FINITE CORRESPONDANCES AND TRANSFERS OVER A REGULAR BASE

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Abstract. In this article, we establish the base of the theory of motivic complexes of V. Voevodsky, more precisely the theory of sheaves with transfers which is exposed in [FSV00], chap. 2 and chap. 4. We intended to give a full treatment of the theory, that is full detailed proofs with the less possible references.

The purpose is twofold. First, we establish the theory over a general regular noetherian base by using the Tor formula for intersection multiplicities of [Ser58]. Secondly we give all the proofs of [FSV00] in the case of a perfect field. Though this relies on the ideas of [FSV00], the exposition differs notably as we consider solely the Nisnevich topology (instead of Zariski) and we use directly correspondances up to homotopy.

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General notations and conventions

Every schemes in this paper are implicitly assumed to be noetherians.

We say simply smooth (resp. étale) where one should say smooth (resp. étale) of finite type for a scheme over a base or a morphism. For any scheme S, we will denote by $\mathscr{L}_S$ the category of smooth S-schemes. We will also consider the category $\mathscr{L}_{\mathrm{cor},S}$ of smooth S-scheme equipped with finite S-correspondances with its canonical graph functor$\gamma : \mathscr{L}_S \to \mathscr{L}_{\mathrm{cor},S}$ (see def. 1.19).

Any presheaf (or sheaf) considered in this paper is assumed to be a presheaf of abelian group, unless explicitly stated.

Let S be a scheme and F be a presheaf over $\mathscr{L}_S$. We will extend the presheaf F to the category of pro-smooth S-schemes obviously : if $X_\bullet = (X_i)_{i \in \mathcal{I}}$ is such a pro-object, we put

$$F(X_\bullet) = \varinjlim_{i \in \mathcal{I}^{op}} F(X_i),$$

the colimit being computed in the category of abelian groups.

We will identify isomorphic pro-smooth S-schemes. This implies that a pro-object $X_\bullet$ which admits a limit in the category of S-schemes is determined uniquely by this limit (because the term of the pro-object are of finite presentation over S). Note in particular that, if for example this limit is affine over S, we can consider $X_\bullet$ as a pro-object of affine S-schemes.

The topology used to compute associated sheaves, cohomology or Čech cohomology, unless explicitly stated, is the Nisnevich topology.

We will denote by $\mathscr{N}_S^{\mathrm{tr}}$ the category of sheaves with transfers over S (see def. 2.3) and by $\mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ the subcategory described by sheaves with transfers which are homotopy invariant. Such sheaves will simply be called "homotopy sheaves". This terminology is inspired by the theory of perverse sheaves. Indeed, at least in the case of a perfect base field k, the category $\mathrm{H}\mathscr{N}_k^{\mathrm{tr}}$ is the heart of the category $DM_-^{eff}(k)$ for the homotopy t-structure[1].

Introduction

The generalisation of the theory of sheaves with transfers over a regular base in principally done in the first two sections.

In the first one, we prove all the basic facts concerning finite correspondances over a regular base, by using only Serre's To formula for intersection multiplicities. The original part here is the study of the functoriality of finite correspondances, the base change and "forget the base" functor, in subsection 1.5.

In the second section, we develop the theory of sheaves with transfers over a regular base. The arguments are very closed to that of [FSV00], chap. 4, apart being more precised. The study of the functoriality on the countrary is new.

In the rest of the paper, we give a proof of the fundamental facts about homotopy invariant sheaves with transfers over a perfect field. This proof follows the principal ideas of [FSV00], chap. 2 but it seems to us more accurate for two reasons.

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[1] Note that the homotopy t-structure is also defined by Morel on the stable homotopy category of schemes over k. Its heart are indeed homotopy invariant sheaves but these ones do not have transfers in general, in the sense given here. Thus the correct terminology for the object of $\mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ should be homotopy sheaves with transfers or may be homotopy oriented sheaves but there is no risk of confusion here.

First, we interpret the constructions about pretheories in the framework of correspondances up to homotopy, in section 3. This shows in particular how the arguments of Voeovdsky are related to cycles and even consist of constructing cycles. See especially proposition 3.21 and theorem 3.23. The first one is a corrected statement of a result of [FSV00], chap. 2 (which is valid only in the case of an infinite field). The second one on the countrary is a generalisation of a result in *loc. cit.* which allows to use Nisnevich topology in what follows. In this part, we use a little the functoriality of finite correspondance.

Secondly, we use only the Nisnevich topology. In the study of the category of homotopy invariance and sheaves with transfers, this allows to give another proof of the fundamental fact, that is the associated sheaf functor preserves homotopy invariance for sheaves with transfers (cf corollary 4.14). The strategy is to use the Nisnevich Čech cohomology functor together with the computation done in proposition 4.10 (this computation is a slightly more precised version of proposition 5.4 of *loc. cit.*).

The fundamental theorem for the theory is that a homotopy invariant sheaf with transfers over a perfect field has homotopy invariant cohomology. We expose the proof of this important theorem of Voevodsky in the last section. The fundamental argument of the proof is completely due to Voevodsky. Our work has only consist to establish the preliminary facts clearly (in full generality), and more precisely the construction of the localisation long exact sequence for homotopy sheaves (see section 5.3). By the way, these facts are slightly different from the analog in *loc. cit.* as we use the Nisnevich topology. In particular, we have to use our generalisation of the theory established in theorem 3.23. We hope that the reader will be able to appreciate the beautiful argument of Voeovdsky in this proof which consist to prove both localisation and homotopy invariance for cohomology (see 5.4 for more details).

Future works

The intention of the author in establishing the theory in that generality is to construct the category of motivic complexes over a general regular base and more importantly to give full functoriality for this category. This will be done in a future work of D.-C. Cisinski and the author (cf [CD]).

The definition of the category is simple : motivic complexes are (unbounded) complexes of sheaves with transfers with homotopy invariant hypercohomology. This full generality if needed for the theory. On the other hand, deriving functors in this setting is not so obvious. The key point is to use the model category theory of Quillen and already, the informed reader could recognize that one can do that following [MV99]. Indeed, the category of motivic complexes is the $\mathbb{A}^1$-localisation of the derived category of sheaves with transfers. We will establish this in a greater generality in [CD].

Note that other a regular base S, the category of motivic complexes which are compact can be described easily, almost as in the case of a perfect base field :
Let $K^b(\mathscr{L}_{\mathrm{cor},S})$ the category of bounded complexes of the category of finite S-correspondances.
Let $\mathcal{T}$ be the localising subcategory generated by the complexes of the form :

(1) $[\mathbb{A}^1_X] \xrightarrow{p_*} [X]$ for a smooth S-scheme X, p the canonical projection.
(2) $[W] \xrightarrow{g_*-k_*} [U] \oplus [V] \xrightarrow{j_*+f_*} [X]$ for a distinguished square $\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$.

The category of compact motivic complexes is equivalent to $K^b(\mathscr{L}_{\mathrm{cor},S})/\mathcal{T}$.

For the functoriality, we intend to use the recent work of J.Ayoub on cross functors to get the six functor formalism for motivic complexes. First of all, we have to construct a non effective version of motivic complexes, which can be done by considering symetric spectra of complexes. Still, there is a technical problem as the theory developped here allows only regular base schemes. However, though it is not obvious, one can do the work of Ayoub with only those regular bases. Nonetheless, the choice of the author is to generalize motivic complexes to arbitrary base by using the theory of relative cycles of [FSV00] - which was surely the original intention of Voevodsky.

1. Finite correspondances

1.1. **Relative cycles.** In this section, we use a particular case of the notion of a relative cycle from [FSV00], chap. 2, by restricting to the case of a regular base. We set up the entire foundations of the theory in this case using [Ser58] and [Ful98].

1.1.1. *Definition.* We begin by recalling a few fact about equidimensionality :

Definition 1.1. Let $f : X \to S$ be a morphism. One says that f is equidimensional if :

(1) f is of finite type.
(2) the relative dimension of f is constant.
(3) Every irreducible component of X is dominant over an irreducible component of S.

If S is regular, normal or more generaly geometrically unibranch, one obtains equivalent conditions to that of the above definition by replacing the third property with the stronger one of being universally open (cf [GD66, 14.4.4]).

The following lemma allows to simplify this notion in a particular case which we are interested in :

Lemma 1.2. *Let X, S be irreducible schemes, S geometrically unibranch. The following conditions on a morphism $f : X \to S$ are equivalent :*

(1) *f is finite equidimensional.*
(2) *f is finite onto.*
(3) *f is proper, equidimensional of dimension 0.*

Proof. Conditions 1 and 2 are equivalent because a finite morphism is of constant relative dimension 0 and universally closed. The equivalence between 1 and 3 follows from the Stein factorisation. □

We will use the following particular notion on cycles from the general theory of [SV00] which is all what we need for our constructions :

Definition 1.3. Let S be a regular scheme and X an S-scheme.

We define the abelian group $\mathrm{c}_0(X/S)$ as the subgroup of the group of cycles on X generated by points x which closure in X is finite equidimensional over S.

The elements of this group are called the finite relative cycles on X/S.

Remark 1.4. In *loc. cit.*, the cycles defined above are denoted by $c_{equi}(X/S, 0)$, and called the equidimensional relative cycles of relative dimension 0 on X/S.

A cycle α on X is a finite relative cycle on X/S if and only if its support is finite equidimensional over S (lemma 1.2).

Remark also that if $S = S_1 \sqcup S_2$, $\mathrm{c}_0(X/S) = \mathrm{c}_0(X/S_1) \oplus \mathrm{c}_0(X/S_2)$. This allows to reduce to the case S irreducible.

1.5. Let S be a regular scheme and X an S-scheme.

Consider a closed subscheme Z of X which is finite equidimensional over S. The irreducible componenents of Z which dominate an irreducible component of S are finite equidimensional over S.

Let $(z_i)_{\{i=1,\dots,n\}}$ be the generic points of Z which dominates an irreducible component of S. One associate to Z a finite relative cycle on X/S :

$$[Z]_{X/S} = \sum_i \mathrm{lg}(\mathcal{O}_{Z,z_i}).z_i.$$

1.1.2. *Pullback.* Let S and T be regular schemes and consider a cartesian square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

with p smooth.

Let α be a finite relative cycle on X/S and U be its support. We show that the pullback cycle $f^*(\alpha)$ is well defined in the sense of [Ser58], that is the hypothesis of V.C.7 (b) are satisfied[2].

As U is finite equidimensional over S, $V = f^{-1}(U)$ is again finite equidimensional over T. Suppose that U is irreducible. Then considering the irreducible component of X containing U and its image on S, we can suppose that X and S are irreducible. As the morphism p is smooth, it is equidimensional of dimension n and the codimension of U in X is n. Moreover, the morphism q is equidimensional of dimension n and the codimension of V in Y is n. This proves $f^*(\alpha)$ is well defined. Moreover, it is a finite relative cycle on Y/T.

Definition 1.6. With the preceding notations, we will put $\Delta^*(\alpha) = f^*(\alpha)$ as a cycle in $c_0(Y/T)$.

Using [Ser58], V.C.7 exercice 1, we obtain this pullback is functorial with respect to composition of cartesian squares.

We note also the following lemma :

Lemma 1.7. *Let S, T be regular schemes and consider a cartesian square*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

such that p is smooth and g is flat.

Then, for any closed subscheme Z in X which is finite equidimensional over S,

$$\Delta^*([Z]_{X/S}) = [Z \times_S T]_{Y/T}.$$

Proof. Put $\alpha = [Z]_{X/S}$ and let Γ_f be the graph of f as a closed subscheme of $Y \times X$. As Y is regular, using [Ser58], V.C.8, $\Delta^*(\alpha) = [\Gamma_f].[Y \times \alpha]$. But Γ_f is isomorphic to Y, thus it is a regular scheme. Its local rings are all Cohen-Macaulay local rings and the result now follows from [Ful98], prop. 7.1 (see also [Ser58], V.C.1 th. 1 for the identification of Serre's intersection multiplicities with Samuel's). □

[2]More precisely, we consider the extension of the theory presented in [Ser58] to the case of arbitrary noetherian regular schemes, as described in V.C.8. Moreover, we don't need theorem 1 of V.B.3 as the correct equality of dimension is already true in our case. Besides, the positivity of intersection multiplicities for arbitrary regular noetherian local rings has been proved recently by O.Gabber.

1.1.3. *Compacity of relative cycles.* Consider $(T_i)_{i\in\mathcal{I}}$ a pro-object of affine regular noetherian S-schemes. It admits a limit $\mathcal{T}$ in the category of affine S-schemes. Indeed, T_i is the spectrum over S of a coherent $\mathcal{O}_S$-algebra $\mathcal{A}_i$. The family $(\mathcal{A}_i)_{i\in I^{op}}$ together with its natural transition morphisms form an ind-$\mathcal{O}_S$-algebra. It admits a limit and we put $\mathcal{A} = \varinjlim_{i\in\mathcal{I}^{op}} \mathcal{A}_i$. Then $\mathcal{T} = \mathrm{Spec}_S(\mathcal{A})$.

Note that $\mathcal{T}$ needs not be noetherian nor regular[3].

Proposition 1.8. *We adopt the notations above and suppose $\mathcal{T}$ is regular noetherian. Let X be a smooth S-scheme. We put $X_i = X \times_S T_i$, $X_\mathcal{T} = X \times_S \mathcal{T}$ and consider the cartesian square*

$$\begin{tikzcd} X_\mathcal{T} \arrow[r, "f_i"] \arrow[d] \arrow[dr, phantom, "\Delta_i"] & X_i \arrow[d] \\ \mathcal{T} \arrow[r, "g_i"'] & T_i. \end{tikzcd}$$

Then the morphism $\delta = \varinjlim_{i\in\mathcal{I}^{op}} \Delta_i^* : \varinjlim_{i\in\mathcal{I}^{op}} \mathrm{c}_0(X_i/T_i) \to \mathrm{c}_0(X_\mathcal{T}/\mathcal{T})$ *is an isomorphism.*

Proof. Let first prove δ is surjective. Let Z be a closed integral subscheme of $X_\mathcal{T}$ finite equidimensional over $\mathcal{T}$. Then Z is defined by a quasi-coherent ideal of $\mathcal{O}_{X_\mathcal{T}}$. As $X_\mathcal{T}$ is noetherian this ideal is coherent, generated by a finite number of sections $f_1, ..., f_n$. The sheaf $\mathcal{O}_{X_\mathcal{T}}$ is the inductive limite of the $\mathcal{O}_{X_i}$ and we can suppose there exists $i_0 \in I$ such that $f_1, ..., f_n$ lift to $\mathcal{O}_{X_i}$. Let Z_i be the closed subscheme of X_i defined by the equations $f_1, ..., f_n = 0$. Note, the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X_\mathcal{T} \arrow[d, "p_i \times_S 1_X"] \\ Z_i \arrow[r] & X_i \end{tikzcd}$$

is cartesian.

Denote by $\mathcal{I}/i$ the category whose objects are the arrows $j \to i$. Considering such an arrow, we let Z_j be the pullback of Z_i along the corresponding morphism. This defines a pro-object $(Z_j)_{j\in\mathcal{I}/i}$ such that the canonical morphism

$$Z \to \varprojlim_{j\in\mathcal{I}/i} Z_j$$

is an isomorphism.

Using [GD66] there exists $j \to i$ such that :

(1) Z_j is integral using cor. 8.4.3 of *loc.cit.* because the transition morphism of the pro-object $(Z_j)_{j\geq i}$ are dominants.
(2) Z_j is finite surjective over a component of T_j using 8.10.5 of *loc.cit.*.

Thus the cycle $[Z_j]$ of X_j associated to Z_j is a finite relative cycle on X_j/T_j. As $f_j^{-1}(Z_j) = Z$ is an integral scheme, we obtain $f_j^*([Z_j]) = [Z]$ that is $\Delta_j^*([Z_j]) = [Z]$.

We show finally δ is injective. Let $i \in \mathcal{I}$ and $\alpha_i \in \mathrm{c}_0(X_i/T_i)$ such that $\delta(\alpha_i) = 0$. For any $j \to i$ in $\mathcal{I}$ with corresponding transition morphism $f_{ji} : X_j \to X_i$, let Z_j be the support of $f_{ji}^*(\alpha)$.

Then $(Z_j)_{j\in\mathcal{I}/i}$ is a pro-object. As $f_i^*(\alpha) = 0$ this pro-object as the empty scheme as limit. This means the canonical morphism

$$\emptyset \to \varprojlim_{j\in\mathcal{I}/i} Z_j$$

[3] Supposing $\mathcal{T}$ is noetherian, the author is aware essentially of two hypothesis that imply $\mathcal{T}$ regular. The first one is when the transition morphisms of $(T_i)_{i\in I}$ are flat. The second one is when S is of equal characteristic.

is an isomorphism. From the first point 8.10.5 of *loc. cit.* there exists $j \to i$ such that $Z_j = \emptyset$. Thus $f_{ji}^*(\alpha) = 0$ which shows α_i is 0 in the colimit $\varinjlim_{i \in \mathcal{I}^{op}} \mathrm{c}_0(X_i/T_i)$. $\square$

1.1.4. *General pushout.* One of the advantage of relative cycles is that pushout by any morphism is always defined and functorial.

Lemma 1.9. *Let S be an irreducible scheme and $f : X \to Y$ be a morphism of finite type S-schemes.*

Let Z be a closed integral subscheme of X. If Z is finite and surjective over S then $f(Z)$ equipped with its reduced structure of subscheme in Y is closed and finite surjective over S. The morphism $Z \to f(Z)$ is finite surjective.

Proof. Indeed as Z/S is proper, $f(Z)$ is closed in Y. With its induced structure of reduced subscheme of Y, it is proper over S, as it can be seen for example from the valuative criteron of properness (cf [Har77]). Moreover using [GD63] 4.4.2, $f(Z)$ is finite over S because its fibers are finite. Thus the induced morphism $Z \to f(Z)$ is finite. $\square$

Definition 1.10. Let S be a scheme, X and Y be finite type S-schemes and $f : Y \to X$ be an S-morphism.

For Z a closed integral subscheme of X which is finite and equidimensional over S, we set according to the preceding lemma

$$f_*([Z]) = d.[f(Z)] \in \mathrm{c}_0(X/S)$$

where d is degree of the extension induced by f between the respective function field of Z and $f(Z)$.

By linearity, this defines a morphism $f_* : \mathrm{c}_0(Y/S) \to \mathrm{c}_0(X/S)$.

1.11. This pushout coincide with the one of [Ser58, V-27.6]. As it is always reduced to a pushout by a proper morphism according to the preceding lemma, it is functorial in f. This is easily seen directly using the transitivity of degree extensions.

Proposition 1.12. *Let S, S' be regular schemes and $q : S' \to S$ be a flat morphism. Consider the two following cartesian squares of schemes :*

$$\begin{tikzcd} X' \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\Theta"] & X \arrow[d] \\ Y' \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\Delta"] & Y \arrow[d] \\ S' \arrow[r, "q"'] & S. \end{tikzcd}$$

We denote by $\Delta \bullet \Theta$ the cartesian square defines by the external arrows of this diagram. Then for all finite relative cycle $\alpha \in \mathrm{c}_0(X/S)$, we have the relation

$$g_*(\Delta \bullet \Theta)^*(\alpha) = \Delta^* f_*(\alpha).$$

Proof. Using linearity, we can assume that α is a closed integral subscheme Z of X. The cycle $f_*(\alpha)$ is supported in $f(Z)$. Thus we can assume Y equals $f(Z)$ and f is proper. Finally considering lemma 1.7, we are exactly reduced to the classical projection formula of [Ful98, 1.7]. $\square$

We state another projection formula involving intersection product which will be suitable for our needs.

Proposition 1.13. *Let X, X' and S be regular schemes and consider the diagram*

$$\begin{tikzcd} & S & \\ Y' \arrow[ur] \arrow[rr, "f"] \arrow[d] & & Y \arrow[ul] \arrow[d, "p"] \\ X' \arrow[rr, "g"'] & \Delta & X \end{tikzcd}$$

where Δ is cartesian and p smooth.

Let $\sigma \in \mathrm{c}_0(Y/X)$, $\epsilon \in \mathrm{c}_0(Y'/S)$.

Then, the following equation holds whenever the intersection involved are proper :

$$f_*\big(\Delta^*(\sigma).\epsilon\big) = \sigma.f_*(\epsilon).$$

Proof. Let V be the support of ϵ. Then, as V is proper over S, the restriction $f|_V : V \to Y$ is proper using the arguments preceding definition 1.10. Thus the formula is simply formula (10) of [Ser58], V.C.8. □

1.2. **Composition of finite correspondances.** Let S be a regular scheme. Generalising the definition of [Voe00a], we introduce the finite S-correspondances.

Definition 1.14. Let X and Y be two smooth S-schemes.

We define the group of finite S-correspondances from X to Y as the abelian group :

$$\mathrm{c}_S\left(X, Y\right) = \mathrm{c}_0(X \times_S Y/X)\,.$$

We adopt the following notation. If X and Y are two S-schemes, we put $XY = X \times_S Y$ and we denote by $p_X^{XY} : XY \to X$ the canonical projection. If X, X', Y, Y' are smooth schemes, we denote by $p_{(X)Y}^{(XX')YY'}$ for the cartesian square

$$\begin{tikzcd} XX'YY' \arrow[r] \arrow[d] & XY \arrow[d] \\ XX' \arrow[r] & X \end{tikzcd}$$

induced by the canonical projections.

The following lemma will show that the composition law of finite correspondances is well defined.

Lemma 1.15. *Let X, Y, Z be smooth S-scheme, $\alpha \in \mathrm{c}_S\left(X, Y\right)$ and $\beta \in \mathrm{c}_S\left(Y, Z\right)$.*

1. *The cycles $p_{(Y)Z}^{(XY)Z^*}(\beta)$ and $p_{(X)Y}^{(X)Y(Z)^*}(\alpha)$ of XYZ intersect each other properly.*
2. *According to the first point, the intersection product in XYZ*
$$p_{(Y)Z}^{(XY)Z^*}(\beta).p_{(X)Y}^{(X)Y(Z)^*}(\alpha)$$
is well defined. It is a finite relative cycle on XYZ/X.

Proof. We can assume that α and β are closed integral subschemes. Then we can assume also that X, Y and Z are irreducible, considering the particular component which α and respectively β dominate.

Consider the following diagram :

$$\begin{tikzcd} \beta \times_Y \alpha \arrow[r] \arrow[d] & \beta \arrow[r] \arrow[d] & Z \\ \alpha \arrow[r] \arrow[d] & Y & \\ X. & & \end{tikzcd}$$

The vertical arrows are all finite equidimensionnal. In particular, $\beta \times_Y \alpha$ is finite equidimensional over X. But there is a canonical isomorphism

$$\beta \times_Y \alpha \to (X\beta) \times_{XYZ} (\alpha Z) = (X\beta) \cap (\alpha Z).$$

In particular, every component of $(X\beta) \cap (\alpha Z)$ is finite equidimensional over X. Thus, they are all of codimension $\dim(Y)+\dim(Z)$ in XYZ, which proves the first point. The second point then follows from the preceding remark. □

Definition 1.16. Let X, Y and Z be smooth S-scheme, and $\alpha \in \mathrm{c}_S\,(X, Y)$ and $\beta \in \mathrm{c}_S\,(Y, Z)$. Let consider the pushout
${p_{(X)Z}^{(X)YZ}}_* : \mathrm{c}_0(XYZ/X) \to \mathrm{c}_0(XZ/X)$ defined in 1.10.

We put

$$\beta \circ \alpha = {p_{(X)Z}^{(X)YZ}}_* \left(p_{(Y)Z}^{(XY)Z^*}(\beta).p_{(X)Y}^{(X)Y(Z)^*}(\alpha)\right)$$

which is a well defined finite S-correspondance from X to Z from the preceding lemma.

Example 1.17. Let $f : X \to Y$ be a S-morphism between smooth S-scheme. As Y/S is separated, the S-graph Γ_f of f is a closed subscheme of XY. As the canonical projection $\Gamma_f \to X$ is an isomorphism, the cycle $[\Gamma_f]_{XY/X}$ belongs to $\mathrm{c}_S\,(X, Y)$. This allows to define an application

$$\mathrm{Hom}_{\mathscr{L}_S}(X, Y) \to \mathrm{c}_S\,(X, Y)$$

which is obviously injective.

We will use the same letter f for the the finite S-correspondance $[\gamma_f]_{XY/X}$. The following lemma will prove that the composition of S-morphisms coincides with that of finite S-correspondances which make this confusion anodyne.

Lemma 1.18. *Let X, Y, Z be smooth S-schemes. Then the following relations hold :*

1. *For all $\alpha \in \mathrm{c}_S\,(X, Y)$, $\beta \in \mathrm{c}_S\,(Y, Z)$, $\gamma \in \mathrm{c}_S\,(Z, T)$, $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.*
2. *For all $\alpha \in \mathrm{c}_S\,(X, Y)$ and all S-morphism $f : Y \to Z$,*
$$f \circ \alpha = (1_X \times_S f)_*(\alpha)$$
using definition 1.10 for the pushout.
3. *For all $\beta \in \mathrm{c}_S\,(Y, Z)$ and all S-morphism $f : X \to Y$, considering the cartesian square* $f_Z :$ $\begin{array}{ccc} XZ & \to & YZ \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$ *, one has*
$$\beta \circ f = f_Z^*(\beta)$$
using definition 1.6 for the pullback.
4. *For all S-morphism $f : X \to Y$ and $g : Y \to Z$, $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{g \circ f}]$.*

Proof. 1. The idea to prove this relation is to show that one can compose the three correspondances by pulling them back all to $XYZT$, make the intersection product in that scheme and then pushout the result to XT. We detail this procedure :

$$\gamma \circ (\beta \circ \alpha) = \gamma \circ \left({p_{XZ}^{XYZ}}_* \left(p_{(Y)Z}^{(XY)Z^*}(\beta).p_{(X)Y}^{(X)Y(Z)^*}(\alpha)\right)\right)$$

$$= {p_{XT}^{XZT}}_* \left(p_{(Z)T}^{(XZ)T^*}(\gamma).p_{(X)Z}^{(X)Z(T)^*} {p_{XZ}^{XYZ}}_* \left(p_{(Y)Z}^{(XY)Z^*}(\beta).p_{(X)Y}^{(X)Y(Z)^*}(\alpha)\right)\right)$$

$$= {p_{XT}^{XZT}}_* \left(p_{(Z)T}^{(XZ)T^*}(\gamma).{p_{XZT}^{XYZT}}_* p_{(X)YZ}^{(X)YZ(T)^*} \left(p_{(Y)Z}^{(XY)Z^*}(\beta).p_{(X)Y}^{(X)Y(Z)^*}(\alpha)\right)\right) \quad (a)$$

$$= {p_{XT}^{XZT}}_* {p_{XZT}^{XYZT}}_* \left(p_{(XZ)T}^{(XYZ)T^*} p_{(Z)T}^{(XZ)T^*}(\gamma).p_{(X)YZ}^{(X)YZ(T)^*} \left(p_{(Y)Z}^{(XY)Z^*}(\beta).p_{(X)Y}^{(X)Y(Z)^*}(\alpha)\right)\right) \quad (b)$$

$$= {p_{XT}^{XYZT}}_* \left(p_{(Z)T}^{(XYZ)T^*}(\gamma).\left(p_{(Y)Z}^{(XY)Z(T)^*}(\beta).p_{(X)Y}^{(X)Y(ZT)^*}(\alpha)\right)\right) \quad (c)$$

$$= {p_{XT}^{XYZT}}_* \left(p_{(Z)T}^{(XYZ)T^*}(\gamma).p_{(Y)Z}^{(XY)Z(T)^*}(\beta).p_{(X)Y}^{(X)Y(ZT)^*}(\alpha)\right) \quad (d)$$

with the following justifications :

(a) This is proposition 1.12 for the cartesian squares
$$\begin{tikzcd} XYZT \arrow[r] \arrow[d] & XYZ \arrow[d] \\ XZT \arrow[r] \arrow[d] & XZ \arrow[d] \\ XT \arrow[r] & X \end{tikzcd}.$$

(b) This is the projection formula 1.13 for
$$\begin{tikzcd} & X & \\ XYZT \arrow[ur] \arrow[rr] \arrow[d] & & XZT \arrow[ul] \arrow[d] \\ XYZ \arrow[rr] & & XZ. \end{tikzcd}$$

(c) This is the functoriality of pullback and pushout, and the compatibility of intersection product with pullback.

(d) This notation is valid using the associativity of intersection product (cf [Ser58], V.C.3.b).

Similarly the same computation works for the right hand side of the equality which concludes.

2. We can assume that α is a closed integral subscheme of XY. Then we must compute the following cycle :

$$\begin{aligned} f \circ \alpha &= p^{XYZ}_{XZ\,*}\left(p^{XYZ\,*}_{YZ}[\Gamma_f].p^{XYZ\,*}_{XY}(\alpha)\right) \\ &= p^{XYZ}_{XZ\,*}\left([X\Gamma_f].[\alpha Z]\right). \end{aligned}$$

The intersection involved in this cycle is particularly simple :

$$\begin{tikzcd} X\Gamma_f \times_{XYZ} \alpha Z \arrow[rrr] \arrow[dr, "b"] \arrow[dd, hook] & & & \alpha Z \arrow[dr] \arrow[dd, hook] & \\ & X\Gamma_f \times_{XY} \alpha \arrow[rrr, "a"] \arrow[dd, hook] & & & \alpha \arrow[dd, hook, "i_\alpha"] \\ X\Gamma_f \arrow[rrr, hook, "i"] \arrow[dr, equal] & & & XYZ \arrow[dr] & \\ & X\Gamma_f \arrow[rrr, "\sim", "p"'] & & & XY \end{tikzcd}$$

where p is induced by the canonical isomorphism $\Gamma_f \to Y$,and i, i_α are the canonical closed immersions.

The front square in this cube is cartesian so a is an isomorphism. But the backward and right squares are both cartesian which implies the left square is also cartesian. Thus b is an isomorphism.

This implies that $W = X\Gamma_f \times_{XYZ} \alpha Z$ is isomorphic to α. Thus it is an integral scheme. In particular, the intersection of $X\Gamma_f$ and αZ is reduced to the single component W. Moreover, the intersection multiplicity of W is 1 using [Ful98], prop. 7.2. Indeed, a base field is not needed here : we only need the comparison of Serre's intersection multiplicities (the Tor formula) with Samuel's intersection multiplicity. This is [Ser58], V.C.4. We have obtained :

$$\begin{aligned} &[X\Gamma_f]_{XYZ}.[\alpha Z]_{XYZ} = [X\Gamma_f \times_{XYZ} \alpha Z]_{XYZ} \\ &= i_*([X\Gamma_f \times_{XYZ} \alpha Z]_{X\Gamma_f}) = i_*([X\Gamma_f \times_{XY} \alpha]_{X\Gamma_f}) = i_* p^*(\alpha). \end{aligned}$$

We now use the factorization

$$\begin{tikzcd} & X\Gamma_f \arrow[dl, "p"'] \arrow[dr, "i"] & \\ XY \arrow[rr, "1_X \times_S \gamma_f"'] & & XYZ \end{tikzcd}$$

where $\gamma_f : Y \to YZ$ is the graph morphism. Thus $i_* = (1_X \times_S \gamma_f)_* p_*$ which means $(1_X \times_S \gamma_f)_* = i_* p^*$ as p is an isomorphism.

And finally : $p^{XYZ}_{XZ\,*}([X\Gamma_f].[\alpha Z]) = p^{XYZ}_{XZ\,*}(1_X \times_S \gamma_f)_*(\alpha) = (1_X \times_S f)_*\alpha.$

3. We compute

$$\beta \circ f = p_{(X)Z}^{(XY)Z*}\left(p_{(Y)Z}^{(XY)Z}(\beta).[\Gamma_f Z]_{XYZ}\right)$$

We consider the cartesian square

$$\begin{tikzcd} \Gamma_f Z \arrow[r, "p"] \arrow[d, "\iota"'] & XZ \arrow[d, "f \times_S 1_Z"] \\ XYZ \arrow[r, "p_{YZ}^{XYZ}"] & YZ \end{tikzcd}.$$

Using the definition of [Ser58], V.C.7,

$$p_{(Y)Z}^{(XY)Z}(\beta).[\Gamma_f Z]_{XYZ} = \iota_* \iota^* p_{YZ}^{XYZ*}(\beta) = \iota_* p^*(f \times_Z 1_Z)^*(\beta).$$

As p is an isomorphism $p^* = (p_*)^{-1}$. Therefore we conclude because $p_{XZ}^{XYZ} \circ \iota = p$.

4. This follows either from point 2 and functoriality of pushout or point 3 and functoriality of pullback. □

It follows from the preceding lemma the product $\circ$ of finite S-correspondances is associative and for any smooth S-scheme X the S-morphism 1_X seen as a correspondance is the neutral element.

Definition 1.19. Let S be a regular scheme.

We denote by $\mathscr{L}_{\mathrm{cor},S}$ the category whose objects are finite type smooth separated S-schemes and morphisms are the finite S-correspondances.

We denote by $\gamma : \mathscr{L}_S \to \mathscr{L}_{\mathrm{cor},S}$ the faithful functor which is the identity on objects and sends a S-morphism to its graph (cf ex. 1.17).

If X is a smooth S-scheme, we denote by $[X]$ the corresponding object of $\mathscr{L}_{\mathrm{cor},S}$.

This category is additive and for any smooth S-schemes X and Y, $[X] \oplus [Y] = [X \sqcup Y]$.

1.3. Monoidal structure.

Lemma 1.20. *Let X, X', Y, Y' be smooth S-scheme.*

Then for any $\alpha \in \mathrm{c}_S(X,Y)$ and $\beta \in \mathrm{c}_S(X',Y')$, the cycles ${p_{XYX'Y'}^{XY}}^*(\alpha)$ and ${p_{XYX'Y'}^{X'Y'}}^*(\beta)$ intersects properly and the intersection cycle is a finite relative cycle on $XYX'Y'/XX'$.

Proof. Let assume α et β are closed integral subscheme. Consider the diagram :

$$\begin{tikzcd} X\beta \times_X \alpha \arrow[r] \arrow[d] & \alpha \arrow[d] \\ X\beta \arrow[r] \arrow[d] & X \\ XX'. & \end{tikzcd}$$

All vertical arrows are finite equidimensional. But $X\beta \times_X \alpha$ is isomorphic to $XY\beta \cap \alpha X'Y'$. Thus this scheme is finite equidimensional on XX' which implies the corresponding intersection is proper and concludes. □

Definition 1.21. Let X, X', Y, Y' be smooth S-scheme.

For all $\alpha \in \mathrm{c}_S(X,Y)$ and $\beta \in \mathrm{c}_S(X',Y')$ we put

$$\alpha \otimes_S^{tr} \beta = {p_{XY}^{XYX'Y'}}^*(\alpha).{p_{X'Y'}^{XYX'Y'}}^*(\beta).$$

From the preceding lemma, this is a well defined cycle and an element of $\mathrm{c}_S(XX',YY')$.

Lemma 1.22. *Suppose given finite S-correspondances :*
$\alpha : X \to Y,\ \alpha' : Y \to Z,\ \beta : X' \to Y',\ \beta' : Y' \to Z'.$

$$(\alpha' \circ \alpha) \otimes_S^{tr} (\beta' \circ \beta) = (\alpha' \otimes_S^{tr} \beta') \circ (\alpha \otimes_S^{tr} \beta).$$

Proof. As for the associativity of composition of finite correspondances, the proof is to show one can pullback all cycles on $XYZX'Y'Z'$ first then take the intersection and pushout the result on $XZX'Z'$. As in the proof of the first point of 1.18, we use the two projection formulas 1.12 and 1.13 and the functoriality of pushout and pullback. We let the details to the reader. □

Proposition 1.23. *The category $\mathscr{L}_{\mathrm{cor},S}$ is monoidal symetric with tensor product*

$$[X] \otimes_S^{tr} [Y] = [X \times_S Y]$$

for smooth S-scheme X and Y and tensor product of finite S-correspondances given by definition 1.21.

The functor $\mathscr{L}_S \xrightarrow{\gamma} \mathscr{L}_{\mathrm{cor},S}$ of definition 1.19 is monoidal where the tensor product on $\mathscr{L}_S$ is the cartesian product over S.

Proof. Commutativity is obvious using commutativity of intersection product (cf [Ser58], V.C.3.a). Associativity follows from the same proof as for associativity of composition product : using the projection formulas (cf prop. 1.12 and 1.13), one reduces to associativity of intersection product.

For the last assertion, one reduces to prove the equality $f \otimes_S^{tr} 1_Y = f \times_S 1_Y$ for smooth S-schemes X, X', Y and an S-morphism $f : X \to X'$. Let Δ_Y be the diagonal of Y/S and Γ_f the S-graph of f. Following now the same line as in the proof of the third point of lemma 1.18 we note that the intersection of $\Gamma_f YY$ and $XX'\Delta_Y$ is isomorphic to the graph of $f \times_S 1_Y$ which is isomorphic to XY and thus reduced. This implies that the intersection multiplicities are 1 and gives the result. □

1.4. **A finiteness property.** Let $(X_i)_{i\in I}$ be a pro-object of affine smooth S-schemes. As we have seen in 1.1.3, this pro-object admits a limit $\mathcal{X}$ in the category of affine S-schemes. We assume $\mathcal{X}$ is regular noetherian.

In that case, for any smooth S-scheme Y, we will put $\bar{\mathrm{c}}_S(\mathcal{X}, Y) = \mathrm{c}_0(\mathcal{X} \times_S Y/\mathcal{X})$ to extend definition 1.14. Moreover, the projection morphisms $p_i : \mathcal{X} \to X_i$ induces by pullback a morphism $p_i^* : \mathrm{c}_S(X_i, Y) \to \bar{\mathrm{c}}_S(\mathcal{X}, Y)$. These morphisms are obviously natural in $i \in I$.

Then proposition 1.8 admits immediately the following corollary :

Proposition 1.24. *Consider the hypothesis and notations above.*

Then the morphism

$$\text{(A)} \qquad \varinjlim_{i\in I^{op}} p_i^* : \varinjlim_{i\in \mathcal{I}^{op}} \mathrm{c}_S(X_i, Y) \to \bar{\mathrm{c}}_S(\mathcal{X}, Y)$$

is an isomorphism.

Remark 1.25. This proposition is implicitly used in the proof of prop. 3.1.3 in [SV00], chap.5.

1.26. Let us loosely remark that, using the arguments of lemma 1.15, we can define a product $\bar{\mathrm{c}}_S(\mathcal{X}, Y) \otimes_{\mathbb{Z}} \mathrm{c}_S(Y, Z) \to \bar{\mathrm{c}}_S(\mathcal{X}, Z), (\bar{\alpha}, \beta) \mapsto \beta\bar{\circ}\bar{\alpha}$. Then, using the argument of the proof for the first point of lemma 1.18, we obtain the relation $(\gamma \circ \beta)\bar{\circ}\bar{\alpha} = \gamma\bar{\circ}(\beta\bar{\circ}\bar{\alpha})$.

In particular, the abelian group $\bar{\mathrm{c}}_S(\mathcal{X}, Y)$ is functorial with respect to the finite S-correspondances in Y. Finally, considering this functoriality, the isomorphism (A) is natural in Y with respect to finite S-correspondances.

1.27. Suppose now we are given a second pro-object $(X'_i)_{i\in I}$ of smooth affine S-schemes and a family of S-morphisms

$$f_i : X'_i \to X_i$$

which are compatible with transition morphisms.

Let $\mathcal{X}'$ be the projective limit of $(X'_i)_{i\in I}$, $p'_i : \mathcal{X}' \to X'_i$ the canonical projection and $\mathfrak{f} : \mathcal{X}' \to \mathcal{X}$ the projective limit of the $(f_i)_{i\in I}$. Then by considering suitable pullbacks of relative cycles, we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{c}_S(X_i, Y) \arrow[r, "p_i^*"] \arrow[d, "f_i^*"'] & \bar{\mathrm{c}}_S(\mathcal{X}, Y) \arrow[d, "\mathfrak{f}^*"] \\ \mathrm{c}_S(X'_i, Y) \arrow[r, "p_i'^*"] & \bar{\mathrm{c}}_S(\mathcal{X}', Y). \end{tikzcd}$$

This commutative diagram express in fact that the isomorphism (A) is natural in $\mathcal{X}$ with respect to morphisms of pro-S-schemes.

1.5. **Functoriality.**

1.5.1. *Base change.* Let $\tau : T \to S$ be a morphism of regular schemes.

If X and Y are smooth S-schemes we identify through the canonical isomorphism the schemes $X_T \times_T Y_T$ and $X \times_T Y$ which we both denote by XY_T.

For any smooth S-schemes X, X', Y, Y' we consider the following cartesian squares :

$$\begin{tikzcd} XX'YY' \arrow[r] \arrow[d] \arrow[dr, phantom, "p_{(X)Y}^{(XX')YY'}"] & XY \arrow[d] \\ XX' \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} XX'YY'_T \arrow[r] \arrow[d] \arrow[dr, phantom, "q_{(X)Y}^{(XX')YY'}"] & XY_T \arrow[d] \\ XX'_T \arrow[r] & X_T \end{tikzcd} \qquad \begin{tikzcd} XY_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\tau_{XY}"] & XY \arrow[d] \\ X_T \arrow[r] & X \end{tikzcd}$$

made up with the obvious projections.

For every finite S-correspondance $\alpha : X \to Y$ we put $\alpha_T = \tau_{XY}^*(\alpha)$ using definition 1.6.

Lemma 1.28. *Soit X et Y des schémas dans $\mathscr{L}_S$. Alors, pour tout $\alpha \in \mathrm{c}_S(X, Y)$ et $\beta \in \mathrm{c}_S(Y, Z)$, on a*

$$\beta_T \circ \alpha_T = (\beta \circ \alpha)_T.$$

Proof. Indeed we can do the following computation :

$$\begin{aligned} (\tau_{YZ}^*\beta) \circ (\tau_{XY}^*\alpha) &= q_{XZ\,*}^{XYZ}\big(q_{(Y)Z}^{(XY)Z^*}(\tau_{YZ}^*\beta).q_{(X)Y}^{(X)Y(Z)^*}(\tau_{XY}^*\alpha)\big) \\ &= q_{XZ\,*}^{XYZ}\Big(\tau_{XYZ}^*(p_{(Y)Z}^{(XY)Z^*}\beta).\tau_{XYZ}^*(p_{(X)Y}^{(X)Y(Z)^*}\alpha)\Big) \quad (1) \\ &= q_{XZ\,*}^{XYZ}\tau_{XYZ}^*\Big((p_{(Y)Z}^{(XY)Z^*}\beta).(p_{(X)Y}^{(X)Y(Z)^*}\alpha)\Big) \quad (2) \\ &= \tau_{XZ}^*\left(p_{XZ\,*}^{XYZ}\Big((p_{(Y)Z}^{(XY)Z^*}\beta).(p_{(X)Y}^{(XY)Z^*}\alpha)\Big)\right). \quad (3) \end{aligned}$$

where equality (1) follows from the functoriality of pullback, equality (2) is compatibility of pullback with intersection product (cf [Ser58], V.C.7 and equality 3 is proposition 1.12. □

Definition 1.29. Let $\tau : T \to S$ be a morphism of regular schemes.

Using the preceding lemma, we define the base change functor

$$\begin{aligned} \tau^* : \mathscr{L}_{\mathrm{cor},S} &\to \mathscr{L}_{\mathrm{cor},T} \\ X/S &\mapsto X_T/T \\ \mathrm{c}_S(X, Y) \ni \alpha &\mapsto \alpha_T. \end{aligned}$$

We sum up the basic properties of base change for correspondances in the following lemma.

Lemma 1.30. (1) *The functor τ^* is symetric monoidal.*

(2) *Let $\tau_0^* : \mathscr{L}_{\mathrm{cor},S} \rightarrow \mathscr{L}_{\mathrm{cor},T}$ be the classical base change functor on smooth schemes. The following diagram is commutative.*

$$\begin{tikzcd} \mathscr{L}_S \arrow[r, "\gamma_S"] \arrow[d, "\tau_0^*"'] & \mathscr{L}_{\mathrm{cor},S} \arrow[d, "\tau^*"] \\ \mathscr{L}_T \arrow[r, "\gamma_T"] & \mathscr{L}_{\mathrm{cor},T}. \end{tikzcd}$$

(3) *If $\sigma : T' \rightarrow T$ is a morphism of regular schemes, we have a canonical isomorphism of functors*

$$(\tau \circ \sigma)^* \simeq \sigma^* \circ \tau^*.$$

Proof. 1. Let $\alpha \in \mathrm{c}_S(X, Y)$, $\beta \in \mathrm{c}_S(X', Y')$. Then,

$$\begin{aligned} (\alpha \otimes^{tr} \beta)_T &= \tau^*_{XX'YY'}\left(p_{XY}^{XX'YY'*}(\alpha).p_{X'Y'}^{XX'YY'*}(\beta)\right) \\ &= \left(\tau^*_{XX'YY'} p_{XY}^{XX'YY'*}(\alpha)\right) . \left(\tau^*_{XX'YY'} p_{X'Y'}^{XX'YY'*}(\beta)\right) \\ &= \left(q_{XY}^{XX'YY'*}(\alpha_T)\right) . \left(q_{X'Y'}^{XX'YY'*}(\beta)\right) . \end{aligned}$$

2. This point follows from the fact that for any S-morphism $f : X \rightarrow Y$, there is a canonical isomorphism $\Gamma_{f_T} \rightarrow \Gamma_f \times_S T$.

3. Indeed there is a canonical isomorphism $X_{T'} \simeq (X_T)_{T'}$. Its naturality against finite correpondances follows from the functoriality of pullback on cycles. □

1.5.2. *Restriction.* Let $\tau : T \rightarrow S$ be a smooth morphism of regular schemes.

Let X, Y be smooth T-schemes. We denote by $\delta_{XY} : X \times_T Y \rightarrow X \times_S Y$ the canonical regular closed immersion deduced by base change from the diagonal immersion of T/S.

Let $\alpha \in \mathrm{c}_T(X, Y)$. We will consider the cycle $\delta_{XY*}(\alpha)$ as an element of $\mathrm{c}_S(X, Y)$ using definition 1.10.

Lemma 1.31. *Let X, Y and Z be smooth T-schemes. The following relations are true :*

(1) *For all T-morphism $f : X \rightarrow Y$, $\delta_{XY*}\big([\Gamma_f]_T\big) = [\Gamma_f]_S$.*

(2) *For all $\alpha \in \mathrm{c}_T(X, Y)$ and $\beta \in \mathrm{c}_T(Y, Z)$,*

$$\delta_{XZ*}(\beta \circ \alpha) = (\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha)).$$

Proof. In this proof, we stop mentionning the extensions of schemes involved to avoid being too tedious.

The first assertion is obvious.

For the second assertion, we start by introducing the following notations :

$$\begin{tikzcd} & X \times_T Y \arrow[dl, equal] & & X \times_T Y \times_T Z \arrow[ll, "q_{XY}^{XYZ}"'] \arrow[rr, "q_{YZ}^{XYZ}"] \arrow[dl, "a"] \arrow[dr, "b"'] \arrow[dd, "\delta_{XYZ}"] & & Y \times_T Z \arrow[dr, equal] & \\ X \times_T Y \arrow[dr, "\delta_{XY}"'] & & X \times_T Y \times_S Z \arrow[ll, "p"'] \arrow[dr, "c"] & & X \times_S Y \times_T Z \arrow[rr, "q"] \arrow[dl, "d"'] & & Y \times_T Z \arrow[dl, "\delta_{YZ}"] \\ & X \times_S Y & & X \times_S Y \times_S Z \arrow[ll, "p_{XY}^{XYZ}"] \arrow[rr, "p_{YZ}^{XYZ}"'] & & Y \times_S Z, & \end{tikzcd}$$

where every horizontal arrows are canonical projections and any other arrows are canonical closed immersions.

The equality is obtained along the following lines :

$$\begin{aligned}
\delta_{XZ*}\left(q^{XYZ}_{XZ}\right)_*\left(q^{XYZ*}_{YZ}(\beta).q^{XYZ*}_{XY}(\alpha)\right) &= p^{XYZ}_{XZ}{}_*\delta_{XYZ*}\left(q^{XYZ*}_{YZ}(\beta).q^{XYZ*}_{XY}(\alpha)\right) \\
&= p^{XYZ}_{XZ}{}_* d_* b_*\left(b^* q^*(\beta).a^* p^*(\alpha)\right) \\
&= p^{XYZ}_{XZ}{}_* d_*\left(q^*(\beta).b_* a^* p^*(\alpha)\right) \\
&= p^{XYZ}_{XZ}{}_* d_*\left(q^*(\beta).d^* c_* p^*(\alpha)\right) \\
&= p^{XYZ}_{XZ}{}_*\left(d_* q^*(\beta).c_* p^*(\alpha)\right) \\
&= p^{XYZ}_{XZ}{}_*\left(p^{XYZ*}_{YZ}\delta_{YZ*}(\beta).p^{XYZ*}_{XY}\delta_{XY*}(\alpha)\right)
\end{aligned}$$

using the functoriality of pullback and pushout and the projection formulas 1.13, 1.12. □

Definition 1.32. Let $\tau : T \to S$ be a smooth morphism of regular schemes.
Using the preceding lemma, we define a functor

$$\begin{array}{rcl}
\tau_\sharp : \mathscr{L}_{\mathrm{cor},T} & \to & \mathscr{L}_{\mathrm{cor},S} \\
X \to T & \mapsto & (X \to T \xrightarrow{\tau} S) \\
\mathrm{c}_T\,(X,Y) \ni \alpha & \mapsto & \delta_{XY*}(\alpha).
\end{array}$$

1.33. Note that from the first point of the preceding lemma, the restriction of τ to $\mathscr{L}_T$ is the classical functor forgetting the base.

Moreover, for a sequence of smooth morphisms $R \xrightarrow{\sigma} T \xrightarrow{\tau} S$ between regular schemes, we evidently have

$$(\tau \circ \sigma)_\sharp = \tau_\sharp \circ \sigma_\sharp \ .$$

1.5.3. *Properties.*

Proposition 1.34. *Let $\tau : T \to S$ be a smooth finite type morphism of regular schemes.*

1. *The functor $\tau_\sharp$ is left adjoint to the functor τ^*.*
2. *For every smooth algebraic T-scheme X (resp. S-scheme Y), the obvious morphism obtained by adjunction*
$$\tau_\sharp(\tau^* X \otimes_T Y) \to X \otimes_S \tau_\sharp Y$$
is an isomorphism.

Proof. For the first assertion, we only remark that for a smooth T-scheme X (resp. S-scheme Y), $(\tau_\sharp X) \times_S Y \simeq X \times_T (\tau^* Y)$.

The second assertion is clear for the case of τ^* and $\tau_\sharp$ (for morphisms of schemes) and we only have to apply the second point of 1.30 and the first point of 1.33. □

2. Sheaves with transfers

In this section, S with no further precisions will be a regular scheme.

2.1. **Nisnevich topology.** We will consider the *Nisnevich topology* on the site $\mathscr{L}_S$. Recall that a cover for the Nisnevich topology is a family of étale maps $p_i : Y_i \to X$ such that for any $x \in X$, there exists $y_i \in Y_i$ satisfying $p_i(y_i) = x$ and the induced map between the residue fields $\kappa(x) \to \kappa(y_i)$ is an isomorphism.

Among such covers, there is the family of covers induced by the *distinguished squares* of [MV99] which are cartesian squares

$$\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array}$$

such that j is an open immersion, p is an étale morphism and the induced morphism $p^{-1}(X-U)_{red} \to (X-U)_{red}$ is an isomorphism. The family (p, j) is indeed a Nisnevich cover. Moreover, recall from [MV99], prop. 1.4 that a presheaf F on $\mathscr{L}_S$ is a Nisnevich sheaf if and only if for any disinguished square as above, the square

$$\begin{array}{ccc} F(X) & \xrightarrow{p^*} & F(V) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ F(U) & \xrightarrow{-k^*} & F(W) \end{array}$$

is cartesian.

Let X be a smooth S-scheme and x a point of X. A *Nisnevich neighbourhood* of x in X is a couple (V, y) where V is an étale X-scheme and y a point of V over x such that the induced morphism $\kappa(x) \to \kappa(y)$ is an isomorphism. We let $\mathcal{V}_x^h(X)$ be the category of Nisnevich neighbourhood of x in X with arrows the morphisms of pointed scheme. This category is non empty, essentially small and left filtering. We define the h-localisation of X in x as the pro-scheme $X_x^h = \tilde{\varprojlim\limits_{V \in \mathcal{V}_x^h(X)}} V$.

Following the general notations of this article, we define the fiber of F at the point x of X as the abelian group $F(X_x^h) = \varinjlim\limits_{V \in \mathcal{V}_x^h(X)^{op}} F(V)$.

It is now classical to show the functor from Nisnevich sheaves to abelian groups $F \mapsto F(X_x^h)$ is exact and commutes with arbitrary sums. Moreover, the family of fiber functors induced by a pointed smooth S-scheme (X, x) is conservative for the category of Nisnevich sheaves over $\mathscr{L}_S$.

Remark 2.1. Let $\mathcal{O}_{X,x}^h$ be the henselisation of the local ring of X at x. Then $\operatorname{Spec}\left(\mathcal{O}_{X,x}^h\right)$ is the limit of the pro-object X_x^h.

Definition 2.2. We will denote by $\mathscr{P}_S$ (resp. $\mathscr{N}_S$) the category of presheaves (resp. sheaves for the Nisnevich topology) on $\mathscr{L}_S$.

2.2. **Definition and examples.** Recall the canonical map $\gamma : \mathscr{L}_S \to \mathscr{L}_{\mathrm{cor},S}$ of definition 1.19.

Definition 2.3. A presheaf with transfers F over S is an additive presheaf of abelian groups over $\mathscr{L}_{\mathrm{cor},S}$. We denote by $\mathscr{P}_S^{\mathrm{tr}}$ the corresponding category.

A sheaf with transfers over S is a presheaf with transfers F such that the functor $F \circ \gamma$ is a Nisnevich sheaf. We denote by $\mathscr{N}_S^{\mathrm{tr}}$ the full subcategory of $\mathscr{P}_S^{\mathrm{tr}}$ of sheaves with transfers.

Let X be a smooth S-scheme. We denote by $\mathrm{L}_S[X]$ the presheaf on $\mathscr{L}_{\mathrm{cor},S}$ represented by X.

Lemma 2.4. *Let X be a smooth S-scheme. The presheaf $\mathrm{L}_S[X]$ restricted to $\mathscr{L}_S$ via γ is an étale sheaf.*

Proof. Let Y be a smooth algebraic S-scheme. As Y is algebraic, it is sufficient to consider a surjective étale morphism $f : V \to Y$. We may assume Y is irreducible by additivity.

Let $W = V \times_X V$ and consider the canonical projections $p, q : W \to V$. Using the third property of lemma 1.18, we have to show the exactness of the sequence

$$0 \to \mathrm{c}_0(Y \times_S X/Y) \xrightarrow{f_X^*} \mathrm{c}_0(V \times_S X/V) \xrightarrow{p_X^* - q_X^*} \mathrm{c}_0(W \times_S X/W).$$

As f_X is faithfully flat, the pullback f_X^* is injective.

Let $\alpha \in \mathrm{c}_0(V \times_S X/V)$ be a cycle such that $p_X^*(\alpha) = q_X^*(\alpha)$. Write α as a linear combination $\alpha = \sum_{i=1}^n \lambda_i.z_i$ with z_i a point of $V \times_S X$ with closure finite and surjective over Y. As p_X and q_X are étale, we get by hypothesis

$$\sum_{i=1}^n \lambda_i. \sum_{x \in p_X^{-1}(z_i)} x = \sum_{j=1}^n \lambda_j. \sum_{y \in q_X^{-1}(z_j)} y.$$

Denote by I the set $\{f_X(z_1), ..., f_X(z_n)\}$. Note that for any $w \in I$ if i and j are integers such that $f(z_i) = w = f(z_j)$ then $\lambda_i = \lambda_j$. Indeed the equality above show the coefficient of $x = (z_i, z_j) \in W \times_S X$ is λ_i and λ_j. For $w \in I$, we put $\lambda(w) = \lambda_i$ for any i such that $w = f(z_i)$.

If we define $\beta = \sum_{w \in I} \lambda(w).w$ then $f_X^*(\beta) = \alpha$ as f_X is étale. Finally, lemma 1.9 shows β is an element of $\mathrm{c}_0(Y \times_S X/Y)$. □

2.3. **Associated sheaf with transfers.** Let $p : U \to X$ be a S-morphism of smooth S-schemes. We denote by U_X^n the n-fold product of U over X.

Consider the Čech simplicial scheme $\check{S}_*(U/X)$ associated to U/X with the convention $\check{S}_n(U/X) = U_X^{n+1}$. We will denote by $\check{C}_*(U/X)$ the associated chain complex considered in the additive category generated by the category of schemes.

Applying the additive functor $\mathrm{L}_S[.]$ to this complex, we get a complex of sheaves with transfers $\mathrm{L}_S\left[\check{C}_*(U/X)\right]$ naturally augmented over $\mathrm{L}_S[X]$.

The following proposition is an obvious generalisation of prop. 3.1.3 in [Voe00b].

Proposition 2.5. *Let X be a smooth S-scheme and $p : U \to X$ be a Nisnevich cover.*

The natural augmentation morphism $\mathrm{L}_S\left[\check{C}_*(U/X)\right] \to \mathrm{L}_S[X]$ is a quasi-isomorphism in the category of Nisnevich sheaves over S.

Proof. We have only to check the assertion on the conservative family of points introduced in section 2.1. Let (Y, y) be a pointed smooth S-scheme. We consider $\mathcal{O}_{Y,y}^h$ the henselian local ring of Y at the point y and put $\mathcal{Y} = \mathrm{Spec}\left(\mathcal{O}_{Y,y}^h\right)$. Using proposition 1.24 the canonical morphism

$$\mathrm{c}_S\left(Y_y^h, X\right) \to \bar{\mathrm{c}}_S(\mathcal{Y}, X)$$

starting from the fiber of $\mathrm{L}_S[X]$ at (Y, y) (cf section 2.1) is an isomorphism.

Thus we have to show the exactness of the sequence

$$C_* = ... \xrightarrow{d_{n*}} \bar{\mathrm{c}}_S\left(\mathcal{Y}, U_X^{n+1}\right) \to ... \xrightarrow{d_{0*}} \bar{\mathrm{c}}_S(\mathcal{Y}, U) \xrightarrow{p_*} \bar{\mathrm{c}}_S(\mathcal{Y}, X) \to 0.$$

Let $\mathcal{F}$ be the set, ordered by inclusion, of reduced closed the subschemes of $\mathcal{Y} \times_S X$ which are finite equidimensional on $\mathcal{Y}$. Consider $Z \in \mathcal{F}$ and put

$$C_n^{(Z)} = \bar{\mathrm{c}}_S\left(\mathcal{Y}, Z \times_X U_X^{n+1}\right) \subset \bar{\mathrm{c}}_S\left(\mathcal{Y}, U_X^{n+1}\right).$$

Then $C_*^{(Z)}$ is a subcomplexe of C^*. Moreover the complex $C_*^{(Z)}$ is increasing with respect to Z and we have $C_* = \bigcup_{Z \in \mathcal{F}} C_*^{(Z)}$. Thus it is sufficient to prove the complex $C_*^{(Z)}$ is contractile.

Following the hypothesis, Z is finite over $\mathcal{Y}$. As $\mathcal{Y}$ is henselian, Z is a direct sum of local henselian schemes. Following [Ray70], the Nisnevich cover $p_Z : Z \times_X U \to Z$ admits a section $s : Z \to Z \times_X U$.

It is now a classical fact that the augmented Čech complex $\check{C}_*(Z \times_X U/Z) \to Z$ is contractile in the additive category generated by the category of schemes. An explicit homtopy is given by the collection of morphism for $n \geq -1$

$$s_n = s \times_X 1_{U_X^{n+1}} : Z \times_X U_X^{n+1} \to Z \times_X U_X^{n+2}.$$

The result follows by application of the functor $\mathrm{c}_S(\mathcal{Y}, .)$. □

Following the idea of the proof of lemma 3.1.6 in [Voe00b], we obtain the next lemma.

Lemma 2.6. *Let F be a presheaf with transfers. Denote by $\check{H}^0 F$ the 0-th Čech Nisnevich cohomology presheaf on $\mathscr{L}_S$ associated with F and $\eta : F \to \check{H}^0 F$ the canonical morphism.*

Then there exists a unique couple $(\check{H}^0_{tr} F, \mu)$ such that

1. *$\check{H}^0_{tr} F$ is a sheaf with transfers satisfying $\check{H}^0_{tr} F \circ \gamma = \check{H}^0 F$.*
2. *μ is natural transformation $F \to \check{H}^0_{tr} F$ of presheaves with transfers which coincide with η.*

Proof. As F is a presheaf with transfers, we have a canonical inclusion

$$F(X) \simeq \mathrm{Hom}_{\mathscr{P}_S^{\mathrm{tr}}}(\mathrm{L}_S[X], F) \subset \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[X], F).$$

Reciprocally, a natural transformation of presheaf on $\mathscr{L}_S$

$$F \xrightarrow{\phi} \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[.], F)$$

is equivalent to a structure of presheaf with transfers on F as soon as it respects the composition product (in which case, it is a monomorphism). The two structures are in one-to-one correspondance using the equation

$$\text{(B)} \qquad \forall \alpha \in \mathrm{c}_S(Y, X),\ a \in F(X),\ F(\alpha).a = \phi_X(a)_Y.\alpha$$

We worh rather with the natural tranformation ϕ than with the structure of a presheaf with transfers.

1) Let suppose $\check{H}^0_{tr} F$ is defined.
Consider $\alpha \in \mathrm{c}_S(Y, X)$ and $a \in F_{\mathrm{Nis}}(X)$. Recall that

$$\check{H}^0 F(X) = \varinjlim_{U \to X} \mathrm{Ker}\big(F(U) \to F(U \times_X U)\big).$$

As the colimit is filering, there exists a Nisnevich cover U of X such that a can be lifted to an element $a_U \in F(U)$. Applying lemma 2.5, $\mathrm{L}_S[U] \to \mathrm{L}_S[X]$ is an epimorphism. Thus there exists a Nisnevich cover V of Y and a correspondance $\alpha_U \in \mathrm{c}_S(V, U)$ such that $p \circ \alpha_U = \alpha|_V$.

We thus have obtained the following commutative diagram (the commutativity of the bottom square is the first condition appearing in the statement of the theorem)

$$\begin{tikzcd}
\check{H}^0 F(X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[X], \check{H}^0 F\big) \arrow[d] \\
\check{H}^0 F(U) \arrow[r] & \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[U], \check{H}^0 F\big) \\
F(U) \arrow[r] \arrow[u, "\eta"] & \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[U], F). \arrow[u]
\end{tikzcd}$$

This in turn can be translated into the following local equation which caracterizes $\check{H}^0_{tr} F$

$$\check{H}^0_{tr} F(\alpha|_V).a = \eta_Y(F(\alpha_U).a_U) \in F(V).$$

2) Reciprocally, we have to prove that the above equality does not depend on the choice of the cover U because it then defines $\check{H}^0_{tr} F$.

Let U be a Nisnevich cover of X and $H = \mathrm{Hom}_{\mathscr{P}_S}\big(.,\check{H}^0F\big)$. As H is left exact, lemma 2.5 implies the following sequence is exact

$$0 \to H(\mathrm{L}_S[X]) \to H(\mathrm{L}_S[U]) \to H(\mathrm{L}_S[U \times_X U]).$$

Let us consider the arrow

$$F(X) \hookrightarrow \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[X], F) \xrightarrow{\eta_X} \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[X], \check{H}^0F\big) = H(\mathrm{L}_S[X]).$$

As it is natural, it induces the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H(\mathrm{L}_S[X]) \arrow[r] & H(\mathrm{L}_S[U]) \arrow[r] & H(\mathrm{L}_S[U \times_X U]) \\
0 \arrow[r] & Ker_U \arrow[r] \arrow[u, dashed] & F(U) \arrow[r] \arrow[u] & F(U \times_X U), \arrow[u]
\end{tikzcd}$$

which is natural in U. Taking the limit over all Nisnevich cover U of X, we obtain the arrow

$$\phi_X^{\mathrm{Nis}} : \check{H}^0F(X) = \varinjlim_{U \to X} \big(Ker_U\big) \longrightarrow H(\mathrm{L}_S[X]) = \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[X], \check{H}^0F\big)$$

which in turn defines transfers on $\check{H}^0F$ according to equation (B) :

$$\check{H}^0_{tr}F(\alpha) : F(X) \to F(Y),\ a \mapsto \phi_X^{\mathrm{Nis}}(a)_Y.\alpha.$$

Consider $a \in \check{H}^0F(X)$ and $\alpha \in \mathrm{c}_S(Y, X)$. As in the first step of the proof, choose a cover U (resp. V) of X (resp. Y) and liftings $a_U \in F(U)$, $\alpha_U \in \mathrm{c}_S(V, U)$. Then tautologically,

$$\check{H}^0_{tr}F(\alpha|_V).a = \eta_Y(F(\alpha_U).a_U).$$

We deduce now easily from this local equation the compatibility of ϕ^{Nis} with the product of correspondances. By the very construction, ϕ^{Nis} extends ϕ. $\square$

The unicity statement in the preceding lemma implies the naturality with respect to F of the transformation $F \to \check{H}^0_{tr}F$ and the following corollary.

Corollary 2.7. *With the notation of the previous lemma, the association*

$$a_{tr} : \mathscr{P}_S^{\mathrm{tr}} \to \mathscr{N}_S^{\mathrm{tr}}, F \mapsto \check{H}^0_{tr}\check{H}^0_{tr}F$$

defines a functor left adjoint to the forgetful functor $\mathscr{N}_S^{\mathrm{tr}} \hookrightarrow \mathscr{P}_S^{\mathrm{tr}}$. Moreover, the following diagram commutes

$$\begin{tikzcd}
\mathscr{P}_S^{\mathrm{tr}} \arrow[r, "a_{tr}"] \arrow[d] & \mathscr{N}_S^{\mathrm{tr}} \arrow[d] \\
\mathscr{P}_S \arrow[r, "a_{\mathrm{Nis}}"] & \mathscr{N}_S.
\end{tikzcd}$$

Recall that a Grothendieck abelian category is an abelian category which admits arbitrary direct sums, has a set of generators (any object is a quotient of a direct sums of the generators) and such that filtering inductive limits are exact.

Proposition 2.8. *The category $\mathscr{N}_S^{\mathrm{tr}}$ is an abelian Grothendieck category. It is complete (i.e. admits arbitrary small projective limits).*

The forgetfull functor $\mathscr{N}_S^{\mathrm{tr}} \to \mathscr{N}_S$ admits a left adjoint $\mathrm{L}_S[.]$.

An essentially small family of generators for $\mathscr{N}_S^{\mathrm{tr}}$ is given by the sheaves $\mathrm{L}_S[X]$ for a smooth S-scheme X.

Proof. The existence of the right exact functor a_{tr} implies that $\mathscr{N}_S^{\mathrm{tr}}$ is cocomplete (as the category $\mathscr{P}_S^{\mathrm{tr}}$). As in the traditional case, an inductive limit of sheaves with transfers is constructed by first computing it in the category of presheaves with transfers then taking the associated sheaf with transfers. This description makes clear that $\mathscr{N}_S^{\mathrm{tr}}$ is a Grothendieck abelian category.

Consider a Nisnevich sheaf F. Then, classically $F = \varinjlim_{X/F \in \mathscr{L}_S/F} \mathbb{Z}_S(X)$, where $\mathbb{Z}_S(X)$ is the free abelian sheaf represented by a smooth S-scheme X and the limit is taken over every morphism $\mathbb{Z}_S(X) \to F$ for such an X.

We simply put $\mathrm{L}_S[F] = \varinjlim_{X/F \in \mathscr{L}_S/F} \mathrm{L}_S[X]$, the inductive limit being calculated in the category of sheaves with transfers.

The construction of the functor $\mathrm{L}_S[.]$ makes clear the last assertion. □

Let X be an smooth S-scheme. The graph morphism induces a morphism of sheaves $\eta_X : \mathbb{Z}_S(X) \to \mathcal{O}_{tr}\mathrm{L}_S[X]$. Using the description of the Ext groups for sheaves and sheaves with transfers has class of extensions we deduce a canonical morphism, natural in X and the sheaf with transfers F

$$\eta_X^i : \mathrm{Ext}^i_{\mathscr{N}_S^{\mathrm{tr}}}(\mathrm{L}_S[X], F) \to \mathrm{Ext}^i_{\mathscr{N}_S}(\mathbb{Z}_S(X), \mathcal{O}_{tr}F) = H^i(X; \mathcal{O}_{tr}F).$$

Proposition 2.9. *Using the notation introduced above, for every smooth S-scheme, every sheaf with transfers F over S, and every integer $i \in \mathbb{N}$, the morphism η_X^i is an isomorphism.*

Proof. Using the Yoneda lemma, the property is evident for $i = 0$. Consider the case $i > 0$.

The category $\mathscr{N}_S^{\mathrm{tr}}$, being a Grothendieck abelian category, has enough injective. In particular, the Ext groups with coefficients in F are calculated by choosing an injective resolution of F in $\mathscr{N}_S^{\mathrm{tr}}$. Consequently, we are reduced to prove that for any sheaf with transfers I which is injective in the category $\mathscr{N}_S^{\mathrm{tr}}$, the sheaf $\mathcal{O}_{tr}I$ is acyclic.

Following [Mil80], prop. III.2.11, this property is again equivalent to the vanishing of all the positive Čech cohomology groups $\check{H}^i(X; \mathcal{O}_{tr}I)$. But this now follows from proposition 2.5. □

Corollary 2.10. *Let F be a presheaf with transfers. Then for all integer $i \in \mathbb{N}$, the presheaf $H^i_{\mathrm{Nis}}(., F_{\mathrm{Nis}})$ has a canonical structure of a presheaf with transfers.*

2.4. **Closed monoidal structure.** Recall we have defined in proposition 1.23 a monoidal structure on $\mathscr{L}_{\mathrm{cor},S}$.

Lemma 2.11. *The category $\mathscr{N}_S^{\mathrm{tr}}$ admits a unique structure of a symetric monoidal category with a right exact tensor product and such that the graph functor $\mathscr{L}_{\mathrm{cor},S} \to \mathscr{N}_S^{\mathrm{tr}}$ is monoidal.*

Proof. Let F and G be sheaves with transfers. Using proposition 2.8, we can write

$$F = \varinjlim_{X/F \in \mathscr{L}_S/F} \mathrm{L}_S[X], \quad G = \varinjlim_{Y/F \in \mathscr{L}_S/G} \mathrm{L}_S[Y].$$

Necessarily, the tensor product of sheaves with transfers must satisfy

$$F \otimes_S^{tr} G = \varinjlim_{X/F, Y/G} (\mathrm{L}_S[X] \otimes_S^{tr} \mathrm{L}_S[Y]).$$

The axioms of a symetric monoidal category then follows from the corresponding property of the category $\mathscr{L}_{\mathrm{cor},S}$ and the unicity is established as well. □

Definition 2.12. We denote by $\otimes_S^{tr}$ the tensor product on $\mathscr{N}_S^{\mathrm{tr}}$ satisfying the conditions of the previous lemma.

Remark 2.13. We can express the difference between the tensor product with transfers and the usual tensor product of abelian sheaves. Indeed, for any sheaf with

transfers F we have an epimorphism of sheaves with transfers

$$\begin{array}{rcl} \bigoplus_{X \in \mathscr{L}_S} F(X) \otimes_{\mathbb{Z}} \mathrm{L}_S\left[X\right] & \to & F \\ \bigoplus_{X \in \mathscr{L}_S} F(X) \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y, X\right) & \to & F(Y) \\ \rho \otimes \alpha & \mapsto & \rho \circ \alpha, \end{array}$$

where we view ρ (resp. α) as a map $\mathrm{L}_S\left[X\right] \to F$ (resp. $\mathrm{L}_S\left[Y\right] \to \mathrm{L}_S\left[X\right]$).

Thus, as $\otimes_S^{tr}$ is right exact, we deduce an epimorphism of sheaves

$$\begin{array}{rcl} \bigoplus_{X, X' \in \mathscr{L}_S} (F(X) \otimes_{\mathbb{Z}} \mathrm{L}_S\left[X\right]) \otimes (G(X') \otimes_{\mathbb{Z}} \mathrm{L}_S\left[X'\right]) & \to & F \otimes_S^{tr} G \\ \bigoplus_{X, X' \in \mathscr{L}_S} F(X) \otimes_{\mathbb{Z}} G(X') \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y, X \times X'\right) & \to & (F \otimes_S^{tr} G)(Y) \\ \rho \otimes \mu \otimes \alpha & \mapsto & (\rho \circ \alpha) \otimes_S^{tr} (\mu \circ \alpha). \end{array}$$

In particular, for any pointed scheme (Y, y), we have on the level of the fiber at Y_y^h (cf section 2.1) an epimorphism of abelian groups

$$\begin{array}{rcl} \bigoplus_{X, X' \in \mathscr{L}_S} F(X) \otimes_{\mathbb{Z}} G(X') \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y_y^h, X \times X'\right) & \to & (F \otimes_S^{tr} G)(Y_y^h) \\ \rho \otimes \mu \otimes \bar{\alpha} & \mapsto & (\rho \circ \bar{\alpha}) \otimes_S^{tr} (\mu \circ \bar{\alpha}). \end{array}$$

Proposition 2.14. *The monoidal category $\mathscr{N}_S^{\mathrm{tr}}$ is closed : the bifunctor $\otimes_S^{tr}$ admits a right adjoint $\underline{\mathrm{Hom}}_{\mathscr{N}_S^{\mathrm{tr}}}(.,.)$.*

Proof. Let F and G be sheaves with transfers. We put

$$\underline{\mathrm{Hom}}_{\mathscr{N}_S^{\mathrm{tr}}}\left(F, G\right)\left(X\right) = \mathrm{Hom}_{\mathscr{N}_S^{\mathrm{tr}}}\left(F \otimes_S^{tr} \mathrm{L}_S\left[X\right], G\right).$$

As a sheaf with transfers is an inductive limit of representable presheaves with transfers (cf prop. 2.8), one obtains easily the expected adjoint property. □

2.5. **Functoriality.**

We fix a morphism $\tau : T \to S$ of regular schemes.

2.5.1. *The abstract case.* Consider an abstract additive functor $\varphi : \mathscr{L}_{\mathrm{cor},S} \to \mathscr{L}_{\mathrm{cor},T}$ which sends a Nisnevich cover of an S-scheme to a Nisnevich cover of a T-scheme.

In this situation, we will define the following two functors :

(1) If F is a sheaf with transfers over S, we define over T the sheaf with transfers $\varphi(F) = \varinjlim_{X/F} \mathrm{L}_T\left[\varphi(X)\right]$.
(2) If G is a sheaf with transfers over T, we define over S the sheaf with transfers $\varphi'(G) = G \circ \varphi$.

Note there is an abuse of notation in (1). This is justified by the fact that the functor φ on sheaves with transfers is an extension of the functor φ on schemes via the associated represented sheaf with transfers functor.

The Yoneda lemma implies immediately that φ' is right adjoint to φ.

Similarly, the same construction applies to the graph functor $\gamma_S : \mathscr{L}_S \to \mathscr{L}_{\mathrm{cor},S}$. Indeed this functor respects tautologically the Nisnevich coverings and we obtain an extension on sheaves $\gamma_S : \mathscr{N}_S \to \mathscr{N}_S^{\mathrm{tr}}$ and a right adjoint which is the forgetfull functor $\mathscr{O}_S^{tr} : \mathscr{N}_S^{\mathrm{tr}} \to \mathscr{N}_S$.

Getting back to the hypothesis of the begining, we suppose given in addition the commutative diagram of functors

$$\begin{array}{ccc} \mathscr{L}_S & \xrightarrow{\gamma_S} & \mathscr{L}_{\mathrm{cor},S} \\ {\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathscr{L}_T & \xrightarrow{\gamma_T} & \mathscr{L}_{\mathrm{cor},T}. \end{array}$$

By hypothesis, φ_0 respects Nisnevich coverings and the same process gives a pair of adjoint functors

$$\varphi_0 : \mathscr{N}_S \to \mathscr{N}_T, \quad \varphi'_0 : \mathscr{N}_T \to \mathscr{N}_S.$$

It is now obvious that these functors are related by the commutative diagrams

$$\begin{tikzcd} \mathscr{N}_S \arrow[r, "\gamma_S"] \arrow[d, "\varphi_0"'] & \mathscr{N}_S^{\mathrm{tr}} \arrow[d, "\varphi"] \\ \mathscr{N}_T \arrow[r, "\gamma_T"] & \mathscr{N}_T^{\mathrm{tr}}, \end{tikzcd} \qquad \begin{tikzcd} \mathscr{N}_S & \mathscr{N}_S^{\mathrm{tr}} \arrow[l, "\mathcal{O}_S"'] \\ \mathscr{N}_T \arrow[u, "\varphi'_0"] & \mathscr{N}_T^{\mathrm{tr}}. \arrow[l, "\mathcal{O}_T"'] \arrow[u, "\varphi'"'] \end{tikzcd}$$

Finally, suppose that φ is monoidal. Then the extension $\varphi : \mathscr{N}_S^{\mathrm{tr}} \to \mathscr{N}_T^{\mathrm{tr}}$ is again monoidal. In addition, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}_{\mathscr{N}_S^{\mathrm{tr}}}\left(F, \varphi'(G)\right) \simeq \underline{\mathrm{Hom}}_{\mathscr{N}_T^{\mathrm{tr}}}\left(\varphi(F), G\right).$$

The same remark applies to the pair of adjoint functors $(\gamma_S, \mathcal{O}_S^{tr})$.

2.5.2. *Base change.* We first apply the abstract case above to the monoidal functor

$$\tau^* : \mathscr{L}_{\mathrm{cor},S} \to \mathscr{L}_{\mathrm{cor},T}$$

defined in 1.29.

This yields the base change functor $\tau^* : \mathscr{N}_S^{\mathrm{tr}} \to \mathscr{N}_T^{\mathrm{tr}}$ and its right adjoint $\tau_* = (\tau^*)' : \mathscr{N}_T^{\mathrm{tr}} \to \mathscr{N}_S^{\mathrm{tr}}$. The first one is monoidal and the second one coincide with the usual pushout for sheaves without transfers.

2.5.3. *Exceptional direct image.* Suppose now the morphism $\tau : T \to S$ is smooth of. We apply the abstract construction to the monoidal functor

$$\tau_\sharp : \mathscr{L}_{\mathrm{cor},T} \to \mathscr{L}_{\mathrm{cor},S}$$

defined in 1.32.

This yields the twisted exceptional direct image functor $\tau_\sharp : \mathscr{N}_T^{\mathrm{tr}} \to \mathscr{N}_S^{\mathrm{tr}}$ which is monoidal.

Remark 2.15. When τ is étale, this functor is really the usual exceptional direct image $\tau_!$. Otherwise we need to twist this functor in order to get the fundamental equality $\tau_! = \tau_*$ when τ is smooth projective.

Lemma 2.16. *If $\tau : T \to S$ is smooth, there exists a canonical isomorphism of functors $\tau^* \simeq (\tau_\sharp)'$.*

Proof. Let F be a sheaf with transfers over S. Following definitions, τ^*F is the sheaf associated with the presheaf with transfers over T

$$Y \mapsto \varinjlim_{X/F} \mathrm{c}_T\left(Y, X \times_S T\right).$$

The canonical isomorphism $Y \times_S X \to Y \times_T (X \times_S T)$ induces an isomorphism $\mathrm{c}_S\left(\tau_\sharp Y, X\right) \to \mathrm{c}_T\left(Y, X \times_S T\right)$. The definition of composition product and base change for finite correspondances shows this isomorphism is natural in X and Y with respect to finite correspondances (the projections involved in the two ways of computing products in the above isomorphic groups coincide).

As $F = \varinjlim_{X/F} \mathrm{L}_S[X]$ in the category of sheaves with transfers, the result follows from the computation of inductive limits in the category of sheaves with transfers over S (cf proof of 2.8). $\square$

In particular, when τ is smooth, τ^* is right adjoint to $\tau_\sharp$. Thus τ^* is exact (and commutes with every inductive and projective limits). Moreover, τ^* coincide with the usual base change functor on sheaves without transfers.

We now set up the projection formula.

Let F (resp. G) be a sheaf with transfers over S (resp. T). We consider the adjunction morphism following from the previous lemma

$$G \to \tau^* \tau_\sharp G.$$

Applying the functor $(\tau^* F) \otimes_S^{tr} (.)$ to this morphism we get

$$(\tau^* F) \otimes_S^{tr} G \to (\tau^* F) \otimes_S^{tr} (\tau^* \tau_\sharp G).$$

Using the monoidal property of τ^* and adjunction we get a morphism

$$\phi : \tau_\sharp((\tau^* F) \otimes_S^{tr} G) \to F \otimes_S^{tr} (\tau_\sharp G).$$

Lemma 2.17. *Under the above hypothesis and notation, the morphism ϕ is an isomorphism.*

Proof. The morphism ϕ is natural in F and G. As every functors involved commute with inductive limits, it is sufficient to check the isomorphism on representable sheaves $F = \mathrm{L}_S[X]$, $G = \mathrm{L}_T[Y]$. Then the morphism is reduced to the canonical isomorphism $(X \times_S T) \times_T Y \to X \times_S Y$ of S-schemes. □

2.5.4. *Pro-smooth morphisms.* Let $(T_i)_{i \in I}$ be a pro-object of smooth affine S-schemes. As in subsection 1.1.3, we write $T_i = \mathrm{Spec}_S(\mathcal{A}_i)$ and put $\mathcal{A} = \varinjlim_{i \in I^{op}} \mathcal{A}_i$. The scheme $\mathcal{T} = \mathrm{Spec}_S(\mathcal{A})$ is the projective limit of $(T_i)_{i\in I}$ in the category of affine S-schemes. We suppose it is regular noetherian.

We denote by $\tau : \mathcal{T} \to S$ the canonical morphism[4].

First, we note that the functoriality constructed above for sheaves with transfers can equally be constructed for presheaves with transfers. In particular, based on the functor $\tau^* : \mathscr{L}_{\mathrm{cor},S} \to \mathscr{L}_{\mathrm{cor},\mathcal{T}}$, we obtain the base change functor $\hat{\tau}^* : \mathscr{P}_S^{\mathrm{tr}} \to \mathscr{P}_{\mathcal{T}}^{\mathrm{tr}}$ and its right adjoint $\hat{\tau}_* : \mathscr{P}_{\mathcal{T}}^{\mathrm{tr}} \to \mathscr{P}_S^{\mathrm{tr}}$.

In fact, when F is a sheaf with transfers over S, we have $\tau^* F = a_{tr}(\hat{\tau}^* F)$ using the associated sheaf with transfers of corollary 2.7. For a sheaf with transfers G over $\mathcal{T}$ we have more simply $\tau_* G = \hat{\tau}_* G$.

Secondly, given a smooth scheme $\mathcal{X}$ over $\mathcal{T}$, as it is in particular of finite presentation, there exists $i \in I$ such that $\mathcal{X}/\mathcal{T}$ can be descended to a finite presentation scheme X_i/T_i. That is, $\mathcal{X} = X_i \times_{T_i} \mathcal{T}$. For any $j \to i$, we put $X_j = X_i \times_{T_i} T_j$. Using now [GD66], 17.7.8, by enlarging i, we can assume X_i/T_i is smooth. We finally have

$$\mathcal{X} = \varprojlim_{j \in I/i} X_j$$

where any X_j is smooth over S.

Proposition 2.18. *Suppose we are in the hypothesis described above.*

Then for any presheaf with transfers F over S, we have a canonical isomorphism

$$\hat{\tau}^* F(\mathcal{X}) \simeq \varinjlim_{j \in I/i^{op}} F(X_j).$$

[4]In general, τ is not necessarily formally smooth but only regular, that is the fibers of τ are geometrically regular.

Proof. According to the definition,

$$\hat{\tau}^* F(\mathcal{X}) = \varinjlim_{U/F} \mathrm{c}_{\mathcal{T}}\left(\mathcal{X}, U \times_S \mathcal{T}\right)$$

where the limits run over all morphisms $\mathrm{L}_S[U] \to F$ of sheaves with transfers for a smooth S-scheme U.

Note that using the notation introduced in 1.1.3, we have a canonical isomorphism $\mathrm{c}_{\mathcal{T}}\left(\mathcal{X}, U \times_S T\right) \simeq \bar{\mathrm{c}}_S\left(\mathcal{X}, U\right)$. Proposition 1.24 now implies

$$\bar{\mathrm{c}}_S\left(\mathcal{X}, U\right) = \bar{\mathrm{c}}_S\left(\varprojlim_{j \in I/i} X_j, U\right) \simeq \varinjlim_{j \in I/i^{op}} \mathrm{c}_S\left(X_j, U\right),$$

the isomorphism being functorial in U with respect to finite S-correspondances from subsection 1.26. Finally, we can conclude as we have

$$\varinjlim_{U/F} \varinjlim_{j \in I/i^{op}} \mathrm{c}_S\left(X_j, U\right) = \varinjlim_{j \in I/i^{op}} \varinjlim_{U/F} \mathrm{c}_S\left(X_j, U\right) = \varinjlim_{j \in I/i^{op}} F(X_j).$$

□

Suppose now we are given a $\mathcal{T}$-morphism $\mathfrak{f} : \mathcal{X}' \to \mathcal{X}$. This morphism is of finite presentation and thus there exists $i \in I$ such that $\mathfrak{f}$ can be descended to T_i. That is there exists X_i/T_i and X_i'/T_i schemes of finite presentation, a T_i-morphism $f_i : X_i' \to X_i$ such that $\mathfrak{f} = f_i \times_{T_i} \mathcal{T}$.

We put $f_j = f_i \times_{T_i} T_j$. Anew by using [GD66], 17.7.8, we can assume X_i' and X_i to be smooth over T_i.

Then using subsection 1.27, the isomorphism of the preceding proposition is functorial with respect to $(f_j)_{j \in I/i}$. As a consequence, we obtain the following proposition.

Proposition 2.19. *Suppose we are in the hypothesis preceding proposition 2.18.*
Then for any sheaf with transfers F over S, $\hat{\tau}^* F$ is a sheaf with transfers.
In particular, $\tau^* F(\mathcal{X}) \simeq \varinjlim_{j \in I/i^{op}} F(X_j)$.

Indeed, using the caracterisation of a Nisnevich sheaf from 2.1, this is a consequence of the following lemma and the fact that filtered inductive limits are exact.

Lemma 2.20. *Consider a distinguish square of smooth $\mathcal{T}$-schemes*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\mathfrak{v}"] \arrow[d, "\mathfrak{g}"'] \arrow[dr, phantom, "\Delta"] & \mathcal{V} \arrow[d, "\mathfrak{f}"] \\ \mathcal{U} \arrow[r, "\mathfrak{u}"'] & \mathcal{X}. \end{tikzcd}$$

Then there exists $i \in I$ and a distinguish square of smooth schemes over T_i

$$\begin{tikzcd} W_i \arrow[r, "v_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\Delta_i"] & V_i \arrow[d, "f_i"] \\ U_i \arrow[r, "u_i"'] & X_i. \end{tikzcd}$$

such that $\Delta = \Delta_i \times_{T_i} \mathcal{T}$.

Proof. We have already seen just before the above proposition that we can find $i \in I$ and a square of smooth T_i-schemes

$$\begin{tikzcd} W_i \arrow[r, "v_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\Delta_i"] & V_i \arrow[d, "f_i"] \\ U_i \arrow[r, "u_i"'] & X_i. \end{tikzcd}$$

For any $j \to i$, we put $\Delta_j = \Delta_i \times_{T_i} T_j$, $Z_j = (X_j - U_j)_{red}$ and $T_j = (V_j \times_{T_j} Z_j)_{red}$. Then Δ is the projective limit of the Δ_j.

By finding a suitable $j \to i$, we can assume :

(1) this square is cartesian, that is the morphism $W_j \to U_j \times_{X_j} V_j$ is an isomorphism (cf [GD66], 8.10.5(i)),
(2) the morphism $T_j \to V_j$ induced by f_j is an isomorphism (cf *loc. cit.*),
(3) the morphism u_j is an open immersion (cf [GD66], 8.10.5(i)),
(4) the morphism f_j is étale (cf [GD66], 17.7.8(ii)).

□

In the course of section 4, we will particularly need the following reinforcement of the preceding proposition, relying on the same lemma :

Lemma 2.21. *Suppose we are in the hypothesis preceding proposition 2.18.*

We denote by $\check{H}^0_{tr}$ the functor constructed in lemma 2.6 either for presheaves with transfers over S or over $\mathcal{T}$.

Then we have a canonical isomorphism of functors $\mathscr{P}^{\mathrm{tr}}_S \to \mathscr{P}^{\mathrm{tr}}_{\mathcal{T}}$:

$$\hat{\tau}^* \check{H}^0_{tr} \simeq \check{H}^0_{tr} \hat{\tau}^*.$$

Proof. Let F be a presheaf with transfers over S and $\mathcal{X}$ be a smooth S-scheme. Fix $i \in I$ and a smooth T_i-scheme X_i such that $\mathcal{X} = X_i \times_{T_i} \mathcal{T}$. We put $X_j = X_i \times_{T_i} T_j$.

For any (noetherian) scheme A, we let $\mathcal{D}_A$ be the subcategory of A-schemes W such that there exists a distinguished square
$$\begin{array}{ccc} U \times_X V & \rightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & A \end{array}$$
such that $W = U \sqcup V$ as an A-scheme. This category is left filtering as any Nisnevich covering admits a refinement of this form.

Then

$$\hat{\tau}^* \check{H}^0_{tr} F(\mathcal{X}) = \varinjlim_{j \in I/i^{op}} \varinjlim_{W \in \mathcal{D}^{op}_{X_j}} \operatorname{Ker}\big(F(W) \to F(W \times_{X_j} W)\big).$$

Moreover the preceding lemma says precisely that the inclusion functor

$$\bigsqcup_{j \in I/i} \mathcal{D}_{X_j} \to \mathcal{D}_{\mathcal{X}}, W_j \mapsto W_j \times_{X_j} \mathcal{X}$$

is surjective, thus final.

This implies that

$$\begin{aligned} \check{H}^0_{tr} \hat{\tau}^* F(\mathcal{X}) &= \varinjlim_{j \in I/i^{op}} \varinjlim_{W_j \in \mathcal{D}_{X_j}} \operatorname{Ker}\big(\hat{\tau}^* F(W_j \times_{X_j} \mathcal{X}) \to \hat{\tau}^* F(W_j \times_{X_j} W_j \times_{X_j} \mathcal{X})\big) \\ &= \varinjlim_{j \in I/i^{op}} \varinjlim_{W_j \in \mathcal{D}_{X_j}} \varinjlim_{k \in I/j^{op}} \operatorname{Ker}\big(F(W_j \times_{X_j} X_k) \to F(W_j \times_{X_j} X_k)\big) \end{aligned}$$

where the second equality follows from proposition 2.18 and the exactness of filtering inductive limits. The lemma then follows. □

3. Homotopy equivalence for finite correspondances

3.1. **Definition.** Consider a regular scheme S.

Definition 3.1. Let X and Y be smooth S-schemes. Consider two correspondances $\alpha, \beta \in c_S(X, Y)$.

A homotopy from α to β is a corresponance $H \in c_S\left(\mathbb{A}^1 \times X, Y\right)$ such that

(1) $H \circ i_0 = \alpha$
(2) $H \circ i_1 = \beta$

where i_0 (resp. i_1) is the closed immersion $X \to \mathbb{A}^1_X$ corresponding to the point 0 (resp. the point 1) of $\mathbb{A}^1_X$.

The existence of a homotopy between two correspondances is obviously a reflexive and symmetric relation. However, transitivity fails. We thus adopt the following definition :

Definition 3.2. Let X and Y be smooth S-schemes and $\alpha, \beta \in \mathrm{c}_S(X, Y)$.

We say α is homotopical to β, denoted by $\alpha \sim_h \beta$ is there exists a sequence of correspondances $\gamma_0, ..., \gamma_n \in \mathrm{c}_S(X, Y)$ such that $\gamma_0 = \alpha$, $\gamma_n = \beta$ and for every integer $0 \leq i < n$, there exists a homotopy from γ_i to γ_{i+1}.

The relation $\sim_h$ is obviously additive and compatible with the composition law of finite correspondances.

Definition 3.3. For two smooth S-scheme X and Y, we denote by $\pi_S(X, Y)$ the quotient abelian group of $\mathrm{c}_S(X, Y)$ by the homotopy relation $\sim_h$.

We denote by $\pi\mathscr{L}_{\mathrm{cor},S}$ the category with objects smooth S-scheme and with morphisms the equivalence classes of finite S-correspondances for the relation $\sim_h$.

3.2. **Compactifications.** The purpose of this section is to give a tool (the good compactifications) which allows to compute the equivalence classes of finite correspondances for the homotopy relation.

3.2.1. *Définition.*

Definition 3.4. Let S be a regular scheme and X be an algebraic S-curve.

(1) A compactification of X/S is a proper normal curve $\bar{X}/S$ containing X as an open subscheme.
(2) Let $\bar{X}/S$ be a compactification of X/S. Put $X_\infty = \bar{X} - X$ seen as a reduced closed subscheme of $\bar{X}$. We say the compactification $\bar{X}/S$ of X/S is *good* if X_∞ is contained in an open subscheme of $\bar{X}$ which is affine over S.

When considering a given compactification $\bar{X}/S$ of a curve X/S, we will always put $X_\infty = \bar{X} - X$.

Remark 3.5. If $\bar{X}/S$ is a good compactification of X/S, X_∞ is finite over S as it is proper and affine over S. If S is irreducible, X_∞ is surjective over S and Chevalley's theorem (cf [GD61], II.6.7.1) implies S is affine.

Definition 3.6. We call closed pair any couple (X, Z) such that X is a scheme and Z is a closed subscheme of X.

A morphism of closed pair $(f, g) : (Y, T) \to (X, Z)$ is a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

which is cartesian on the corresponding topological spaces. The morphism is said to be cartesian if it is cartesian as a square of schemes.

Let (X, Z) be a closed pair such that X is an S-curve. A good compactification of (X, Z) over S is an S-scheme $\bar{X}$ which is a good compactification for both X/S and $(X - Z)/S$.

3.2.2. *The case of a base field.* We suppose here S is the spectrum of a field k.

Proposition 3.7. *Let C/k be a quasi-affine regular algebraic curve.*

There exists a projective regular curve $\bar{C}$ over k such that for all closed subscheme Z of C nowhere dense in C, $\bar{C}$ is a good compactification of (X, Z) over k.

Proof. We can restrict to the case C is affine and integral.

As C/k is algebraic, we can find a closed immersion $C \to \mathbb{A}^n_k$. Let $\bar{C}$ be the reduced closure of C in $\mathbb{P}^n_k$. It is an integral projective curve over k.

Consider the normalisation $\tilde{C}$ of $\bar{C}$. Then $\tilde{C}$ is finite over $\bar{C}$. Is is then a proper algebraic k-curve. As it is normal, it is then a projective regular curve over k from [GD61], 7.4.5 et 7.4.10.

The curve C is a dense open subscheme of $\bar{C}$. As it is normal it is again a dense open subscheme of $\tilde{C}$.

Let Z be a closed subscheme of C of dimension 0. Then $(\tilde{C} - C) \sqcup Z$ is a finite closed subset of $\tilde{C}$. As $\tilde{C}/k$ is projective, it admits an open affine neighbourhood if $\tilde{C}$. □

3.2.3. *Semi-local case.*

Theorem 3.8 (Walker)**.** Let k be an infinite field.

Let (X, Z) be a closed pair such that X is a smooth affine k-scheme and Z is nowhere dense in X.

Let $\{x_1, ..., x_n\}$ be a finite set of points of X.

Then there exists

1. *a smooth affine k-scheme S,*
2. *an open affine neighbourhood of $x_1, ..., x_n$ in X,*
3. *a smooth k-morphism $f : U \to S$ of relative dimension 1*

such that $(U, U \cap Z)$ admits a good compactification over S.

Proof. For the commodity of the reader, we include the following proof which follow the guideline of [Wal96], remark 4.13.

1) <u>Reduction</u> : We can assume X is irreducible.

Moreover, we can assume all the x_i are closed taking if necessary specialisations. If we can find a good compactification in a neighbourhood V_i of each x_i separately, we can define a good compactification in a neighbourhood of all the x_i by first reducing the neighbourhoods V_i such that they become disjoint then taken their dijoint union. We are thus reduced to the case of a single point $x_1 = x$.

Finally, as we can enlarge Z, we assume it is a divisor in X.

2) <u>Construction of S</u> : Let r be the dimension of Z.

As X is an affine algebraic k-scheme, we can find a closed immersion $X \hookrightarrow \mathbb{A}^n_k$. We identify X to its image in $\mathbb{A}^n_k$ under this embedding. Let us denote by :

1. $\bar{X}$ (resp. $\bar{Z}$) the reduced closure of X (resp. Z) in $\mathbb{P}^n_k$
2. $\dot{X} = \bar{X} - X$, intersection of $\bar{X}$ with hyperplane at infinity, thus a scheme of dimension less than r.

We allow us to increase n by arbitrarily considering an embedding $\mathbb{A}^n_k \hookrightarrow \mathbb{A}^{n'}_k$.

We find f by considering the orthognal projection of $\mathbb{A}^n_k$ with center in *general position* among the linear subvariety of $\mathbb{A}^n_k$ of codimension r.

Parametrisation of the orthogonal projections $\mathbb{A}^n_k \to \mathbb{A}^r_k$.

In fact these projections are parametrized by the points of $\mathbb{A}^{nr}_k$. Indeed, let λ be a point of $\mathbb{A}^{nr}_k$ and $\kappa(\lambda)$ its residual field. It is in fact an element $(\lambda_{i,j})_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$ de $\kappa(\lambda)^{nr}$.

We associate to λ the linear projection $p_\lambda : \mathbb{A}^n_{\kappa(\lambda)} \to \mathbb{A}^r_{\kappa(\lambda)}$ defined as the spectrum of the $\kappa(\lambda)$-linear morphism

$$\begin{array}{rcl} \kappa(\lambda)[t_1,...,t_r] & \to & \kappa(\lambda)[X_1,...,X_n] \\ t_i & \mapsto & \sum_{j=1}^n X_j - \lambda_{i,j}. \end{array}$$

We denote by L_λ the center of this projection. It is the intersection of the r hyperplanes corresponding to the zeros of each projections of $\mathbb{A}^r_{\kappa(\lambda)}$ to $\mathbb{A}^1_{\kappa(\lambda)}$ composed with p_λ.

Moreover if $\dot{L}_\lambda$ denotes the border of L_λ as a reduced sub-scheme of $\mathbb{P}^n_{\kappa(\lambda)}$, the morphism p_λ can be extended to a morphism $\bar{p}_\lambda : \mathbb{P}^n_{\kappa(\lambda)} - \dot{L}_\lambda \to \mathbb{P}^r_{\kappa(\lambda)}$. These notations being fixed we state the following lemma which allows to construct f :

Lemma 3.9. *Let Ω_n be the open subset of $\mathbb{A}^{nr}_k$ defined by the points λ such that :*

1. *$p_\lambda|_{Z_{\kappa(\lambda)}}$ is finite,*
2. *$\dot{X}_{\kappa(\lambda)} \cap \dot{L}_\lambda$ is a finite set of closed point,*
3. *p_λ is smooth at all the points of $X_{\kappa(\lambda)} \cap p_\lambda^{-1}(p_\lambda(x))$.*

Then, for n large enough, Ω_n is dense in $\mathbb{A}^{nr}_k$.

Proof. The fact Ω_n is open is easy. To prove it is dense, we proceed in two steps :

i) Let first suppose x is a *rational point* of X. Then we can assume further $x = 0$.

The first condition defines a dense subset as Z is closed in $\mathbb{A}^n_k$ of dimension r.

The second condition defines a dense subset as the intersection in $\mathbb{P}^n_k$ of the projective subvariety $\dot{X}$, of dimension less than r, with a linear subvariety of codimension r in general position is finite.

For the third condition one has only to suppose the intersection of L_λ and X is transversal in 0. We finally use the following theorem of [MAJ73], exposé XI, théorème 2.1 :

Theorem 3.10. *The intersection in $\mathbb{A}^n_k$ of X with r hypersurfaces of degree 2 containing 0 in general position is transversal.*

Through the Veronese embedding of $\mathbb{A}^n_k$ in $\mathbb{A}^{n^2}_k$, a linear subvariety of $\mathbb{A}^n_k$ corresponds to a quadric in $\mathbb{A}^{n^2}_k$ and the preceding theorem can be applied to our case replacing n by n^2.

ii) *General case.* Let k'/k be a finite extension such that the fiber of x in $X \otimes_k k'$ is made up with rational points x'_i. For each i, the preceding lemma gives a dense open subset $\Omega'_{n,i}$ of $\mathbb{A}^{nr}_{k'}$. As $\mathbb{A}^r_{k'}/\mathbb{A}^r_k$ is faithfully flat, the direct image of $\cap_i \Omega'_{n,i}$ in $\mathbb{A}^{nr}_k$ is included in Ω_n as the three conditions of the lemma satisfy faithfully flat descent. This implies Ω_n is dense. □

As k is infinite, Ω_n admits a rational point λ. We set $\dot{L} = \dot{X} \cap \dot{L}_\lambda$ which is a finite k-scheme. Let $p : X \to \mathbb{A}^r_k$ (resp. $\bar{p} : \bar{X} - \dot{L} \to \mathbb{P}^r_k$) be the restriction of p_λ (resp. $\bar{p}_\lambda$).

To extend $\bar{p}$ into a projective morphism, we consider $\tilde{X}$ the closure of the graph of $\bar{p}$ in $\bar{X} \times_k \mathbb{P}^r_k$. Then $\bar{X} - \dot{L}$ is a dense open subscheme of $\tilde{X}$ and the canonical projection $\tilde{p} : \tilde{X} \to \mathbb{P}^r_k$ extends $\bar{p}$. As $\bar{X}/k$ is projective, $\tilde{p}$ is projective. We have

obtained the following diagram :

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "p"'] & \bar{X} - \dot{L} \arrow[r, hook] \arrow[d, "\bar{p}"] & \tilde{X} \arrow[dl, "\tilde{p}"] \\
A^r_k \arrow[r, hook] & \mathbb{P}^r_k .
\end{tikzcd}$$

3) Construction of the compactification :

As the square in the above diagram is cartesian and L is finite over k, the fibers of $\tilde{p}$ in $\tilde{X} - X$ above $\mathbb{A}^r_k$ are finite.

Thus there exists an open affine neighbourhood S of $p(x)$ in $\mathbb{A}^r_k$ such that $\tilde{p}^{-1}(S) \cap (\tilde{X} - X)$ is finite over S. Reducing S if necessary we can assume $p^{-1}(S) \to S$ is smooth using the third condition imposed on Ω_n in the preceding lemma.

Finally we put $U = p^{-1}(S)$ and we denote by $f : U \to S$ the restriction of p to U. Then the morphism f is smooth of relative dimension 1. Moreover the restriction $Z \cap U \to S$ of p is finite following from the first condition imposed on the points of Ω_n.

We set $\bar{U} = \tilde{p}^{-1}(S)$ so that the restriction $\bar{f} : \bar{U} \to S$ of $\tilde{p}$ is projective. From the choice of S, $\bar{U} - U$ is finite over S.

To conclude the following lemma shows that by reducing S near $p(x)$ we can assume $(\bar{U} - U) \sqcup Z \cap U$ admits an affine neighbourhood :

Lemma 3.11. *Let $\bar{p} : \bar{U} \to S$ be a projective curve and F be a closed subscheme of $\bar{U}$ such that F/S is finite. Let x be a point of F and $s = \bar{p}(x)$.*

Then there exists an open affine neighbourhood S' of s in S and an effective divisor D in $\bar{X}$ such that :

(1) *$F_{S'} \subset \bar{U}_{S'} - D$.*
(2) *$\bar{U}_{S'} - D_{S'}$ is affine.*

Proof. Let F_s be the fiber of F above s. As a set F_s is finite. As $\bar{U}/S$ is projective there exists for i large enough a section f in $\Gamma(\bar{U}, \mathcal{O}_{\bar{U}}(i))$ whose divisor D is disjoint of F_s. Thus there exists an open affine neighbourhood S' of s in S such that D is disjoint of $F_{S'}$ which guarantee the first condition. As S' is affine and $D_{S'}$ is the divisor associated to a global section of a very ample fiber bundle over $\bar{U}_{S'}$, the scheme $\bar{U}_{S'} - D_{S'}$ is affine. □

□

3.3. Relative Picard group.

Definition 3.12. Let (X, Z) be a closed pair.

We denote by $\mathrm{Pic}\,(X, Z)$ the group of couples $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf on X and $s : \mathcal{O}_Z \xrightarrow{\sim} \mathcal{L}|_Z$ is a trivialisation of $\mathcal{L}$ over Z, modulo isomorphism of invertible sheaves compatible with the trivialisation.

The group structure is induced by the tensor product of $\mathcal{O}_X$-modules.

There is a canonical morphism $\mathrm{Pic}\,(X, Z) \to \mathrm{Pic}(X)$.

Definition 3.13. let X be a scheme and Z a closed subscheme of X.

(1) If Z' is closed subscheme of Z, we define the restriction morphism from Z to Z'
$$\begin{array}{ccc} \mathrm{Pic}\,(X, Z) & \xrightarrow{r_{Z'}} & \mathrm{Pic}\,(X, Z') \\ (\mathcal{L}, s) & \mapsto & (\mathcal{L}, s|_{Z'}). \end{array}$$
(2) Let $(f, g) : (Y, W) \to (X, Z)$ be a morphism of closed pairs (cf def. 3.6). We define the pullback morphism
$$\begin{array}{rcl} (f, g)^* : \mathrm{Pic}\,(X, Z) & \to & \mathrm{Pic}\,(Y, W) \\ (\mathcal{L}, s) & \mapsto & (f^*(\mathcal{L}), g^*(s))\,. \end{array}$$

3.14. Let S be a regular affine scheme. We consider a smooth quasi-affine curve X/S and suppose it admits a good compactification $\bar{X}/S$.

Let α be a relative cycle on X/S. Then, considered as a codimension 1 cycle of $\bar{X}$, it corresponds to an invertible sheaf $\mathcal{L}(\alpha)$ on $\bar{X}$ whose isomorphism class is unique. Moreover, if Z is the support of α, this sheaf has a canonical trivialisation on $\bar{X} - Z$. Let $s(\alpha)$ be its restriction to X_∞. We thus have defined a canonical morphism

$$\mathrm{c}_0(X/S) \xrightarrow{\lambda_{X/S}} \mathrm{Pic}\left(\bar{X}, X_\infty\right), \alpha \mapsto (\mathcal{L}(\alpha), s(\alpha)).$$

Lemma 3.15. *Consider the above notations.*

(1) *Let S' be a regular scheme and $\tau : S' \to S$ be a flat morphism.*
Put $X' = X \times_S S'$ and $\bar{X}' = \bar{X} \times_S S'$ and $\Delta : X'/S' \to X/S$ be the morphism induced by τ.
We consider $(\bar{f}, f_\infty) : (\bar{X}', X'_\infty) \to (\bar{X}, X_\infty)$ the cartesian morphism of closed pairs induced by τ. Then the following diagram is commutative :

$$\begin{tikzcd}
\mathrm{c}_0(X/S) \arrow[r, "\lambda_{X'/S'}"] \arrow[d, "\Delta^*"'] & \mathrm{Pic}\left(\bar{X}, X_\infty\right) \arrow[d, "(\bar{f}, f_\infty)^*"] \\
\mathrm{c}_0(X'/S') \arrow[r, "\lambda_{X/S}"] & \mathrm{Pic}\left(\bar{X}', X'_\infty\right).
\end{tikzcd}$$

(2) *Let Z be a closed subscheme of X and suppose $\bar{X}$ is a good compactification of (X, Z) over S. Then the following diagram commutes :*

$$\begin{tikzcd}
\mathrm{c}_0(X - Z/S) \arrow[r, "\lambda_{X-Z/S}"] \arrow[d, "j_*"'] & \mathrm{Pic}\left(\bar{X}, X_\infty \sqcup Z\right) \arrow[d, "r_{(Y \times_S X_\infty)}"] \\
\mathrm{c}_0(X/S) \arrow[r, "\lambda_{X/S}"] & \mathrm{Pic}\left(\bar{X}, X_\infty\right).
\end{tikzcd}$$

Proof. The second point is obvious by construction.

For the first point, let $f : X' \to X$ be the flat morphism induced by τ. Let α be a finite relative cycle on $\bar{X}/S$. Suppose α is the class of a closed subscheme Z in X. Then by proposition 1.7, $\Delta^*\alpha$ is the cycle associated to the closed subscheme $f^{-1}(Z)$ of X'. Thus the conclusion follows by construction of λ. □

Proposition 3.16. *Consider the notations above the previous lemma and let Y be a smooth affine S-scheme.*

Then the morphism $\lambda_{Y \times_S X/Y}$ factors through the homotopy relation. The induced morphism

$$\pi_S\left(Y, X\right) \to \mathrm{Pic}\left(Y \times_S \bar{X}, Y \times_S X_\infty\right)$$

is an isomorphism.

Proof. Let $i_0 : Y \to \mathbb{A}^1_Y$ (resp. $i_1 : Y \to \mathbb{A}^1_Y$) be the zero section (resp. unit section) of $\mathbb{A}^1_Y/Y$.

Note that i_0 and i_1 are inclusions of a cycle associated to a principal Cartier divisor. Then the pullback maps $\mathrm{c}_0\left(\mathbb{A}^1_Y \times_S X/\mathbb{A}^1_Y\right) \to \mathrm{c}_0(Y \times_S X/Y)$ induced by i_0 and i_1 coincide with the operartion of intersecting with divisors defined in [Ful98], 2.3 (see also remark 2.3 of *loc. cit.*). This allows to extend the first case of the previous lemma to the case where τ is i_0 or i_1. Finally, using the homotopy invariance of the Picard group for regular schemes, λ indeed factors through the homotopy relation of S-correspondances.

To prove the induced morphism is an isomorphism, we construct its reciprocal. It is sufficient to treat the case $Y = S$. Let $(\mathcal{L}, s)$ be a couple in $\mathrm{Pic}\left(\bar{X}, X_\infty\right)$. Consider an open affine neighbourhood V of X_∞ in $\bar{X}$. The trivialisation s of $\mathcal{L}$ then extends to a trivialisation $\tilde{s}$ of $\mathcal{L}$ over V. To the pseudo-divisor $(\mathcal{L}, \bar{X} - V, \tilde{s})$

is associated a unique Cartier divisor $D(\mathcal{L}, \bar{X} - V, \tilde{s})$ on $\bar{X}$ following [Ful98], lemma 2.2. Let α be the associated cycle. The support of α lies in $(\bar{X} - V)$. Moreover, as X/S is quasi-affine and $\bar{X}/S$ is proper, V is dense in all the fibers of the curve $\bar{X}/S$ which implies $\bar{X} - V$ is finite over S. Finally, the support of α is finite over S and α is in fact a finite relative cycle on X/S.

We prove now that the homotopy class of α in $\mathrm{c}_S(S, X)$ does not depend on the choice of $\tilde{s}$. Suppose given two extensions $\tilde{s}_0$ and $\tilde{s}_1$ of s to V. Let α_0 and α_1 be the respective cycles obtained in the process described above. Define $\mathcal{L}'$ as the pullback of $\mathcal{L}$ along the morphism $\pi : \mathbb{A}^1_{\bar{X}} \to \bar{X}$. For $i = 0, 1$, we obtain a trivialisation $\pi^* \tilde{s}_i$ of $\mathcal{L}'$ over $\mathbb{A}^1_V$. Let H be the cycle associated to the pseudo-divisor $D(\mathcal{L}', \mathbb{A}^1_{\bar{X}-V}, t\pi^*\tilde{s}_0 + (1-t)\pi^*\tilde{s}_1)$ where t is the canonical parameter of $\mathbb{A}^1_{\bar{X}}$. Then, using the begining of the proof, we obtain $H \circ i_0 = \alpha_0$ and $H \circ i_1 = \alpha_1$. □

Remark 3.17. The previous proposition is a particular case of the computation of the Suslin singular homology of the curve X/S in [SV96], th. 3.1.

3.4. **Constructing usefull correspondances up to homotopy.**

3.4.1. *Factorisations.*

Proposition 3.18. *Let S be an affine regular scheme, and (X, Z) a closed pair such that X is a smooth affine S-curve. Put $U = X - Z$ and denote by $i : U \to X$ the canonical open immersion. Suppose that (X, Z) admits a good compactification $\bar{X}$ over S.*

Let $\mathcal{L}(1_X)$ be the invertible sheaf corresponding to $1_X \in \mathrm{c}_S(X, X)$ in the notation of 3.14.

The following conditions are equivalent :

1. *For any smooth affine S-scheme Y, the morphism*
$$\pi_S(Y, U) \xrightarrow{i\circ} \pi_S(Y, X)$$
is surjective.
2. *The morphism*
$$\pi_S(X, U) \xrightarrow{i\circ} \pi_S(X, X)$$
is surjective.
3. *The invertible sheaf $\mathcal{L}(1_X)|_{X \times_S Z}$ is trivial.*

Proof. Conditions 1 and 2 are equivalent to the existence of a section of i up to homotopy. Thus the proposition is implied by the more precise lemma :

Lemma 3.19. *Consider the hypothesis of the preceding proposition. Let Y be a smooth affine S-scheme and $\beta : Y \to X$ a finite S-correspondance. The following conditions are equivalent :*

1. *There exists a finite S-correspondance α which makes the following diagram of S-correspondances commutative up to homotopy*
$$\begin{tikzcd} & X - Z \arrow[dr, "i"] & \\ Y \arrow[ur, dashed, "\alpha"] \arrow[rr, "\beta"] & & X. \end{tikzcd}$$
2. *The invertible sheaf $\mathcal{L}(\beta)|_{Y \times_S Z}$ is trivial, with the notation of 3.14.*

Moreover, the finite S-correspondances which satisfy condition 1 are in one-to-one correspondance with the trivialisations of $\mathcal{L}(\beta)|_{Y \times_S Z}$.

We use proposition 3.16 applied first to the affine curve X/S and secondly to the quasi-affine curve U/S.

$2 \Rightarrow 1$: Consider a trivialisation s of $\mathcal{L}(\beta)|_{Y\times_S Z}$. Then the class of the couple $(\mathcal{L}(\beta), s(\beta)\oplus s)$ in $\mathrm{Pic}\left(Y\times_S \bar{X}, Y\times_S X_\infty \sqcup Y\times_S Z\right)$ defines a finite S-correspondance α which, according to lemma 3.15, satisfies $i\circ\alpha = \beta$ as required.

$1 \Rightarrow 2$: Reciprocally, the finite S-correspondance α corresponds to an element of $\mathrm{Pic}\left(Y\times_S \bar{X}, Y\times_S X_\infty \sqcup Y\times_S Z\right)$ which is the class of the couple $(\mathcal{L}(\alpha), s(\alpha))$. Thus, as $i\circ\alpha = \beta$, there exists an isomorphism $\phi : \mathcal{L}(\beta)\to\mathcal{L}(\alpha)$ making the following diagram commutative

$$\begin{tikzcd} \mathcal{L}(\beta)|_{Y\times_S X_\infty} \arrow[rr, "\phi|_{Y\times_S X_\infty}"] \arrow[dr, "s(\beta)"'] & & \mathcal{L}(\alpha)|_{Y\times_S X_\infty}. \arrow[dl, "s(\alpha)|_{Y\times_S X_\infty}"] \\ & \mathcal{O}_{Y\times_S X_\infty} & \end{tikzcd}$$

Then $s(\alpha)|_{Y\times_S Z}\circ\phi^{-1}|_{Y\times_S Z}$ is indeed a trivialisation of $\mathcal{L}(\beta)|_{Y\times_S Z}$.

The last point of the lemma is clear from the proof. □

Example 3.20. As an easy application of this proposition, we can consider two open subschemes X and U of the affine line $\mathbb{A}^1_k$ over a field k such that $U\subset X$. Put $Z=(X-U)_{red}$.

Then the open immersion $i:U\to X$ admits a section in $\pi\mathscr{L}_{\mathrm{cor},k}$ as $\mathbb{P}^1_k$ is a good compactification of (X,Z), X is affine and $\mathrm{Pic}\,(X\times_k Z)=0$.

Moreover, choosing a trivialisation of $\mathcal{L}(1_{\mathbb{A}^1_k})$ once and for all, we define trivialisations for all open immersions $i:U\to X$ which are functorial with respect to open immersions in X and U.

3.4.2. *Local section of open immersions in $\pi\mathscr{L}_{\mathrm{cor},k}$.* The following proposition is directly inspired by proposition 4.17 of [Voe00a] :

Proposition 3.21. *Let k be a field, X a smooth k-scheme, U a dense open subscheme of X, and x a point of X. Then there exists*

1. *an open neighbourhood V of x in X,*
2. *a finite k-correspondance $\alpha : V\to U$,*

such that the following diagram is commutative up to homotopy

$$\begin{tikzcd} & V \arrow[dl, dashed, "\alpha"'] \arrow[d, "j"] \\ U \arrow[r, "i"] & X. \end{tikzcd}$$

where i and j are the obvious open immersions.

Proof. Suppose first k is infinite.

Put $Z=(X-U)_{red}$. Using theorem 3.8, there exists an affine smooth k-scheme S, an affine open neighbourhood V of x in X, a smooth morphism $f:V\to S$ of relative dimension 1, and an S-scheme $\bar{X}$ such that $\bar{X}/S$ is a good compactification of $(V, V\cap Z)$.

From the commutative diagram

$$\begin{tikzcd} V\cap U \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X, \end{tikzcd}$$

on sees the theorem holds for V, if it holds for X. Thus we can assume $X=V$ which implies (X,Z) has a good compactification $\bar{X}$ over S.

Let $\mathcal{L}(1_X)$ be an invertible shead over $X\times_S\bar{X}$ which corresponds to $1_X\in\pi_S(X,X)$ according to proposition 3.16. As Z is affine and closed in the proper curve $\bar{X}/S$, it is finite over S. The scheme $\mathrm{Spec}\,(\mathcal{O}_{X,x})\times_S Z$ is finite over the local scheme $\mathrm{Spec}\,(\mathcal{O}_{X,x})$ thus it is semi-local which implies $\mathrm{Pic}(\mathrm{Spec}\,(\mathcal{O}_{X,x})\times_S Z)=$

0. In particular, $\mathcal{L}(1_X)$ is trivial over $\mathrm{Spec}\,(\mathcal{O}_{X,x}) \times_S Z$. Thus it exists an open neighbourhood V of x in X such that $\mathcal{L}(1_X)$ is trivial over $V \times_S Z$.

From lemma 3.19 applied to $Y = V$ and to the finite S-correspondance $V \xrightarrow{j} X$ there exists a finite S-correspondance $\alpha : V \to U$ which makes the following diagram commutative :

$$\begin{tikzcd} & V \arrow[dl, bend right, "\alpha"'] \arrow[d, "j"] \\ U \arrow[r, "i"] & X. \end{tikzcd}$$

Let $\tau : S \to k$ be the canonical morphism. As τ is smooth, the restriction functor $\tilde{\tau}_\sharp$ of definition 1.32 is well define. Applying this functor to the preceding diagram, we see the k-finite correspondance $\tilde{\tau}_\sharp(\alpha)$ is appropriate.

When k is finite, we consider $L = k(t)$. We put $X_L = X \times_k \mathrm{Spec}\,(L)$ and similarly for any k-scheme. The point x corresponds canonically to a point of X_L still denoted by x.

Applying the preceding case to the open immersion $i_L : U_L \to X_L$ and to the point x, we find a neighbourhood Ω of x in X_L and a finite L-correspondance $\alpha : \Omega \to U_L$ such that $i_L \circ \alpha$ is the open immersion $\Omega \to X_L$. As x comes from a point of X, we can always find an open neighbourhood V of x in X such that $V_L \subset \Omega$. The following diagram

$$\begin{tikzcd} & V_L \arrow[dl, "\alpha|_{V_L}"'] \arrow[d, "j_L"] \\ U_L \arrow[r, "i_L"] & X_L. \end{tikzcd}$$

is commutative in $\pi\mathscr{L}_{\mathrm{cor},L}$, for $j : V \to X$ the canonical immersion.

Applying proposition 1.8, we obtain a canonical isomorphism

$$\mathrm{c}_L\left(V_L, Y_L\right) = \varinjlim_{W \subset \mathbb{A}^1_k} \mathrm{c}_k\left(V \times_k W, Y\right),$$

for any k-scheme Y, where the limit runs over the non empty open subschemes W of $\mathbb{A}^1_k$. It is functorial in Y.

In particular, we can lift both the finite L-correspondance $\alpha|_{V_L}$ and the homotopy making the above diagram commutative for a sufficiently small W in $\mathbb{A}^1_k$. We thus obtain a finite k-correspondance $\alpha_0 : V \times_k W \to U$ such that the diagram over k

$$\begin{tikzcd} & V \times_k W \arrow[dl, "\alpha_0"'] \arrow[d, "j \times_k p"] \\ U \arrow[r, "i"] & X. \end{tikzcd}$$

is commutative up to homotopy, for $p : W \to \mathrm{Spec}\,(k)$ the canonical projection.

Finally, we factor out $j \times_k p$ as $V \times_k W \xrightarrow{1 \times_k p} V \xrightarrow{j} X$. The example 3.20 gives a section of the open immersion $W \to \mathbb{A}^1_k$, which shows $1 \times_k p$ admits a section in $\pi\mathscr{L}_{\mathrm{cor},k}$ and allows to conclude. $\square$

Corollary 3.22. *Let k be a field, X a smooth k-scheme and U a dense open subscheme of X. Then there exists*

(1) *an open covering $p : W \to X$ of X,*
(2) *a finite k-correspondance $\alpha : W \to U$*

such that the following diagram is commutative up to homotopy

$$\begin{tikzcd} & W \arrow[dl, dashed, "\alpha"'] \arrow[d, "j"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where i and j are the canonical open immersions.

Proof. We simply apply the preceding lemma to every points of X and use its quasi-compacity. □

3.4.3. *Homotopy excision.* The following proposition is one of the central point in our interpretation of Voevodsky's theory. It is a generalisation of lemma 4.6 of [Voe00a].

Theorem 3.23. *Let S be an affine regular scheme.*
Consider a distinguished square (cf section 2.1) of smooth affine S-scheme

$$\begin{tikzcd} W \arrow[r, "l"] \arrow[d, "h"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & C. \end{tikzcd}$$

We put $Z = C - U$ and $T = V - W$ with their reduced structure and assume there exists good compatifications $\bar{C}/S$ of (C, Z) and $\bar{V}/S$ of (V, T) which fits into the commutative square

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "f"'] & \bar{V} \arrow[d, "\bar{f}"] \\ C \arrow[r, hook] & \bar{C} \end{tikzcd}$$

and satisfy $V_\infty \subset \bar{f}^{-1}(C_\infty)$.
Assume finally $\mathrm{Pic}\,(C \times_S Z) = 0$. *Then the complex*

$$0 \to [W] \xrightarrow{h-l} [U] \oplus [V] \xrightarrow{(j,f)} [C] \to 0$$

is contractile in the additive category $\pi\mathscr{L}_{\mathrm{cor},S}$.

Proof. In the following lemma, we will construct the chain homotopy between the complex above and the zero complex. Indeed, with the notations of this lemma, the chain homotopy is given by the two morphisms

$$\begin{tikzcd} & {[U] \oplus [V]} \arrow[dl, "(\gamma,-\beta)"'] & & {[C]} \arrow[dl, "\alpha"] \\ {[W]} & & {[U] \oplus [V].} & \end{tikzcd}$$

The necessary relations are stated and proved in the lemma.

Lemma 3.24. *Suppose we are in the hypothesis of the preceding proposition.*
Then there exists finite correspondances

$$\begin{tikzcd} W & V \arrow[l, "\beta"'] \\ U \arrow[u, "\gamma"] & C \arrow[l, "\alpha"'] \end{tikzcd}$$

which satisfy the following relations in $\pi\mathscr{L}_{\mathrm{cor},k}$:

$$\begin{cases} j \circ \alpha = 1_C & (1) \\ l \circ \beta = 1_V & (2) \\ \alpha \circ f = h \circ \beta & (3) \\ l \circ \gamma = 0 & (4) \\ h \circ \gamma = 1_U - \alpha \circ j & (5) \\ \gamma \circ h = 1_W - \beta \circ l & (6) \end{cases}$$

We first apply proposition 3.16 to the morphism 1_C, as an element of $\pi_S\,(C, C)$. It corresponds to the class of a couple $(\mathcal{L}(1_C), s(1_C))$ in $\mathrm{Pic}\,(C\bar{C}, CC_\infty)$[5].

[5]In this proof, we sometimes omit the symbol $\times_S$ when it facilitates the notation

By hypothesis, $\mathrm{Pic}\,(C \times_S Z) = 0$ which implies the invertible sheaf $\mathcal{L}(1_C)$ is trivial on $C \times_S Z$. Let t be such a trivialisation. We define α corresponding to the following element in $\mathrm{Pic}\,\big(C\bar{C}, CC_\infty \sqcup CZ\big)$

$$\alpha \qquad \leftrightarrow \qquad (\mathcal{L}(1_C), s(1_C) + t).$$

Relation (1) simply follows from lemma 3.19 as in the preceding applications.

Using again proposition 3.16, the morphism 1_V, as an element of $\pi_S\,(V, V)$, corresponds to the class of a couple $(\mathcal{L}(1_V), s(1_V))$ in $\mathrm{Pic}\,\big(V\bar{V}, VV_\infty\big)$.

By construction, the sheaf $\mathcal{L}(1_C)$ (resp. $\mathcal{L}(1_V)$) corresponds to the diagonal Δ_C (resp. Δ_V) of C/k (resp. V/k) seen as a closed subscheme of $C \times_S \bar{C}$ (resp. $V \times_S \bar{V}$). Because the morphism $g = f \times_X Z : T \to Z$ is an isomorphism, we obtain

$$(f \times_S g)^{-1}(\Delta_X \cap (X \times_S Z)) = \Delta_V \cap (V \times_S T)$$

which finally gives

$$(f \times_S g)^*(\mathcal{L}(1_C)|_{X \times_S Z}) = \mathcal{L}(1_V)|_{V \times_S T}.$$

In particular, the section $\tau = (f \times_S g)^*(t)$ is a trivialisation of $\mathcal{L}(1_V)$ on $V \times_S T$. Let us define β corresponding to the couple in $\mathrm{Pic}\,\big(V\bar{V}, VV_\infty \sqcup VT\big)$

$$\beta \qquad \leftrightarrow \qquad (\mathcal{L}(1_V), s(1_V) + \tau).$$

Relation (2) is again a consequence of lemma 3.19.

It remains to construct γ. We consider the invertible sheaf $\mathcal{M} = (1_C \times_S \bar{f})^* \mathcal{L}(1_C)$ on $C \times_S \bar{V}$. It corresponds to the divisor $D = (1_C \times_S \bar{f})^{-1}(\Delta_C)$. Let u be the canonical trivialisation of $\mathcal{M}$ on $C \times_S V - D$. As g is an isomorphism, $v = (1_C \times_S g)^* t$ is a trivialisation of $\mathcal{M}|_{C \times_S Z}$. Note that $1 + uv^{-1}$ is a regular invertible section of $\mathcal{O}_{C\bar{V}}$ over $CV_\infty \sqcup CT$. We define γ corresponding to the class of the couple in $\mathrm{Pic}\,\big(C\bar{V}, CV_\infty \sqcup CT\big)$ which follows :

$$\gamma \qquad \leftrightarrow \qquad (\mathcal{O}_{C\bar{V}}, 1 + uv^{-1}).$$

By construction and lemma 3.15, $l \circ \gamma$ corresponds to the couple $(\mathcal{O}_{C\bar{V}}, 1)$, which is the zero correspondance. This is relation (4).

Consider an open affine neighbourhood Ω of $C_\infty \sqcup Z$ in $\bar{C}$. Put $\Omega_0 = \bar{f}^{-1}(\Omega)$ and let $\nu : \Omega_0 \to \Omega$ be the finite morphism induced by $\bar{f}$. Then Ω_0 is an open affine neighbourhood of $V_\infty \sqcup T$. Thus the invertible regular function $1 + uv^{-1}$ admits an extension w to $U \times_S \Omega_0$. Following the computation of [Ful98], 1.4, we see that the correspondance $h \circ \gamma$ corresponds through the isomorphism of prop. 3.16 to the couple in $\mathrm{Pic}\,\big(U\bar{C}, UC_\infty \sqcup UT\big)$

$$\Big(\mathcal{O}_{U\bar{C}}, N(w)|_{UC_\infty \sqcup UT}\Big)$$

where N is the norm associated to the extension ring corresponding to $U\Omega_0/U\Omega$. As $w|_{UV_\infty} = 1$, we obtain easily that $N(w)|_{UC_\infty} = 1$. A more accurate computation shows moreover $N(w)|_{UT} = s(1_U).t^{-1}$, as g is an isomorphism and f is étale.

Besides, the finite correspondance $1_U - \alpha \circ j$ corresponds to the couple

$$\Big(\mathcal{L}(1_U) \otimes (\mathcal{L}(1_C)|_{U\bar{C}})^{-1}, s(1_U).(s(1_C) + t)^{-1}\Big).$$

Thus relation (5) is now clear.

Finally, using again lemma 3.15, $\gamma \circ h$ corresponds to the couple

$$\Big(\mathcal{O}_{W \times_S \bar{V}}, 1 + s(1_V).\tau^{-1}\Big).$$

Indeed, by definition, the pullback of v over $W \times_S \bar{V}$ is τ.

Relation (6) now follows from the fact the finite correspondance $1_W - \beta \circ l$ corresponds to the couple

$$\Big(\mathcal{L}(1_W) \otimes (\mathcal{L}(1_V)|_{W\bar{V}})^{-1}, s(1_W).(s(1_V) + \tau)^{-1}\Big).$$

Only the relation (3) remains. We consider the trivialisation $s(1_V)$ (resp. τ)) of the invertible sheaf $\mathcal{L}(1_V)$ over VV_∞ (resp. VT). As Ω_0 is an affine neighbourhood of $V_\infty \sqcup T$, the trivialisation $s(1_V)$ (resp. τ) admits an extension w_1 (resp. w_2) to $V \times_S \Omega_0$. Using a computation we have already seen, relation (3) is equivalent to show the following couples of $\operatorname{Pic}\big(V\bar{C}, VC_\infty \sqcup VZ\big)$ are equal :

$$\Big((f \times_S 1_{\bar{C}})^* \mathcal{L}(1_C), (f \times_S 1_{\bar{C}})^*(s(1_C) + t)\Big)$$
$$\Big((1_V \times_S \bar{f})_* \mathcal{L}(1_V), N'(w_1 + w_2)|_{VC_\infty \sqcup VZ}\Big)$$

We have denoted by N' the norm associated to the finite extension $V\Omega_0/V\Omega$. Using again that g is an isomorphism and f is étale, we obtain $N'(w_2)|_{VZ} = (f \times_S 1_{\bar{C}})^*(t)$.

But the equality $1_C \circ f = f \circ 1_V$ implies the following couples coincide

$$\Big((f \times_S 1_{\bar{C}})^* \mathcal{L}(1_C), (f \times_S 1_{\bar{C}})^*(s(1_C))\Big)$$
$$\Big((1_V \times_S \bar{f})_* \mathcal{L}(1_V), N'(w_1)|_{VC_\infty}\Big),$$

and this concludes. □

To finish, we give a simple example where we can construct compactifications which appears in the above theorem. Suppose we are only given only the distinguished square in the hypothesis of the preceding proposition and assume S is the spectrum of a field k.

Then, according to proposition 3.7, there exists a smooth projective curve $\bar{C}/k$ which is a good compactification of (C, Z).

The morphism $V \xrightarrow{f} C \to \bar{C}$ is quasi-affine. Applying Zariski's main theorem (cf [GD63], chap. III, 4.4.3), it can be factored as $V \xrightarrow{\tilde{j}} \tilde{V} \xrightarrow{\tilde{f}} \bar{C}$ where $\tilde{j}$ is an open immersion and $\tilde{f}$ a finite morphism.

As $\tilde{V}/k$ is algebraic, its normalisation $\bar{V}$ is finite over $\tilde{V}$, and still contains V as an open subscheme since V is normal. Thus $\bar{V}$ is a good compactification of (V, Z) and we have the following commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] & \bar{V} \arrow[d, "\bar{f}"] \\ C \arrow[r] & \bar{C}. \end{tikzcd}$$

4. Homotopy sheaves with transfers

4.1. **Homotopy invariance.**

Definition 4.1. Let S be a scheme.

A presheaf F on $\mathscr{L}_S$ is said to be homotopy invariant if for all smooth S-scheme X, the morphism induced by the canonical projection $F(X) \to F(\mathbb{A}^1_X)$ is an isomorphism.

When S is regular, we denote by $\mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ (resp. $\mathrm{H}\mathscr{P}_S^{\mathrm{tr}}$) the category of sheaves (resp. presheaves) with transfers over S which are homotopy invariants. Such sheaves (resp. presheaves) will simply be called homotopy sheaves (resp. presheaves).

The following lemma will connect homotopy presheaves with correspondances up to homotopy.

Lemma 4.2. *Let F be a presheaf with transfers over a regular scheme S. The following conditions are equivalent :*

(1) *F is homotopy invariant.*
(2) *For all smooth S-scheme X, considering $s_0 : X \to \mathbb{A}^1_X$ (resp. $s_1 : X \to \mathbb{A}^1_X$) the zero (resp. unity) section of $\mathbb{A}^1_X$, $s_0^* = s_1^*$.*
(3) *F can be factored through the canonical morphism $\mathscr{L}_{\mathrm{cor},S} \to \pi\mathscr{L}_{\mathrm{cor},S}$.*

Proof. For a smooth S-scheme X, we denote by $p_X : \mathbb{A}^1_X \to X$ (resp. $\mu_X : \mathbb{A}^1_X \times_X \mathbb{A}^1_X \to \mathbb{A}^1_X$) the canonical projection (resp. multiplication) of the ringed X-scheme $\mathbb{A}^1_X$.

The lemma now follows easily from the relations $p_X \circ s_0 = p_X \circ s_1 = 1_X$ and the fact that μ_X defines a homotopy from s_0 to s_1. □

4.3. In particular, a homotopy presheaf (resp. sheaf) is nothing but a presheaf on $\pi\mathscr{L}_{\mathrm{cor},S}$ (resp. such that the restriction of F to $\mathscr{L}_S$ is a Nisnevich sheaf).

As a corollary, the forgetfull functor $\mathrm{H}\mathscr{P}^{\mathrm{tr}}_S \to \mathscr{P}^{\mathrm{tr}}_S$ admits a left adjoint $\hat{h}_0\left(\right)$ constructed as follows. Let F be a presheaf with transfers, and define $\hat{h}_0\left(F\right)(X)$ as the cokernel of the morphism $F(\mathbb{A}^1_X) \xrightarrow{s_0^* - s_1^*} F(X)$. The preceding lemma implies $\hat{h}_0\left(F\right)$ is homotopy invariant and the adjunction property.

Consider now a sheaf with transfers F. We denote by $h_0^{(1)}F$ the sheaf with transfers associated with the presheaf $\hat{h}_0\left(F\right)$ using corollary 2.7. In general, this sheaf is not homotopy invariant - unless S is the spectrum of a perfect field (see 4.14). For a natural integer n, we denote by $h_0^{(n)}$ the n-th composition power of $h_0^{(1)}$.

We deduce a sequence of morphisms

$$F \to h_0^{(1)}F \to \ldots \to h_0^{(n)}F \to \ldots$$

We define

$$h_0(F) = \varinjlim_{n \in \mathbb{N}} h_0^{(n)}F,$$

where the limit is taken in the category of sheaves with transfers.

Proposition 4.4. *Let S be a regular scheme and F a sheaf with transfers over S.*

Then the sheaf with transfers $h_0(F)$ defined above is homotopy invariant. Moreover, the functor $h_0 : \mathscr{N}^{\mathrm{tr}}_S \to \mathrm{H}\mathscr{N}^{\mathrm{tr}}_S$ is left adjoint to the obvious forgetfull functor.

Proof. Let X be a smooth S-scheme, s_0 and s_1 respectively the zero and unit section of $\mathbb{A}^1_X/X$. According to the preceding lemma, we have to show that $s_0^* = s_1^*$ on $h_0(F)(\mathbb{A}^1_X)$. Let $\mathbf{x}$ be an element of

$$h_0(F)(\mathbb{A}^1_X) = \varinjlim_{n \in \mathbb{N}} h_0^{(n)}F(\mathbb{A}^1_X).$$

By definition, it is represented by a section x_n in $h_0^{(n)}F(\mathbb{A}^1_X)$ for an integer $n \in \mathbb{N}$.

The transition morphism of level n in the above inductive limit can be factored out as

$$h_0^{(n)}F \xrightarrow{\ a\ } \hat{h}_0\left(h_0^{(n)}F\right) \xrightarrow{\ b\ } h_0^{(n+1)}F.$$

From what we saw before, the sheaf $\hat{h}_0\left(h_0^{(n)}F\right)$ is homotopy invariant. Thus $s_0^*(ax_n) = s_1^*(ax_n)$. As a is a natural transformation, we deduce that $as_0^*(x_n) = as_1^*(x_n)$, thus $bas_0^*(x_n) = bas_1^*(x_n)$ and $s_0^*(\mathbf{x}) = s_1^*(\mathbf{x})$. □

4.2. **Fibers along function fields.** In this subsection, we fix a field k.

4.2.1. *Open immersions.* The following proposition is analog to cor. 4.19 of [Voe00a] and uses the same arguments.

Proposition 4.5. *Let F be a presheaf over $\pi\mathscr{L}_{\mathrm{cor},k}$.*

Let G be one of the following presheaves over $\mathscr{L}_k$:

1. *the Zariski sheaf F_{Zar} over $\mathscr{L}_k$ associated with F,*
2. *the 0-th Čech cohomology presheaf $\check{H}^0 F$ over $\mathscr{L}_k$ associated with F for the Nisnevich topology.*

Then for any smooth k-scheme X, any dense open subscheme U of X, the restriction morphism $G(X) \to G(U)$ is a monomorphism.

Proof. Consider $a \in G(X)$ such that $a|_U = 0$. We prove $a = 0$.

We may assume there exists an element $b \in F(X)$ such that a is the image of b by the canonical morphism $F(X) \to G(X)$. Indeed, there exists a Nisnevich covering (and even Zariski in the first case) of X such that $a|_W$ can be lifted along the morphism $F(W) \to G(W)$. The open scheme $W \times_X U$ of W is still dense and we have $a|_{W\times_X U} = 0$ in $G(W \times_X U)$, thus we can replace X by W and make the above asumption.

Moreover, in the two cases, there exists by hypothesis a Nisnevich covering $W \xrightarrow{p} U$ such that $b|_W = 0$.

As W is a Nisnevich covering of U, there exists a dense open subscheme U_0 of U and an open subscheme W_0 of W such that p induces an isomorphism between W_0 and U_0. Thus, $b|_{U_0} = 0$.

Applying corollary 3.22, we find a Zariski cover W' of X and a finite k-correspondance $\alpha : W' \to U_0$ such that the diagram
$$\begin{tikzcd} & W' \arrow[dl, "\alpha"'] \arrow[d] \\ U_0 \arrow[r] & X \end{tikzcd}$$
commutes up to homotopy.

Applying F to this diagram, we thus obtain that $b|_{W'} = 0$ in $F(W')$ which implies $a = 0$. □

Corollary 4.6. *Let F be a homotopy sheaf over k. Consider a smooth k-scheme X and a dense open subscheme U of X.*

Then the restriction morphism $F(X) \to F(U)$ is a monomorphism.

4.2.2. *Generic points.* Let X be a smooth S-scheme, and x be a generic point of X. The local ring $\mathcal{O}_{X,x}$ of X in x is a field. Thus it is henselian.

If we let $\mathcal{V}_x(X)$ be the category of open neighbourhoods of X, and define the localisation of X in x as the pro-object $X_x = \underset{U\in\mathcal{V}_x(X)}{\tilde{\varprojlim}} U$. Its limit is $\mathrm{Spec}(\mathcal{O}_{X,x})$. Then, as $\mathcal{O}_{X,x} = \mathcal{O}^h_{X,x}$, we have a canonical isomorphism $F(X_x) = F(X^h_x)$.

Note that $\mathcal{O}_{X,x}$ is a separable finite type extension field of k. We call such an extension a function field. We let $\mathscr{E}_k$ be the category of function fields with arrows the k-algebras morphisms.

When E/k is a function field, we put
$$\mathcal{M}^{lis}(E/k) = \left\{A \subset E \mid \mathrm{Spec}(A) \in \mathscr{L}_k, \mathrm{Frac}(A) = E\right\}$$
as an ordered set, the order coming from inclusion. This set is in fact non empty and right filtering.

We define the pro-scheme $(E) = \underset{A\in\mathcal{M}^{lis}(E/k)^{op}}{\tilde{\varprojlim}} \mathrm{Spec}(A)$. Thus, according to our general conventions, for any presheaf F over $\mathscr{L}_k$,
$$F(E) = \varinjlim_{A\in\mathcal{M}^{lis}(E/k)} F(\mathrm{Spec}(A)).$$

Moreover, for any $A \in \mathcal{M}^{lis}(E/k)$, if x denotes the generic point of $X = \operatorname{Spec}(A)$, we have canonical isomorphism $F(E) = F(X_x^h) = F(X_x)$ as $\operatorname{Spec}(E)$ is the limit of all the pro-schemes (E), X_x^h and X_x. In particular, the morphism $F \mapsto F(E)$ from Nisnevich sheaves to abelian groups is a fiber functor.

The following proposition due to Voevodsky shows the fiber functors defined above form a conservative family of "fiber functors" for homotopy sheaves.

Proposition 4.7. *Let F, G be homotopy sheaves over k, and $\eta : F \to G$ be a morphism of sheaves with transfers.*

If for any field E/k in $\mathscr{E}_k$ the induced morphism $\eta_E : F(E) \to G(E)$ is a monomorphism (resp. isomorphism), then η is a monomorphism (resp. isomorphism).

PREUVE : Indeed is is sufficient to apply the next lemma to the morphism η.

Lemma 4.8. *Let F, G be presheaves over $\pi\mathscr{L}_{\mathrm{cor},k}$ and $\eta : F \to G$ be a natural transformation.*

The following conditions are equivalent :

1. *The morphism $\eta_{\mathrm{Zar}} : F_{\mathrm{Zar}} \to G_{\mathrm{Zar}}$ between the associated Zariski sheaves over $\mathscr{L}_k$ is a monomorphism (resp. isomorphism).*
2. *For all extension E/k in $\mathscr{E}_k$, $\eta_E : F(E) \to G(E)$ is a monomorphism (resp. isomorphism).*

Proof. The fact that 1 implies 2 is evident.

Reciprocally, consider N the kernel of η in the category of presheaves with transfers. It is homotopy invariant.

Let X be a smooth irreducible k-scheme with residue field E. Obviously, we have a canonical isomorphism

$$N(E) = \varinjlim_{U \subset X} N_{\mathrm{Zar}}(U)$$

where the limit runs over the open dense subscheme of X. Then proposition 4.5 implies the canonical morphism

$$N_{\mathrm{Zar}}(X) \to \varinjlim_{U \subset X} N_{\mathrm{Zar}}(U) = N(E)$$

is a monomorphism. But $N(E)$ is the kernel of $\eta_E : F(E) \to G(E)$, thus $N(E) = 0$ and $N(X) = 0$.

We now conclude the proof by applying the same reasoning to the cokernel of η. ☐

4.3. Associated homotopy sheaf.

4.3.1. *Čech cohomology of curves.* Let k be a field and C/k be an algebraic curve. We introduce the following property for the curve C :

(N) For all finite extension L/k, $\operatorname{Pic}(C \otimes_k L) = 0$.

Remark 4.9. If this property is true for C, it is true for any open subscheme of C. If C is affine with function ring A, property (N) is equivalent (cf [GD66], 21.7.6 et 21.7.7) to the property that for any finite extension L/k, the ring $A \otimes_k L$ is factorial.

Note this property implies that for any closed subscheme Z of C nowhere dense, $\operatorname{Pic}(C \times_k Z) = 0$. We deduce from that fact the following proposition which is in fact a generalisation of [Voe00a], 5.4 :

Proposition 4.10. *Let k be a field. Consider C/k a smooth affine curve satisfying property (N), and F a presheaf over $\pi\mathscr{L}_{\mathrm{cor},k}$.*

Then for all integer $n \geq 0$, the n-th Čech cohomology group of C with coefficients in F for the Nisnevich topology is

$$\check{\mathrm{H}}^n(C; F) = \begin{cases} F(C) & \textit{if } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. First we remark that for any Nisnevich covering $W \to C$ there exists a distinguished square
$$\begin{tikzcd} U \times_X V \arrow[r, "l"] \arrow[d, "h"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$
with U and V affine such that the covering $U \sqcup V \to C$ is a refinement of $W \to C$.

Indeed, we may assume W is affine. As W/C is a Nisnevich covering, there exists a dense open subset U of X such that $W \times_C U \to U$ admits a section. As this morphism is étale, we have $W \times_C U = U \sqcup U'$. Put $Z = (C - U)_{red}$. Then $W \times_C Z$ is a finite set of closed points of W. As $W \times_C Z \to Z$ is a Nisnevich covering, any point of Z has a preimage in $W \times_C Z$ which is isomorphic to it ; that it $W \times_C Z = Z \sqcup Z'$. If we put now $V = W - Z'$, V is affine as W is regular and Z' is a finite set of points ; we have obtained our distinguished square.

Consider now a distinguished square as above. As C/k satisfies property (N), theorem 3.23 implies the complex

$$0 \to F(C) \xrightarrow{j^*+f^*} F(U) \oplus F(V) \xrightarrow{h^*-l^*} F(U \times_X V) \to 0$$

is contractile.

This implies the Čech cohomology groups associated with the covering $U \sqcup V/C$ is $F(C)$ in degree 0 and 0 in other degrees and this concludes the proposition from the remark at the begining of the proof. □

Corollary 4.11. *Let C/k be a smooth curve satisfying property (N) and F be a presheaf over $\pi\mathscr{L}_{\mathrm{cor},k}$.*

Then for all integer $n > 0$, the Nisnevich cohomology groups

$$\mathrm{H}^n(C; F_{\mathrm{Nis}}) = 0.$$

Proof. Indeed, the Nisnevich cohomology of C/k vanishes in dimension strictly greater than 1, and the Čech cohomology coincide with the usual cohomology in degree 1. □

Remark 4.12. In the hypothesis of this proposition it is not only sufficient but also necessary that C/k satisfies property (N).

Let us assume C/k satisfies $\mathrm{H}^1(C; F_{\mathrm{Nis}}) = 0$ for every presheaf F over $\pi\mathscr{L}_{\mathrm{cor},k}$.

Let $\mathbb{G}_m$ be the sheaf over $\mathscr{L}_k$ represented by $\mathbb{G}_m$. It has a canonical structure of a sheaf with transfers : let X and Y be smooth k-schemes, and α be a finite k-correspondance from X to Y. We assume X is integral and α is an integral closed subscheme of $X \times_k Y$. Let $\kappa(X)$ and $\kappa(Z)$ be the respective function fields of X and Z. Then $\kappa(Z)/\kappa(X)$ is a finite extension as $Z \to X$ is finite surjective. Let $N_{\kappa(Z)/\kappa(X)}$ be the associated norm morphism. Then we construct α^* by the commutative diagram :

$$\begin{tikzcd} \mathcal{O}_Y(Y)^\times \arrow[r] \arrow[rr, bend left, "\alpha^*"] & \mathcal{O}_Z(Z)^\times \arrow[r, dashed] \arrow[d, hook] & \mathcal{O}_X(X)^\times \arrow[d, hook] \\ & k(Z)^\times \arrow[r, "N_{k(Z)/k(X)}"] & k(X)^\times. \end{tikzcd}$$

The dotted arrow exists as $\mathcal{O}_X(X)$ is integral. These transfers are compatible with the composition of finite correspondances using the property of the norm homomorphism.[6]

Let L/k be a finite extension, $j : \mathrm{Spec}\,(L) \to \mathrm{Spec}\,(k)$ the canonical morphism. Then the sheaf with transfers $j_*j^*\mathbb{G}_m$ is still homotopy invariant and we have $H^1(C; j_*j^*\mathbb{G}_m) = \mathrm{Pic}\,(C \otimes_k L)$.

4.3.2. *The* 0*-th Čech cohomology presheaf.* Recall that from lemma 2.6, for any presheaf with transfers F over k, the 0th Čech cohomology presheaf associated to F for the Nisnevich topology has a canonical structure of a presheaf with transfers. We denot by $\check{H}^0_{tr}F$ this presheaf with transfers. Recall that for any smooth k-scheme X, $\Gamma(X; \check{H}^0_{tr}F) = \check{H}^0(X; F)$

The following proposition is the very point where our proof of the technical results concerning homotopy sheaves differs from that of [Voe00a] (especially 4.26 and 5.5).

Proposition 4.13. *Let k be field and F be a presheaf over $\pi\mathscr{L}_{\mathrm{cor},k}$.*
Then the presheaf $\check{H}^0_{tr}F$ is homotopy invariant.

Proof. Let X/k be a smooth scheme. If $s : X \to \mathbb{A}^1_X$ is the 0-section, we have to prove in fact that $s^* : \check{H}^0_{tr}F(\mathbb{A}^1_X) \to \check{H}^0_{tr}F(X)$ is a monomorphism.

We may assume X is irreducible. Applying proposition 4.5, for any non empty open subscheme U of X, the morphism $\check{H}^0_{tr}F(X) \to \check{H}^0_{tr}F(U)$ is a monomorphism. Thus in the commutative diagram below

$$\begin{tikzcd} \check{H}^0_{tr}F(\mathbb{A}^1_X) \arrow[r] \arrow[d, "s^*"'] & \varinjlim_{U\subset X} \check{H}^0_{tr}F(\mathbb{A}^1_U) \arrow[d, "\sigma"] \\ \check{H}^0_{tr}F(X) \arrow[r] & \varinjlim_{U\subset X} \check{H}^0_{tr}F(U) \end{tikzcd}$$

where U runs over the non empty open subschemes of X, the horizontal arrows are injectives and we have only to prove that σ is injective.

Denote by E the function field of X and let $\tau : \mathrm{Spec}\,(E) \to \mathrm{Spec}\,(k)$ be the canonical morphism. We let $\hat{\tau}^* : \mathscr{P}^{\mathrm{tr}}_k \to \mathscr{P}^{\mathrm{tr}}_E$ be the base change functor for presheaves with transfers (cf section 2.5.4).

Let $s_E : \mathrm{Spec}\,(E) \to \mathbb{A}^1_E$ be the 0-section. Then from proposition 2.18 and the remark that follows about functoriality, we deduce that the morphism

$$\sigma : \varinjlim_{U\subset X} \check{H}^0_{tr}F(\mathbb{A}^1_U) \to \varinjlim_{U\subset X} \check{H}^0_{tr}F(U)$$

is isomorphic to

$$s_E^* : \hat{\tau}^*\check{H}^0_{tr}F(\mathbb{A}^1_E) \to \hat{\tau}^*\check{H}^0_{tr}F(\mathrm{Spec}\,(E)).$$

Let us recall that from lemma 2.21 we have $\hat{\tau}^*\check{H}^0_{tr} = \check{H}^0_{tr}\hat{\tau}^*$. To conclude that σ is an isomorphism, it remains to apply proposition 4.10 to the curve $\mathbb{A}^1_E/E$ and to the homotopy presheaf $\hat{\tau}^*F$ over E. □

Corollary 4.14. *Let k be a field.*
For any presheaf over $\pi\mathscr{L}_{\mathrm{cor},k}$, the sheaf F_{Nis} is homotopy invariant.
In particular, the functor $a_{tr} : \mathscr{P}^{\mathrm{tr}}_k \to \mathscr{N}^{\mathrm{tr}}_k$ of corollary 2.7 induces an exact functor $a_{Htr} : \mathrm{H}\mathscr{P}^{\mathrm{tr}}_k \to \mathrm{H}\mathscr{N}^{\mathrm{tr}}_k$ which is left adjoint to the inclusion functor $\mathrm{H}\mathscr{N}^{\mathrm{tr}}_k \hookrightarrow \mathrm{H}\mathscr{P}^{\mathrm{tr}}_k$.

[6]It is also a consequence of [Dég05], 6.5 and 6.6 applied to $\mathbb{G}_m = A^0(.; K^M_*)_1$.

Corollary 4.15. *The category* $\mathrm{H}\mathscr{N}_k^{\mathrm{tr}}$ *is a Grothendieck abelian category which admits arbitrary limits.*

The inclusion functor $\mathrm{H}\mathscr{N}_k^{\mathrm{tr}} \hookrightarrow \mathscr{N}_k^{\mathrm{tr}}$ *is exact.*

Let us consider the notations introduced for proposition 4.4 in the case where the base is a field k. Then the corollary above implies that $h_0^{(1)} = h_0$. The generators of $\mathrm{H}\mathscr{N}_k^{\mathrm{tr}}$ are the elements of the (essentially small) family $\big(h_0\mathrm{L}[X]\big)_{X\in\mathscr{L}_k}$.

Note finally that we obtain also the analog results of [Voe00a] in the case of presheaves with transfers for the Zariski topology :

Corollary 4.16. *Let* F *be a homotopy presheaf over a field* k.

Then the canonical morphism $F_{\mathrm{Zar}} \to F_{\mathrm{Nis}}$ *is an isomorphism.*

Proof. From what preceds, we know F_{Nis} is a homotopy invariant sheaf over k. Thus the result follows from lemma 4.8 applied to the morphism of homotopy presheaves with transfers $F \to F_{\mathrm{Nis}}$. □

5. Homotopy invariance of cohomology

The aim of this section is to prove the following theorem :

Theorem 5.1 (Voevodsky)**.** *Let* k *be a perfect field and* F *be a homotopy sheaf over* k.

Then the Nisnevich cohomology presheaf $H^*(.;F)$ *is homotopy invariant over* $\mathscr{L}_k$.

5.1. Lower graduation.

Definition 5.2. Let S be a regular scheme and F be a homotopy presheaf with transfers over S.

We associate to F the homotopy presheaf with transfers F_{-1} over S, such that for all smooth S-scheme X,

$F_{-1}(X) = \mathrm{coKer}\left(F(\mathbb{A}^1 \times X) \to F(\mathbb{G}_m \times X)\right)$.

With this definition, we always have a split short exact sequence

$$0 \to F(\mathbb{A}^1 \times X) \xrightarrow{j^*} F(\mathbb{G}_m \times X) \to F_{-1}(X) \to 0$$

using the homotopy invariance of F ; a canonical retraction of j^* is induced by the morphism $\mathbb{A}^1 \times X \to X \xrightarrow{s_1} \mathbb{G}_m \times X$ given by projection followed by the unit section of $\mathbb{G}_m \times X/X$. By using this canonical splitting, we will always consider that $F_{-1}(X) \subset F(\mathbb{G}_m \times X)$.

As a consequence, we deduce that if F is a homotopy sheaf, F_{-1} is a homotopy sheaf.

5.2. Local purity.

5.2.1. *Relative closed pairs.* In the definition below, we introduce the analog of the definitions in 3.6 over a base.

Definition 5.3. Let S be a scheme.

A closed pair over S is a couple (X,Z) such that X is a smooth S-scheme and Z is a closed subscheme of X. We will say (X,Z) is *smooth* (resp. *has codimension* n) if Z is smooth (resp. Z is of pure codimension n in X).

A morphism of closed pair $(Y,T) \to (X,Z)$ is a couple of morphisms (f,g) which fits into the commutative diagram of schemes

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

which is cartesian on the corresponding topological spaces.

We will say (f,g) is *cartesian* (resp. *excisive*) if the preceding square is cartesian in the category of schemes (resp. $g_{red} : T_{red} \to Z_{red}$ is an isomorphism).

Remark 5.4. When given a closed immersion $i : Z \to X$ into a smooth S-scheme, we will usually identify the schemes Z and the closed subscheme $i(Z)$ of X when no confusion can arise.

5.5. The following method gives a general process to construct excisive morphisms.

Let S be a scheme. Consider (X,Z) and (X',Z) two closed pairs over S such that X and X' are étale over S.

Let Δ be the diagonal of Z over S. It is a closed subscheme of $Z \times_S Z$ and thus we identify it as a closed subscheme of $X \times_S X'$. Similarly, Δ is a closed subset of $X \times_S Z$ and $Z \times_S X'$.

Lemma 5.6. *We adopt the hypothesis and notations above.*

Define the set $\Omega = X \times_S X' - [(X \times_S Z - \Delta) \cup (Z \times_S X' - \Delta)]$, *and consider the canonical projections* $X \xleftarrow{p} \Omega \xrightarrow{q} X'$.

Then Ω *is an open subscheme of* $X \times_S X'$ *and contains* Δ *as a closed subscheme. Thus, identifying* Δ *with* Z, (Ω, Z) *is a closed pair over* S *such that* Ω/S *is étale. Moreover, the projections* p, q *induce cartesian excisive morphisms*

$$(X,Z) \xleftarrow{\ p\ } (\Omega, Z) \xrightarrow{\ q\ } (X',Z).$$

Proof. We have only to prove that Ω is open in $X \times_S X'$.

Consider the closed immersion $\iota : \Delta \to X \times_S Z$. Identifying Δ with Z, ι is a section of the étale morphism $f \times_S Z : X \times_S Z \to Z$. In particular, ι is an open immersion and $X \times_S Z - \Delta$ is a closed subscheme of $X \times_S X'$. Symetrically, we get the conclusion. $\square$

Remark 5.7. The reader can check that the preceding construction is functorial with respect to cartesian étale morphisms $(Y,T) \to (X,Z)$ and $(Y',T) \to (X',Z)$ of closed pairs over S.

Let X be a S-scheme and $s_0 : X \to \mathbb{A}^n_X$ be the 0-section. We will often consider the closed pair $(\mathbb{A}^n_X, s_0(X))$ which we will always denote by $(\mathbb{A}^n_X, X)$.

Definition 5.8. Let S be a scheme and (X,Z) a closed pair over S.

A parametrisation of (X,Z) over S is a cartesian étale morphism $(f,g) : (X,Z) \to (\mathbb{A}^{c+n}_S, \mathbb{A}^n_S)$ for a couple of integers (n,c).

Suppose given a parametrisation $(X,Z) \to (\mathbb{A}^{c+n}_S, \mathbb{A}^n_S)$. Then (X,Z) is smooth of codimension c. Moreover, X has pure dimension n over S. In particular, the integer (n,c) are uniquely determined by (X,Z).

Reciprocally, when the closed pair (X,Z) is smooth, for any point s of Z there always exists an open neighbourhood U of s in X and a parametrisation of $(U, Z \cap U)$ over S. We will speak loosely of a local parametrisation of (X,Z) at s.

5.9. The following process is the geometric base for the local purity theorem.

Suppose given a closed pair (X,Z) and a parametrisation $(u,v) : (X,Z) \to (\mathbb{A}^{c+n}_S, \mathbb{A}^n_S)$. We associate to this parametrisation a commutative square

$$\begin{tikzcd} Z \arrow[r, "s_0"] \arrow[d] & \mathbb{A}^c_Z \arrow[d, "1 \times_S v"] \\ X \arrow[r] & \mathbb{A}^{c+n}_S. \end{tikzcd}$$

where s_0 is 0-section of $\mathbb{A}^c_Z$.

This gives two closed pairs (X, Z) and $(\mathbb{A}^c_Z, Z)$ over $\mathbb{A}^{n+c}_S$.

Then from the preceding lemma, we obtain cartesian excisive morphisms

$$(X, Z) \to (\Omega, Z) \leftarrow (\mathbb{A}^c_Z, Z).$$

Remark 5.10. Consider in addition a cartesian étale morphism $(Y, T) \xrightarrow{(f,g)} (X, Z)$. Then we associate to the induced parametrisation of (Y, T) over S a closed pair (Π, T) which fits into the commutative diagram of closed pairs over S

$$\begin{tikzcd} (Y,T) \arrow[d, "(f,g)"'] & (\Pi,T) \arrow[l] \arrow[r] \arrow[d] & (\mathbb{A}^c_T, T) \arrow[d, "(1\times_S g, g)"] \\ (X,Z) & (\Omega, Z) \arrow[l] \arrow[r] & (\mathbb{A}^c_Z, Z) \end{tikzcd}$$

We note that this process allows to deduce the following structure theorem for points in the Nisnevich topology :

Corollary 5.11. *Let S be a scheme and (X, Z) a smooth pair over S.*

Then for any point s of Z, the exist an isomorphism

$$X^h_s \simeq \mathbb{A}^c_Z)^h_s$$

of pro-schemes over S which is the identity on Z^h_s. The integer c is the codimension of Z in X at s.

Proof. Indeed, as we can find a local parametrisation of (X, Z) at s, we construct from what preceds an open neighbourhood U of s in X, and excisive morphisms

$$(U, Z \cap U) \to (\Omega, Z \cap U) \leftarrow (\mathbb{A}^c_{Z\cap U}, Z \cap U).$$

This implies Ω is a Nisnevich neighbourhood of s in U (resp. $\mathbb{A}^c_{Z\cap U}$), thus in X (resp. $\mathbb{A}^c_Z$). This concludes. □

5.2.2. *The case of homotopy presheaves.* Let (X, Z) be a pair over a regular scheme S. For any point $s \in Z$, we have a canonical isomorphism

$$Z^h_s = \tilde{\varprojlim_{V \in \mathcal{V}^h_s(X)}} V \times_X Z.$$

It is natural to consider the pro-object

$$X^h_s - Z^h_s = \tilde{\varprojlim_{V \in \mathcal{V}^h_s(X)}} (V - V \times_X Z).$$

We thus have canonical morphisms of pro-objects (pro-immersions) :

$$X^h_s - Z^h_s \xrightarrow{\iota} X^h_s \leftarrow Z^h_s.$$

For any presheaf over $\mathscr{L}_S$, we consider the induced morphism

$$F(X^h_s) \to F(X^h_s - Z^h_s),$$

and denote by $F(X^h_s - Z^h_s)/F(X^h_s)$ the cokernel of ι. Note this is a little abuse of notation as ι is not necessarily a monomorphism.

Proposition 5.12. *Let S be a regular excellent scheme, (X, Z) be a smooth closed pair over S and F be a homotopy presheaf over S.*

Let s be a point of X such that Z is of codimension 1 in X at s.

Then any local parametrisation of (X, Z) at s induces a canonical isomorphism

$$F(X^h_s - Z^h_s)/F(X^h_s) \simeq F_{-1}(Z^h_s).$$

Proof. Indeed, a local parametrisation of (X, Z) at s induces from corollary 5.11 a canonical isomorphism $X_s^h \to \mathbb{A}_Z^1)_s^h$ which is the identity on Z_s^h. Thus we are reduced to the case of the closed pair $(\mathbb{A}_Z^1, Z)$.

Let V be a Nisnevich neighbourhood of s in Z. Then $\mathbb{A}_V^1$ is a Nisnevich neighbourhood of s in $\mathbb{A}_Z^1$. Thus we get a canonical morphism

$$\varinjlim_{V \in \mathcal{V}_s^h(Z)} F(\mathbb{A}_V^1 - V)/F(V) \xrightarrow{(*)} \varinjlim_{W \in \mathcal{V}_s^h(\mathbb{A}_Z^1)} F(W - W_Z)/F(W).$$

Lemma 5.13. *Let S be a regular excellent scheme and Z a smooth S-scheme. For every point s in Z, the canonical morphism described above*

$$F_{-1}(\mathbb{A}^1_{Z_s^h}) \xrightarrow{(*)} F(\mathbb{A}_Z^1)_s^h - Z_s^h)/F(\mathbb{A}_Z^1)_s^h).$$

is an isomorphism.

Let $\mathcal{Z} = \operatorname{Spec}(\mathcal{O}_{Z,s}^h)$ be the limit of Z_s^h, and $\tau : \mathcal{Z} \to S$ the canonical morphism. Note that $\mathcal{Z}$ is regular and noetherian.

As filtered inductive limits are exact, we obtain using proposition 2.18 a canonical isomorphism

$$(\hat{\tau}^* F)_{-1}(\mathbb{A}_{\mathcal{Z}}^1) \simeq F_{-1}(\mathbb{A}^1_{Z_s^h}).$$

Moreover, we can write

$$F(\mathbb{A}_Z^1)_s^h - Z_s^h)/F(\mathbb{A}_Z^1)_s^h) = \varinjlim_{V \in \mathcal{V}_s^h(Z)} \varinjlim_{W \in \mathcal{V}_s^h(\mathbb{A}_V^1)} F(W - W_Z)/F(W).$$

This writing, together with proposition 2.18, gives us a canonical isomorphism

$$\hat{\tau}^* F(\mathbb{A}_{\mathcal{Z}}^1)_s^h - \mathcal{Z})/\hat{\tau}^* F(\mathbb{A}_{\mathcal{Z}}^1) \simeq F(\mathbb{A}_Z^1)_s^h - Z_s^h)/F(\mathbb{A}_Z^1)_s^h).$$

Thus we are reduced to the case where $S = \mathcal{Z}$ is a local henselian scheme with closed point s. Indeed, the two isomorphisms just constructed are compatible with the morphism $(*)$ (cf the remark after proposition 2.18). Note also that $\mathcal{Z}$ is still a regular excellent scheme (cf [GD66], 18.6.10 and 18.7.6).

We consider the category $\mathcal{I}$ of étale morphisms $V \xrightarrow{g} \mathbb{A}_S^1$ such that V is affine and $g^{-1}(S) \to S$ is an isomorphism, with arrows the $\mathbb{A}_S^1$-morphisms. Then $\mathcal{I}$ is a final subcategory of $\mathcal{V}_s^h(\mathbb{A}_S^1)$. Indeed, let $V \xrightarrow{f} \mathbb{A}_S^1$ be a Nisnevich neighbourhood of s in $\mathbb{A}_S^1$. As S is henselian, the morphism $g : V_S \to S$ induced by f admits a section. Thus there exists an open subscheme V' of V such that $V' \cap V_S = S$. As we can always reduce V' in a neighbourhood of s, we can assume V' is affine that is $V' \in \mathcal{I}$.

Let $V \xrightarrow{f} \mathbb{A}_S^1$ be an object in $\mathcal{I}$. To conclude the lemma, we will prove that the morphism

$$F(\mathbb{A}_S^1 - S)/F(\mathbb{A}_S^1) \to F(V - V_S)/F(V)$$

induce by f is an isomorphism.

The main theorem of Zariski implies there exists an S-scheme $\bar{V}$ and morphisms

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "f"'] & \bar{V} \arrow[d, "\bar{f}"] \\ \mathbb{A}_S^1 \arrow[r, "j"] & \mathbb{P}_S^1 \end{tikzcd}$$

such that $\bar{f}$ is finite and k is an open immersion. Replacing $\bar{V}$ by the reduced closure of V in $\bar{V}$, we can assume V is dense in $\bar{V}$. As S is excellent and V is normal, we can assume furthermore that $\bar{V}$ is normal, replacing it by its normalisation.

We claim $\bar{V}/S$ is a good compactification of (V, S). Let $W = \bar{f}(\bar{V} - V)$ as a reduced closed subscheme of $\mathbb{P}_S^1$, and $W_s \subset \mathbb{P}_s^1$ be the special fiber of W. Then W_s is a finite set as it is nowhere dense in $\mathbb{P}_s^1$: we can find a regular function h

on $\mathbb{P}^1_s$ such that $D(h) \cap (W_s \cup \{0\}) = \emptyset$. Let l be an extension of h to $\mathbb{P}^1_S$. As the projection $\bar{V} \to S$ is proper, necessarily $W \cap D(l) = \emptyset$. Thus, $W \cup \{0_S\}$ is included in the affine open subset $V(h) = \mathbb{P}^1_S - D(h)$. Then $(\bar{V} - V) \cup V_S$ is included in the affine open subset $\bar{f}^{-1}(D(h))$ as required.

Finally, we can apply theorem 3.23 to the square
$$\begin{tikzcd} V - V_S \arrow[r, "l"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ \mathbb{A}^1_S - S \arrow[r, "j"] & \mathbb{A}^1_S \end{tikzcd} :$$

The short sequence

$$0 \to F(\mathbb{A}^1_S) \xrightarrow{j^*+f^*} F(\mathbb{A}^1_S - S) \oplus F(V) \xrightarrow{(h^*,-l^*)} F(V - V_S) \to 0$$

is split exact which implies the needed result by an easy diagram chase. □

5.2.3. *The case of sheaves.* For any scheme X, we let X_{Nis} be the site of étale X-schemes with the Nisnevich topology, and $\tilde{X}_{\mathrm{Nis}}$ be the topos associated to X_{Nis}.

When $f : Y \to X$ is any morphism of schemes, we have, following [MAJ73], exp. IV, a couple of adjoint functors

$$(f_*, f^*) : \tilde{Y}_{\mathrm{Nis}} \to \tilde{X}_{\mathrm{Nis}}$$

with f^* exact. When we restrict our attention to the category of abelian sheaves, the functor f_* can be classically derived on the right and induces a functor

$$Rf_* : D^b(\mathbb{Z}.\tilde{Y}_{\mathrm{Nis}}) \to D^b(\mathbb{Z}.\tilde{X}_{\mathrm{Nis}}).$$

Recall that for any $q \in \mathbb{N}$, and any (abelian) Nisnevich sheaf F_Y on Y, $R^q f_* F_Y$ is the Nisnevich sheaf associated to $U/X \mapsto H^q(Y \times_X U; F_Y)$. This implies that f_* is exact whenever f is finite as a finite scheme over a local henselian scheme is a disjoint union of local henselian schemes.

Let S be a regular scheme and F be a homotopy sheaf over S.

For any smooth S-scheme X, we will denote by F_X the restriction of F to X_{Nis}.

Remark 5.14. (1) For any U in X_{Nis}, we obviously have $H^n(U; F_X) = H^n(X; F)$.

(2) When $f : Y \to X$ is a smooth morphism, we have $f^* F_X = F_Y$.

Let more generally $(X_i)_{i \in I}$ be a pro-object of étale X-schemes affines over $\mathbb{Z}$, and $\mathcal{X}$ be its limit.

We can consider $X_\bullet = (X_i)_{i \in I}$ as a pro-object of smooth affine S-schemes. Let $\tau : \mathcal{X} \to S$ be the canonical morphism. We have defined the sheaf with transfers $\tau^* F$ over $\mathcal{X}$ in section 2.5. Recall that proposition 2.19 implies it is homotopy invariant.

Let now $\tau_X : \mathcal{X} \to X$ the canonical morphism. Then as another application of proposition 2.19, we obtain $\tau_X^* F_X = (\tau^* F)_{\mathcal{X}}$.

Remark that the coherence of the Nisnevich topos together with [MAJ73], VI, thus implies

$$H^n(X_\bullet; F) = H^n(\mathcal{X}; \tau^* F).$$

Let (X, Z) be a smooth closed pair over S of codimension 1, $j : X - Z \to X$ and $i : Z \to X$ the canonical immersions.

Applying corollary 4.6, the adjunction morphism $F_X \to j_* j^* F_X$ is a monomorphism. Let C be its cokernel in $\mathbb{Z}.\tilde{X}_{\mathrm{Nis}}$. Then the adjunction morphism $C \to i_* i^* C$ is an isomorphism as for any point $s \in X - Z$, $C(X_s^h) = 0$.

Definition 5.15. With the above notations, we define $F_{(X,Z)}$ to be the Nisnevich sheaf on Z_{Nis} equal to $i^* C$.

Thus, by the above construction, we have a canonical exact sequence of sheaves on X_{Nis}

$$0 \to F_X \to j_* F_U \to i_* F_{(X,Z)} \to 0. \tag{C}$$

For any cartesian morphism $(f,g) : (Y,T) \to (X,Z)$ there is an induced morphism

$$F_{(X,Z)} \to g_* F_{(Y,T)}$$

which makes the above exact sequence functorial.

Lemma 5.16. *Let F be a homotopy sheaf over a regular scheme S, and (X,Z) be a smooth codimension 1 closed pair over S.*

Then for any $s \in Z$, we have a canonical isomorphism

$$F_{(X,Z)}(Z_s^h) = F(X_s^h - Z_s^h)/F(X_s^h)$$

using the notations preceding proposition 5.12.

Proof. By definition, $F_{(X,Z)}(Z_s^h) = (i_* F_X)(X_s^h)$. Thus the result follows by taking fibers along X_s^h in the exact sequence (C). □

Lemma 5.17. *Let F be a homotopy sheaf over a regular scheme S, and $(f,g) : (Y,T) \to (X,Z)$ be an excisive morphism of smooth codimension 1 closed pairs over S.*

Then the canonical morphism $F_{(X,Z)} \to g_* F_{(Y,T)}$ is an isomorphism.

Proof. Let s be a point of Z. We have only to check the assertion by evaluating the sheaves on Z_s^h. As (f,g) is excisive, Y is a Nisnevich neighbourhood of s in X. Thus f induces an isomorphism $Y_t^h \to X_s^h$ with t being the point of Y such that $g(t) = s$. The result now follows from the preceding lemma. □

Let F be a homotopy sheaf over S and Z be a smooth S-scheme. For any étale Z-scheme V, let $\mathcal{I}_V$ be the category with objects the couple (U,η) such that U is an étale $\mathbb{A}^1_Z$-scheme and $\eta : V \to U_Z$ is a Z-morphisms. The arrows in $\mathcal{I}_V$ are the X-morphisms in U compatible with η. Then, by contruction of the pullback on sheaves over small Nisnevich sites, $F_{(\mathbb{A}^1_Z,Z)}$ is the sheaf associated to the presheaf

$$G : V/Z \mapsto \varinjlim_{(U,\eta)\in\mathcal{I}_V} F(U - U_Z)/F(U).$$

There is an obvious morphism $F_{-1}|_Z \to G$ of presheaves over Z_{Nis}, which thus induces a morphism of sheaves over Z_{Nis}

$$\epsilon_{(\mathbb{A}^1_Z,Z)} : F_{-1,Z} \to F_{(\mathbb{A}^1_Z,Z)}.$$

Lemma 5.18. *The morphism $\epsilon_{(\mathbb{A}^1_Z,Z)} : F_{-1}|_Z \to F_{(\mathbb{A}^1_Z,Z)}$ described above is an isomorphism.*

Proof. We have only to check the assertion on fibres. Then using the computation of lemma 5.16, it is just lemma 5.13 using the homotopy invariance of F_{-1}. □

The two previous lemma implies the important result of this section :

Corollary 5.19. *Let (X,Z) be a closed pair over S and $\rho : (X,Z) \to (\mathbb{A}^{n+1}_S, \mathbb{A}^n_S)$ a parametrisation over S.*

Let $(X,Z) \xrightarrow{p} (\Omega,Z) \xleftarrow{q} (\mathbb{A}^1_Z,Z)$ be the morphisms constructed in paragraph 5.9.

Let T be a smooth S-scheme. For any smooth S-scheme Y, we put $Y' = Y \times_S T$; this defines an endomorphism of smooth S-schemes.

Then, all the morphisms in the sequence

$$F_{(X',Z')} \xleftarrow{(p')^*} F_{(\Omega',Z')} \xrightarrow{(q')^*} F_{(\mathbb{A}^1_{Z'},Z')} \xrightarrow{\epsilon_{(\mathbb{A}^1_{Z'},Z')}} F_{-1}|_{Z'}$$

are isomorphisms.

We thus have associated to the parametrisation ρ an isomorphisms of sheaves over Z_{Nis},

$$\epsilon_{\rho,T} : F_{(X\times_k T, Z\times_k T)} \to F_{-1}|_{Z\times_k T}.$$

This isomorphism is obviously functorial in the smooth S-scheme T, using the naturality of ϕ_T with respect to T.

Remark 5.20. This isomorphism is more generally functorial in ρ in a suiteable sense, but we will not use functoriality. Note moreover that by using deformation to the normal cone, we can show at this point that ϵ_ρ does not depend on the choice of ρ. This could be used to construct such an isomorphism for any smooth closed pair of codimension 1 without requiring the existence of a global parametrisation.

5.3. **Localisation long exact sequences.** Let S be a regular scheme.

Let (X_0, Z_0) be a smooth closed pair over S of codimension 1 and $\rho : (X_0, Z_0) \to (\mathbb{A}^{n+1}_S, \mathbb{A}^n_S)$ a parametrisation over S. Let T be a smooth S-scheme and put $(X, Z) = (X_0 \times_S T, Z_0 \times_S T)$.

Let $j : U \to X$ and $i : Z \leftarrow X$ be the canonical closed embedding. The isomorphism constructed in corollary 5.19 induces a canonical exact sequence of sheaves on X_{Nis}

$$0 \to F_X \to j_*F_U \to i_*F_{-1}|_Z \to 0.$$

Recall that i_* is exact. Thus, for any étale X_0-scheme V_0, taking cohomology on $V = V_0 \times_S T$ we get a localisation long exact sequence,

$$\begin{aligned}(\mathrm{D}) \qquad \ldots \to &H^{n-1}(V; F_X) \to H^{n-1}(V; j_*F_U) \to H^{n-1}(V_Z; F_{-1}|_Z)\\ &\to H^n(V; F_X) \to H^n(V; j_*F_U) \to \ldots\end{aligned}$$

This sequence is functorial in V_0 with respect to étale X_0-morphisms and in T with respect to any S-morphisms.

Remark 5.21. We could also have considered the closed pair $(V_0, V_0 \times_X Z)$ and the induced parametrisation $(V_0, V_0 \times_X Z) \to (\mathbb{A}^{n+1}_S, \mathbb{A}^n_S)$. By the very construction, the sequence obtained is exactly the above sequence.

To conclude, we note that if we know the vanishing of $R^m j_*$ for $m > 0$, this exact sequence as the form

$$\ldots \to H^{n-1}(V; F) \xrightarrow{j^*} H^{n-1}(V - V_Z; F) \to H^{n-1}(V_Z; F_{-1}) \to H^n(V; F_X) \to \ldots$$

which is in fact the localisation exact sequence associated to (V, V_Z).

5.4. **Proof.** Let now S be the spectrum of a perfect field k.

The proof of Voevodsky consist now to recursively both prove the homotopy invariance of $H^n(.; F)$ and establish the localisation long exact sequence (for smooth divisors) in dimension less than n which, according to the above, amounts to prove the vanishing of $R^m j_*$ for $m < n$ and for any open immersion j of the complementary of a smooth divisor.

Reduction step 1.–

Let n be a positive integer.

If for any smooth k-scheme X, any point $s \in X$, and any integer $0 < i \leq n$, $H^i(\mathbb{A}^1_{X^h_s}; F) = 0$,

then for any smooth k-scheme X and any integer $0 < i \leq n$, $H^i(\mathbb{A}^1_X; F) = H^i(X; F)$.

Indeed, we let $\pi : \mathbb{A}^1_X \to X$ be the canonical projection and apply the Leray spectral sequence to π and to $F_{\mathbb{A}^1_X}$:

$$E_2^{p,q} = H^p(X; R^q\pi_*F_{\mathbb{A}^1_X}) \Rightarrow H^{p+q}(\mathbb{A}^1_X; F_{\mathbb{A}^1_X}).$$

It suffices to note now that the hypothesis implies $R^q\pi_*F_{\mathbb{A}^1_X} = 0$ if $0 < q \leq n$. Indeed, for any point $s \in X$, $R^q\pi_*F_{\mathbb{A}^1_X}(X_s^h) = H^q(\mathbb{A}^1_{X_s^h}; F)$.

Reduction step 2.– The following step is exactly the point where we need the base to be perfect a perfect field.

Let n be a positive integer.

For a closed pair (X,Z) over k, we consider the property
$P(X,Z)$: the induced morphism $H^n(\mathbb{A}^1_{X_s^h}; F) \to H^n(\mathbb{A}^1_{X_s^h - Z_s^h}; F)$ is a monomorphism.

Suppose that for any smooth closed pair (X,Z) of codimension 1 the property $P(X,Z)$ is true,

then for any smooth k-scheme X, and any point $s \in X$, $H^n(\mathbb{A}^1_{X_s^h}; F) = 0$.

We start by proving that under the assumption, $P(X,Z)$ is true for any closed pair (X,Z) over k. For this, we use precisely the following lemma of Voevodsky (cf [Voe00a], lem. 4.31) :

Lemma 5.22. *Let X be a smooth k-scheme over a perfect field k and Z a nowhere dense closed subscheme of X.*

Then for every point $s \in X$, there exists an open neighbourhood U of x in X and an increasing sequence of closed subscheme $\emptyset = Y_0 \subset \ldots \subset Y_r$ of U for $r > 0$ such that

1. *for any $1 \leq i \leq r$, $Y_i - Y_{i-1}$ is a smooth divisor of $U - Y_{i-1}$.*
2. *$Z \cap U \subset Y_r$.*

Proof. We assume X is connected and use induction on the dimension $n \geq 1$ of X. The case $n = 1$ is trivial as k is perfect.

Let $n \geq 2$ and X be a smooth n-dimensional scheme. Necessarily, there exists an open subscheme U_0 of X and a smooth relative dimension 1 morphism $p : U_0 \to Y$ to a smooth k-scheme Y.

Let Z_{sing} be the singular locus of Z. As k is perfect, $\dim(Z_{sing}) < \dim(Z) < n$. Let T be the reduced closure of $p(Z_{sing})$ in Y. We thus have $\dim(T) \leq n-2$. As Y has pure dimension $n-1$, T is nowhere dense in Y. By the inductive assumption applied to Y and T in a neighbourhood of $p(s)$, there is a neighbourhood V of $p(s)$ in Y and an increasing sequence $Y_0' \subset \ldots \subset Y_r'$ of closed subscheme of V satisfying conditions 1 and 2 for T and V.

Put $U = p^{-1}(V)$, $Y_i = p^{-1}(Y_i')$ for $0 \leq i \leq r$ and $Y_{r+1} = Y_r \cup (Z \cap U)$. Then the sequence $Y_0 \subset \ldots \subset Y_{r+1}$ of closed subscheme of U satisfies conditions 1 and 2 for Z and U. $\square$

With this lemma, we obtain easily $P(X,Z)$ for any pair (X,Z) as if $Z \subset T \subset X$, $P(X,T) \Rightarrow P(X,Z)$ and $P(X,Z)$ is a local property on X.

Fix now a smooth k-scheme X and a point $s \in X$. Let E be the fraction field[7] of $\mathcal{O}^h_{X,s}$. The pro-object $(E) = \underset{Z \subset X}{\tilde{\varprojlim}}\ X_s^h - Z_s^h$ of étales X-schemes has the scheme Spec(E) for limit.

The property $P(X,Z)$ for any Z implies the canonical morphism

$$H^n(\mathbb{A}^1_{X_s^h}; F) \to H^n(\mathbb{A}^1_{(E)}; F)$$

is a monomorphism. Let $\tau : \mathrm{Spec}\,(E) \to \mathrm{Spec}\,(k)$ be the canonical morphism. Then remark 5.14 implies $H^n(\mathbb{A}^1_{(E)}; F) = H^n(\mathbb{A}^1_E; \tau^* F)$. Thus finally, this reduction step follows using corollary 4.11.

[7]The extension field E/k, though of finite transcendantal degree, is not necessarily of finite type.

We are ready now to prove recursively on an integer $n \geq 1$ the following assertions :

(i) For any smooth closed pair (X, Z) of codimension 1, $j : X - Z \to X$ the open immersion, we have : $\forall 0 < m < n, R^m j_*(F_{X-Z}) = 0$.
(ii) For any smooth closed pair (X, Z) of codimension 1 with a given parametrisation ρ, $j : X - Z \to X$ the open immersion, and V an étale X-scheme, the localisation exact sequence (D) induces an exact sequence

$$H^{n-1}(V - V_Z; F) \to H^{n-1}(V_Z; F_{-1}) \to H^n(V; F) \xrightarrow{j^*} H^n(V - V_Z; F)$$

functorial in X with respect to étale morphisms (the parametrisation of an étale X-scheme being the parametrisation induced by ρ).
(iii) $H^n(.; F)$ is homotopy invariant.

Suppose $n = 1$. Let (X, Z) be a smooth pair of codimension 1 and ρ a parametrisation of (X, Z). Let $T = \operatorname{Spec}(k)$ or $\mathbb{A}^1_k$, $(X', Z') = (X \times_k T, Z \times_k T)$, $U' = X' - Z'$ and $j : U' \to X'$ the canonical immersion. The localisation exact sequence (D) associated with these data is

$$0 \to F(X' - Z') \xrightarrow{j^*} F(X') \to F_{-1}(X') \to H^1(X'; F) \xrightarrow{\nu} H^1(X'; j_* F_{X'-Z'}).$$

Applying the Leray spectral sequence for j and $F_{X'-Z'}$, we obtain a canonical morphism $H^1(X'; j_* F_{X'-Z'}) \xrightarrow{b} H^1(X' - Z'; F_{X'-Z'})$ which is a monomorphism and such that $j^* = b \circ \nu$. This implies that $\operatorname{Ker}(\nu) = \operatorname{Ker}(j^*)$ and we thus obtain the sequence of property (ii) for (X', Z'). Functoriality in X with respect to étale morphisms now follows from the functoriality of the sequence (D).

Let s be a point of Z. We now consider the limit of this exact sequence by replacing X by arbitrary Nisnevich neighbourhood od s in X. The functoriality in T of the sequence (D) implies that the following diagram, in which lines are exact, is commutative :

$$\begin{tikzcd}
F(X^h_s - Z^h_s) \arrow[r] \arrow[d, "\sim"'] & F_{-1}(Z^h_s) \arrow[rr] \arrow[d, "\sim"] & & 0 \\
F(\mathbb{A}^1_{X^h_s - Z^h_s}) \arrow[r] & F_{-1}(\mathbb{A}^1_{Z^h_s}) \arrow[r] & H^1(\mathbb{A}^1_{X^h_s}; F) \arrow[r, "(1)"] & H^1(\mathbb{A}^1_{X^h_s - Z^h_s}; F)
\end{tikzcd}$$

Thus (1) is a monomorphism. As any smooth closed pair (X, Z) admits locally a parametrisation, we are done by reduction steps 2 and 1.

Consider $n > 1$ and the inductive hypothesis. To prove (i) we consider $s \in X$ and prove for any $0 < q < n$, the fiber of $R^q j_* F_U$ at X^h_s is zero.

By the inductive hypothesis and corollary 2.10, $H = H^m(.; F)$ is a homotopy presheaf over k. Then proposition 5.12 implies we have an isomorphism

$$R^q j_* F_U(X^h_s) = H(X^h_s - Z^h_s) = H(X^h_s - Z^h_s)/H(X^h_s) \simeq H_{-1}(Z^h_s).$$

Let E be the fraction field of $\mathcal{O}_{Z^h_s}$. Put $(E) = \underset{T \subset Z}{\tilde{\varprojlim}}\ Z^h_s - T^h_s$. The canonical morphism $(E) \to Z^h_s$ is a pro-immersion. Thus, applying corollary 4.6, the induced morphism $H_{-1}(Z^h_s) \to H_{-1}(E)$ is a monomorphism. Indeed, Z^h_s is a point and, though E is not necessarily of finite type over k, it is the filtering union of its sub-k-extensions E' of finite type. Thus $F \mapsto F(E)$ is still a fiber functor for the Nisnevich topology on $\mathscr{L}_k$.

Let $\tau : \operatorname{Spec}(E) \to \operatorname{Spec}(k)$ be the canonical morphism. We obtain finally the following inclusion

$$H_{-1}(E) \subset H(\mathbb{G}_m \times (E)) = H^q(\mathbb{G}_m \times (E); F) = H^q(\mathbb{G}_{m,E}; \tau^* F)$$

using remark 5.14. Since the latter group is zero by corollary 4.11, we are done for (i).

For (ii) it is now sufficient to apply the same reasoning than in the case $n = 1$. Indeed property (i) and the Leray spectral sequence for j give the edge monomorphism $H^n(X; R^0 j_* F_X) \xrightarrow{b} H^n(X - Z; F_{X-Z})$.

For (iii) now, we consider a smooth closed pair (X, Z) of codimension 1 and a point $s \in Z$. It admits a parametrisation in a neighbourhood V of s, which induces a parametrisation of $(\mathbb{A}^1_V, \mathbb{A}^1_{V \cap Z})$. This parametrisation bieng fixed, we can consider the exact sequence of property (ii) for any étale V-scheme. If we take the colimit of these sequences with respect to the Nisnevich neighbourhoods of s in V, we obtain the following exact sequence

$$H^{n-1}(\mathbb{A}^1_{Z^h_s}; F_{-1}) \to H^n(\mathbb{A}^1_{X^h_s}; F) \to H^n(\mathbb{A}^1_{X^h_s - Z^h_s}; F).$$

This concludes by the inductive assumption as we can now use again reduction steps 2 and 1.

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