{ij} \) denotes the projection on the \( i \)-th and \( j \)-th factors of \( X \times Y \times Z \) respectively and \( pr_{ij}^*, pr_{ij}' \) denote the induced push-forwards and pull-backs for Chow groups.

The pseudo-abelian completion of \( Cor_k(S) \) is called the category of *Chow motives with \( S \)-coefficients* and is denoted by \( \mathcal{M}_k(S) \). The objects of \( \mathcal{M}_k(S) \) are pairs \( (X, p) \), where \( X \) is a smooth projective variety and \( p \) is an idempotent, that is, \( p \circ p = p \). The morphisms between two objects \( (X, p) \) and \( (Y, q) \) are the compositions \( q \circ \text{Mor}(X, Y) \circ p \).

2.3. By construction, \( \mathcal{M}_k(S) \) is a tensor additive category with self-duality, where the self-duality is given by the transposition of cycles \( \alpha \mapsto \alpha^! \), and the tensor product is given by the usual fiber product \( (X, p) \otimes (Y, q) = (X \times Y, p \times q) \).

2.4. Observe that the composition product \( \circ \) induces the ring structure on the abelian group \( \text{Ch}^{\dim X}(X \times X) \). The unit element of this ring is the class of the diagonal map \( \Delta_X \), which is defined by \( \Delta_X \circ \alpha = \alpha \circ \Delta_X = \alpha \) for all \( \alpha \in \text{Ch}^{\dim X}(X \times X) \). The motive \( (X, \Delta_X) \) will be denoted by \( \mathcal{M}(X) \).

2.5. Consider the morphism \( (e, \text{id}) : \{pt\} \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \). Its image by means of the induced push-forward \( (e, \text{id})_* \), does not depend on the choice of the point \( e : \{pt\} \to \mathbb{P}^1 \) and defines the projector in \( \text{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1) \) denoted by \( p_1 \). The motive \( \mathbb{Z}(1) = (\mathbb{P}^1, p_1) \) is called *Lefschetz motive*. For a motive \( M \) and a nonnegative integer \( i \) we denote its twist by \( M(i) = M \otimes \mathbb{Z}(1)^{\otimes i} \).

2.6. Let \( X \) be a projective cellular variety. The abelian group structure of \( \text{CH}(X) \) is well-known. Namely, \( X \) has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of \( \text{CH}(X) \). Note that the product of two cellular varieties \( X \times Y \) has a cellular filtration as well, and \( \text{CH}^*(X \times Y) \cong \text{CH}^*(X) \otimes \text{CH}^*(Y) \) as graded rings. The correspondence product of two cycles \( \alpha = f_\alpha \times g_\alpha \in \text{Ch}(X \times Y) \) and \( \beta = f_\beta \times g_\beta \in \text{Ch}(Y \times X) \) is given by (cf. [Bo03, Lemma 5])

\[
(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \text{deg}(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta),
\]

where \( \text{deg} : \text{Ch}(Y) \to \text{Ch}(\{pt\}) = S \) is the degree map.

2.7. Let \( X \) be a projective variety of dimension \( n \) over a field \( k \). Let \( k_s \) be the separable closure of \( k \) and \( X_s = X \times_{\text{Spec} k} \text{Spec} k_s \). We say a cycle \( J \in \text{Ch}(X_s) \) is *rational* if it lies in the image of the pull-back homomorphism
Ch(X) → Ch(Xₜ). For instance, there is an obvious rational cycle Δₓₜ in Chⁿ(Xₜ × Xₜ) that is given by the diagonal class. Clearly, all linear combinations, intersections and correspondence products of rational cycles are rational.

2.8 (Rost Nilpotence). Finally, we shall also use the following fact (see [CGM05, Theorem 8.2]) called Rost Nilpotence theorem. Let X be a projective homogeneous variety over k. Then for any field extension l/k the kernel of the natural ring homomorphism End(M(X)) → End(M(Xₜ)) consists of nilpotent elements.

3 Decomposition

From now on we assume the characteristic of the base field k is 0.

It is well-known (see [GH, Ch. 1, § 5, p. 193]) that the Grassmann variety Gr(l,n) can be represented as the variety of l × n matrices of rank l modulo an obvious action of the group GLₙ. Having this in mind we give the following definition.

3.1 Definition. Let A be a central simple algebra of degree 3 over a field k, c ∈ k*. Fix an isomorphism (A ⊕ A)ₜ ≃ M₃ₙ(kₜ). Consider the variety D = D(A, c) obtained by Galois descent from the variety

\[ \{ \alpha \oplus \beta \in (A \oplus A)ₜ \mid \text{rk}(\alpha \oplus \beta) = 3, \text{Nrd}(\alpha) = c \cdot \text{Nrd}(\beta) \} / \text{GL₁}(Aₜ), \]

where GL₁(Aₜ) acts on Aₜ ⊕ Aₜ by the left multiplication.

This variety was first considered by M. Rost.

Consider the Plücker embedding of Gr(3,6) into projective space (see [GH, Ch. 1, § 5, p. 209]). It is obvious that under this embedding for all c the variety D(M₃(kₜ), c) is a hyperplane section of Gr(3,6).

3.2 Lemma. The variety D is smooth.

Proof. (M. Florence) We can assume k is separably closed. Consider first the variety

\[ V = \{ \alpha \oplus \beta \in M₃(k) \oplus M₃(k) = M₃ₙ(k) \mid \text{rk}(\alpha \oplus \beta) = 3, \text{det}(\alpha) = c \cdot \text{det}(\beta) \}. \]
An easy computation of differentials shows that $V$ is smooth. The variety $V$ is a GL$_3$-torsor over $D$ and to prove its smoothness we can assume that this torsor is split.

Since $D \times_k \text{GL}_3$ is smooth, $D \times_k M_3$ is also smooth. Therefore it suffices to prove that if $D \times_k \mathbb{A}^1$ is smooth, then $D$ is smooth. But this is true for any variety. Indeed, for any point $x$ on $D$ we have $T_{(x,0)}(D \times_k \mathbb{A}^1) = T_x D \oplus T_0 \mathbb{A}^1 = T_x D \oplus k$ and $\dim T_x D = \dim T_{(x,0)}(D \times_k \mathbb{A}^1) - 1 = \dim(D \times_k \mathbb{A}^1) - 1 = \dim D$. \hfill $\square$

3.3 Remark. One can associate to the variety $D$ a Serre-Rost invariant $g_3(D) = [A] \cup [c] \in H^3(k, \mathbb{Z}/3)$ (see [Inv, § 40]). This invariant is trivial if and only if $D$ is isotropic.

It is easy to see that $D^0 := \text{MS}(A, c) := \{a \in A \mid \text{Nrd}(a) = c\}$ is an open orbit under the natural SL$_1(A)$- or SL$_1(A) \times \text{SL}_1(A)$-action on $D$. Namely, the open orbit consists of all $\alpha \oplus \beta$ with $\text{rk}(\alpha) = 3$. $D^0$ is called a Merkurjev-Suslin variety. In other words, the variety $D(A, c)$ is a smooth SL$_1(A)$-equivariant compactification of the Merkurjev-Suslin variety MS$(A, c)$.

Denote as $i : D \to \text{SB}_3(M_2(A))$ the corresponding closed embedding.

3.4 Lemma. For the variety $D_s$ the following properties hold.

1. There exists a $\mathbb{G}_m$-action on $D_s$ with 18 fixed points. In particular, $D_s$ is a cellular variety.

2. The generating function for CH$(D_s)$ is equal to $g = t^8 + t^7 + 2t^6 + 3t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$.

3. Picard group Pic$(D_s)$ is rational.

Proof. 1. We can assume $c = 1$. The right action of $\mathbb{G}_m$ on $D_s$ is induced by the following action:

$$(M_3(k_s) \oplus M_3(k_s)) \times \mathbb{G}_m \to M_3(k_s) \oplus M_3(k_s)$$

$$(\alpha \oplus \beta, \lambda) \mapsto \alpha \text{diag}(\lambda, \lambda^5, \lambda^6) \oplus \beta \text{diag}(\lambda^2, \lambda^3, \lambda^7)$$

Note that this action is compatible with the left action of GL$_3(k_s)$.

The 18 fixed points of $D$ are the $\binom{6}{3} = 20$ 3-dimensional standard subspaces of Gr$(3, 6)$ minus 2 subspaces, generated by the first and by the last 3 basis vectors.

2. By the Lefschetz hyperplane theorem (see [GH]) the pull-back $i_s^*$ is an isomorphism in codimensions $i < \frac{\dim(\text{Gr}(3, 6)) - 1}{2}$. Therefore $\text{rk}\text{CH}^i(D_s) =$
rk \text{CH}^i(\text{Gr}(3, 6)) for such \(i\)'s. Since Poincaré duality holds, we have \(\text{rk} \text{CH}_i(D_s) = \text{rk} \text{CH}_i(\text{Gr}(3, 6))\) for \(i < \frac{\dim(\text{Gr}(3, 6)) - 1}{2} = 4\).

It remains to determine only the rank in the middle codimension. To do this observe that \(\text{rk} \text{CH}^*(D_s) = 18\) (see [BB73]). Therefore \(\text{rk} \text{CH}^4(D_s) = 2\text{rk} \text{CH}^4(\text{Gr}(3, 6)) - 2 = 4\).

3. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Pic}(\text{SB}_3(M_2(A))) & \xrightarrow{\iota^*} & \text{Pic}(D) \\
\downarrow & & \downarrow \text{res}^* \\
\text{Pic}(\text{Gr}(3, 6)) & \xrightarrow{\iota'_*} & \text{Pic}(D_s)
\end{array}
\]

where the vertical arrows are the morphisms of scalar extension. By the Lefschetz hyperplane theorem the map \(\iota^*_*\) restricted to \(\text{Pic}(\text{Gr}(3, 6))\) is an isomorphism. Since \(\text{Pic}(\text{SB}_3(M_2(A)))\) is rational (see [MT95] and [NSZ05, Lemma 4.3]), i.e., the left vertical arrow is an isomorphism, the restriction map \(\text{res}^*\) is surjective. On the other hand, it follows from a Hochschild-Serre spectral sequence (see [Ar82, § 2]) that \(\text{Pic}(D)\) can be identified with a subgroup of \(\mathbb{Z}\). We are done.

\[\square\]

3.5 **Remark.** It immediately follows from this Lemma that the variety \(D\) is not a twisted flag variety. Indeed, the generating functions of all twisted flag varieties over a separably closed field are well-known and all of them are different from the generating function of \(D_s\).

3.6. We must determine a (partial) multiplicative structure of \(\text{CH}(D_s)\). By Lefschetz hyperplane theorem the generators in codimensions 0, 1, 2, and 3 are pull-backs of the canonical generators \(\Delta_{(0,0,0)}, \Delta_{(1,0,0)}, \Delta_{(1,1,0)}, \Delta_{(2,0,0)}, \Delta_{(1,1,1)}, \Delta_{(2,1,0)}, \Delta_{(3,0,0)}\) of \(\text{Gr}(3, 6)\) (see [Ful, 14.7]). We denote these pull-backs as \(1, h_1, h_2, h_3, h_4, h_5\), and \(h_6\) respectively. In the codimension 4 the pull-back is injective and the pull-backs \(h_{(1)}^i := \iota^*_*(\Delta_{(2,1,1)}), h_{(2)}^i := \iota^*_*(\Delta_{(2,2,0)}), h_{(3)}^i := \iota^*_*(\Delta_{(3,1,0)}),\) where \(i\) is as above, form a subbasis of \(\text{CH}^4(D_s)\).
Consider the following diagram:

\[
\begin{array}{c}
\hfill h_3^{(1)} \hfill \\
\hline
h_2^{(1)} \quad h_2^{(1)} \quad h_2^{(1)} \\
\hline
1 \quad h_1 \quad h_3^{(2)} \\
\hline
h_2^{(2)} \quad h_2^{(2)} \quad h_2^{(2)} \\
\hline
h_3^{(3)} \quad h_3^{(3)} \quad h_3^{(3)} \\
\end{array}
\]

Since pull-backs are ring homomorphisms, it immediately follows that

\[ h_1 \cdot u = \sum_{u \sim u} v, \]

where \( u \) is a vertex on the diagram, which corresponds to a generator of codimension less than 4, and the sum runs through all the edges going from \( u \) one step to the right.

Next we compute some products in the middle codimension.

Since \( \Delta_{(3,1,0)} \Delta_{(2,1,1)} = \Delta_{(2,2,0)}^2 = 0 \) and \( \Delta_{(2,1,1)}^2 = \Delta_{(3,1,0)}^2 = \Delta_{(2,2,0)} \Delta_{(3,1,0)} = \Delta_{(3,3,2)} \) (see [Ful, 14.7]), we have \( h_4^{(1)} h_4^{(3)} = (h_4^{(2)})^2 = 0 \) and \( (h_4^{(1)})^2 = (h_4^{(3)})^2 = h_4^{(2)} h_4^{(3)} = h_4^{(1)} h_4^{(2)} = \iota^*(\Delta_{(3,3,2)}) = \text{pt}, \) where \( \text{pt} \) denotes the class of a rational point on \( D_s \).

The next theorem shows that the Chow motive of \( D \) with \( \mathbb{Z}/3 \)-coefficients is decomposable. Note that for any cycle \( h \) in \( \text{CH}(D_s) \) or in \( \text{CH}(D_s \times D_s) \) the cycle \( 3h \) is rational.

\[ 3.7 \text{ Theorem.} \text{ Let } A \text{ denote a central simple algebra of degree } 3 \text{ over a field } k, c \in k^*, \text{ and } D = D(A, c). \text{ Assume that } D \text{ is an anisotropic variety. Then}
\]

\[ \mathcal{M}(D) \simeq R \oplus (\oplus_{i=1}^{5} r(i)), \]

where \( R \) is a indecomposable motive such that over a separably closed field it becomes isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8) \) and \( R' \simeq \mathcal{M}(\text{SB}(A)) \).

\[ \text{Proof.} \text{ Consider the following commutative diagram (see [CPSZ05, 5.5]):}
\]

\[ D_s \times \mathbb{P}^2 \xrightarrow{\iota \times \text{id}_s} \text{Gr}(3, 6) \times \mathbb{P}^2 \xrightarrow{\text{Seg}_s} \text{Gr}(3, 18) \]

\[ D \times \text{SB}(A^{\text{op}}) \xrightarrow{\iota \times \text{id}} \text{SB}_3(M_2(A)) \times \text{SB}(A^{\text{op}}) \xrightarrow{\text{Seg}} \text{SB}_3(M_2(A) \otimes A^{\text{op}}) \]
where the right horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension \( k_s/k \).

This diagram induces the commutative diagram of rings

\[
\begin{array}{c}
\text{Ch}^*(D_s \times \mathbb{P}^2) \xrightarrow{(\iota_s \times \text{id}_s)^*} \text{Ch}^*(\text{Gr}(3,6) \times \mathbb{P}^2) \xrightarrow{\text{Seg}_s^*} \text{Ch}^*(\text{Gr}(3,18)) \\
\text{Ch}^*(D \times \text{SB}(A^{\text{op}})) \xrightarrow{(\text{id} \times \text{id})^*} \text{Ch}^*(\text{SB}_3(M_2(A)) \times \text{SB}(A^{\text{op}})) \xrightarrow{\text{Seg}_s^*} \text{Ch}^*(\text{SB}_3(M_2(A) \otimes A^{\text{op}}))
\end{array}
\]

where all maps are the induced pull-backs. Observe that the right vertical arrow is an isomorphism since \( M_2(A) \otimes A^{\text{op}} \) splits.

Let \( \tau_3 \) and \( \tau_1 \) be tautological vector bundles on \( \text{Gr}(3,6) \) and \( \mathbb{P}^2 \) respectively and let \( e \) denote the Euler class (the top Chern class). By [CPSZ05, Lemma 5.7] the cycle \((\iota_s \times \text{id}_s)^*\left( e(\text{pr}_1^*\tau_3 \otimes \text{pr}_2^*\tau_1) \right) \in \text{Ch}(D_s \times \mathbb{P}^2)\) is rational. A straightforward computation (cf. [CPSZ05, 5.10 and 5.11]) shows that

\[
r := - (\iota_s \times \text{id}_s)^*\left( e(\text{pr}_1^*\tau_3 \otimes \text{pr}_2^*\tau_1) \right) = h_3^{(1)}h_1^{(1)} + h_2^{(1)}h_1^{(1)} + H + H^2 \in \text{Ch}^3(D_s \times \mathbb{P}^2),
\]

where \( H \) is the class of a smooth hyperplane section of \( \mathbb{P}^2 \).

Define five rational cycles \( \rho_i = r(h_1^i \times 1) \in \text{Ch}^{3+i}(D_s \times \mathbb{P}^2) \) for \( i = 0, \ldots, 4 \). A straightforward computation using multiplication rules 3.6 shows that \((-\rho_{4-i}) \circ \rho_i \in \text{Ch}^2(\mathbb{P}^2 \times \mathbb{P}^2)\) is the diagonal \( \Delta_{\mathbb{P}^2} \).

To finish the proof of the theorem it remains by [CPSZ05, 5.4] to lift all these rational cycles \( \rho_i, \rho_i^j \) to \( \text{Ch}(D \times \text{SB}(A^{\text{op}})) \) and to \( \text{Ch}(\text{SB}(A^{\text{op}}) \times D) \) respectively in such a way that the corresponding compositions of their preimages would give the diagonal \( \Delta_{\text{SB}(A^{\text{op}})} \).

Fix an \( i = 0, \ldots, 4 \). Consider first any preimage \( \alpha \in \text{Ch}(D \times \text{SB}(A^{\text{op}})) \) of \(-\rho_{4-i}\) and any preimage \( \beta \in \text{Ch}(\text{SB}(A^{\text{op}}) \times D) \) of \( \rho_i^j \). The image of the composition \( \alpha \circ \beta \) under the restriction map is the diagonal \( \Delta_{\mathbb{P}^2} \). Therefore by Rost Nilpotence theorem for Severi-Brauer varieties (see 2.8) \( \alpha \circ \beta = \Delta_{\text{SB}(A^{\text{op}})} + n \), where \( n \) is a nilpotent element in \( \text{End}(\mathcal{M}(\text{SB}(A^{\text{op}}))) \). Since \( n \) is nilpotent \( \alpha \circ \beta \) is invertible and \((\Delta_{\text{SB}(A^{\text{op}})} + n)^{-1} \circ \alpha \circ \beta = \Delta_{\text{SB}(A^{\text{op}})} \). In other words, we can take \((\Delta_{\text{SB}(A^{\text{op}})} + n)^{-1} \circ \alpha \) as a preimage of \(-\rho_{4-i}\) and \( \beta \) as a preimage of \( \rho_i^j \).

Denote as \( R \) the remaining direct summand of the motive of \( D \). Comparing the left and the right hand sides of the decomposition over \( k_s \) it is easy to see that \( R_s \simeq \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8) \). An easy computation (see Lemma 3.8 and [NSZ05, 5.8]) shows also the indecomposability of \( R \).

\[ \square \]
3.8 Lemma. If the variety $D = D(A, c)$ is anisotropic over $k$ and $l/k$ is a separable extension of degree not divisible by 3, then $D_l$ is anisotropic.

Proof. The variety $D_l$ has a rational point if and only if $c$ lies in the image of the reduced norm $\text{Nrd}(A^*_l)$. By [Inv, Theorem 39.14(3)] this happens if and only if the corresponding Albert algebra $J = J(A, c)$ obtained by the first Tits construction is a division algebra. Now the result follows by [PR94, Corollary on p. 205].

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References


