

Diagrams and torsors

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Introduction

Suppose that I is a small category, and that $X : I \rightarrow \mathbf{S}$ is a functor taking values in simplicial sets. Such functors are also commonly called I -diagrams of simplicial sets.

Suppose further that X is a diagram of equivalences in the sense that any morphism $\alpha : i \rightarrow j$ of I is mapped to a weak equivalence $\alpha_* : X_i \rightarrow X_j$ of simplicial sets. Then it is a fundamental and well known result of Quillen [11], [4, IV.5.7] that each pullback square

$$\begin{array}{ccc} X_i & \xrightarrow{\quad} & \operatorname{holim}_I X \\ \downarrow & & \downarrow c \\ * & \xrightarrow[i]{} & BI \end{array} \quad (1)$$

is homotopy cartesian in the sense that the simplicial set X_i is weakly equivalent to the homotopy fibre of the canonical map c over the vertex i of the nerve BI of the category I . Quillen's Theorem B, which is the original homotopy theoretic foundation for much of algebraic K -theory, is an easy consequence of this result.

The category \mathbf{S}^I of I -diagrams of simplicial sets has a model structure (originally introduced by Bousfield and Kan [1]) for which a natural transformation of $f : X \rightarrow Y$ of I -diagrams is a weak equivalence (respectively fibration) if all constituent maps $f : X_i \rightarrow Y_i$ are weak equivalences (respectively fibrations) of simplicial sets.

Let I/i be the category of morphisms $j \rightarrow i$ in I . Then the functor $I/i \rightarrow I$ which sends $j \rightarrow i$ to j determines a pullback square of simplicial set maps

$$\begin{array}{ccc} \operatorname{pb}(\operatorname{holim}_I X)_i & \xrightarrow{\quad} & \operatorname{holim}_I X \\ \downarrow & & \downarrow c \\ B(I/i) & \xrightarrow{\quad} & BI \end{array} \quad (2)$$

which is equivalent to the square (1), so that Quillen's result could be rephrased to assert that the diagram (2) is homotopy cartesian if X is a diagram of equivalences.

The category \mathbf{S}/BI of simplicial sets over BI has for objects all simplicial set maps $X \rightarrow BI$, and the commutative triangles

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & BI & \end{array}$$

are its morphisms. It is an elementary observation that \mathbf{S}/BI inherits a model structure from simplicial sets for which a morphism as above is a weak equivalence (respectively fibration, cofibration) if and only if the map $f : X \rightarrow Y$ is a weak equivalence (respectively fibration, cofibration) of simplicial sets.

It is a consequence of the fact that the model structure for simplicial sets is proper that if $p : Z \rightarrow BI$ is a fibration, and the I -diagram $\text{pb}(Z)$ is defined by the pullback

$$\begin{array}{ccc} \text{pb}(Z)_i & \longrightarrow & Z \\ \downarrow & & \downarrow p \\ B(I/i) & \longrightarrow & BI \end{array}$$

then $\text{pb}(Z)$ is a diagram of equivalences.

Write $\text{Ho}(\mathbf{S}^I)_e$ for the full subcategory of the homotopy category $\text{Ho}(\mathbf{S}^I)$ arising from the Bousfield-Kan model structure on the diagrams of equivalences. It follows in the standard way that the pullback functor pb induces a homotopy derived functor

$$R\text{pb} : \text{Ho}(\mathbf{S}/BI) \rightarrow \text{Ho}(\mathbf{S}^I)_e.$$

In effect, $R\text{pb}(X)$ is the I -diagram $\text{pb}(Z)$ where $Z \rightarrow BI$ is a (natural) fibrant replacement of $X \rightarrow BI$. The homotopy colimit functor $\underline{\text{holim}}_I$ preserves weak equivalences, and hence induces a functor

$$\underline{\text{holim}}_I : \text{Ho}(\mathbf{S}^I)_e \rightarrow \text{Ho}(\mathbf{S}/BI),$$

and it is essentially a consequence of Quillen's result (ie. that the diagrams (2) are homotopy cartesian) that the derived pullback and homotopy colimit functors together determine an equivalence of categories

$$\text{Ho}(\mathbf{S}/BI) \simeq \text{Ho}(\mathbf{S}^I)_e. \quad (3)$$

There's a new concept that can now be introduced: say that an I -diagram X is an I -torsor if X is a diagram of equivalences and the canonical simplicial set map $\underline{\text{holim}}_I X \rightarrow *$ is a weak equivalence. Write $I - \mathbf{Tors}$ for the category of I -torsors, and let $\pi_0(I - \mathbf{Tors})$ for its class of path components. Write \mathbf{Triv}/BI for the subcategory of \mathbf{S}/BI on those objects $Z \rightarrow BI$ such that Z is weakly equivalent to a point, and let $\pi_0(\mathbf{Triv}/BI)$ be its class of path components. Then the construction giving the equivalence of categories (3) specializes to a bijection

$$\pi_0(\mathbf{Triv}/BI) \cong \pi_0(I - \mathbf{Tors}).$$

At the same time, we know from [9] that there is a bijection

$$\pi_0(\mathbf{Triv}/BI) \xrightarrow{\cong} [*, BI]$$

which is defined by taking the diagram

$$* \xleftarrow{\cong} Z \rightarrow BI$$

to the obvious map $* \rightarrow BI$ in the homotopy category of simplicial sets. We have, in other words, a natural identification

$$\pi_0(I - \mathbf{Tors}) \cong [*, BI] \quad (4)$$

relating the set of morphisms from $*$ to BI in the homotopy category with path components of a suitably defined category of I -torsors.

The essential point of this paper is that all of the foregoing can be greatly generalized: simplicial sets can be replaced by the category $s\mathrm{Pre}(\mathcal{C})$ on an arbitrary small Grothendieck site \mathcal{C} , and the category of I -diagrams above can be replaced by the category $s\mathrm{Pre}(\mathcal{C})^A$ of enriched A -diagrams on a presheaf of small categories A which is enriched in simplicial sets. One can go even further, and replace the standard model structure on the category of simplicial presheaves by an f -local theory for a set of cofibrations $f : A \rightarrow B$, provided that the corresponding f -local model structure is proper. The main results are Theorem 17 and Theorem 18 in the f -local case, and Theorem 20 and Theorem 21 for the ordinary (ie. “unlocalized”) model structure on enriched diagrams of simplicial presheaves. Theorems 17 and 20 are analogues of the equivalence (3) while Theorems 18 and Theorem 21 give analogues of (4) which identify path components of suitably defined categories of A -torsors with morphisms $[*, dBA]$ in the respective homotopy categories. Here, dBA is the diagonal of the bisimplicial nerve BA which is canonically associated to the simplicial category A .

Theorems 20 and 21 are consequences of the arguments for Theorems 17 and 18, respectively, but the unlocalized case is important in its own right. In particular, the concept of A -torsor specializes to the definition of G -torsor for a presheaf of groupoids enriched in simplicial sets given in [9], which itself is the higher analogue of the classical notion of torsor for a sheaf of groups. The arguments of [9] are given within the framework of a model structure for presheaves of groupoids enriched in simplicial sets whose associated homotopy category is equivalent to the standard homotopy category of simplicial presheaves. I don’t know of a corresponding model structure for presheaves of categories enriched in simplicial sets, but it doesn’t matter: the argument for Theorem 21 which is presented here requires no such ambient model structure. As such, the result was unexpected.

The localized structure theorems Theorem 17 and Theorem 18 specialize to the case of motivic homotopy theory. Thus, one can now speak of motivic A -torsors for a presheaf of categories enriched in simplicial sets on the smooth Nisnevich site of a scheme, and identify the path components of the corresponding

category with the set of morphisms $[\ast, dBA]$ in the motivic homotopy category. Note that the whole question of what a motivic stack or a higher motivic stack should be has been avoided. The model structures which describe such objects can be written down, but they are irrelevant for the arguments presented here.

The basic outline of the argument for the main results is fairly simple, but there are technical issues to overcome. One such issue is that fibres over points lose their meaning in the simplicial presheaf context, and so one has to work relative to the full presheaf $\text{Ob}(A)$ of objects of A , rather than one fibre at a time. Thus, for example, the assertion that an enriched A -diagram X is a diagram of local equivalences has to be expressed in the requirement that the diagram (6) below is locally homotopy cartesian. Also, one uses an f -local model structure for enriched A -diagrams which is bootstrapped up from an f -local model structure on the category $s\text{Pre}(\mathcal{C})/\text{Ob}(A)$ in a predictable way, but the f -local model structure that is used for $s\text{Pre}(\mathcal{C})/\text{Ob}(A)$ does not have a naive definition. The required structure is defined “internally” here by methods which were introduced recently by Cisinski [2], [7], although one has the option of making it by hand.

1 Diagrams of simplicial sets

Suppose that A is a small category which is enriched in simplicial sets, and that X is an enriched diagram on A taking values in simplicial sets. This means that X consists of set-valued functors $X_n : A_n \rightarrow \mathbf{Set}$ which respect the simplicial identities in the obvious way, or equivalently that X consists of a simplicial set map $\pi : X_0 \rightarrow \text{Ob}(A)$ and an action

$$\begin{array}{ccc} X_0 \times_s \text{Mor}(A) & \xrightarrow{\phi} & X_0 \\ \downarrow & & \downarrow \pi \\ \text{Mor}(A) & \xrightarrow{t} & \text{Ob}(A) \end{array}$$

which is associative and respects the composition law and identities of A . Here $s, t : \text{Mor}(A) \rightarrow \text{Ob}(A)$ are source and target maps, respectively, and the pull-back diagram

$$\begin{array}{ccc} X_0 \times_s \text{Mor}(A) & \longrightarrow & \text{Mor}(A) \\ \downarrow & & \downarrow s \\ X_0 & \xrightarrow{\pi} & \text{Ob}(A) \end{array}$$

defines the object $X_0 \times_s \text{Mor}(A)$.

Natural transformations $f : X \rightarrow Y$ between enriched A -diagrams are the

obvious thing, namely commutative diagrams of simplicial set maps

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & Y_0 \\ & \searrow \pi & \swarrow \pi \\ & \text{Ob}(A) & \end{array}$$

which respect the actions of A on X and Y . The category of enriched A -diagrams with natural transformations between them will be denoted by $s\mathbf{Set}^A$.

Say that a map $f : X \rightarrow Y$ of enriched A -diagrams is a weak equivalence (respectively fibration) if the simplicial set map $f : X_0 \rightarrow Y_0$ is a weak equivalence (respectively fibration) of simplicial sets over $\text{Ob}(A)$. A cofibration in the category of enriched A -diagrams is a map which has the left lifting property with respect to all trivial fibrations.

Lemma 1. *With these definitions, $s\mathbf{Set}^A$ has the structure of a proper closed simplicial model category.*

Proof. It is easy to verify the claim when A is discrete in the sense that it consists entirely of identity morphisms on the elements of $\text{Ob}(A)$. In that case, diagrams are simplicial sets defined over $\text{Ob}(A)$ and the model structure of the statement of the Lemma is just the standard model structure for simplicial sets over $\text{Ob}(A)$.

Write $j : \text{Ob}(A) \rightarrow A$ for the inclusion of the object set in the simplicial category A , and interpret $\text{Ob}(A)$ to be a discrete simplicial category, so that j is a functor. $\text{Ob}(A)$ -diagrams, from this point of view, are just simplicial set maps $Z \rightarrow \text{Ob}(A)$. Write j^*Z for the left Kan extension of the functor Z along j . Then the standard construction of the left Kan extension functor implies directly that the “object set” of j^*Z has the form

$$j^*Z_0 = Z \times_s \text{Mor}(A).$$

over $\text{Ob}(A)$. It follows that the left Kan extension functor preserves cofibrations and weak equivalences. The usual small object arguments give the desired closed model structure. \square

Say that an enriched A -diagram X is a diagram of equivalences if all morphisms $\alpha : i \rightarrow j$ of the category A_0 induce weak equivalences $X_i \rightarrow X_j$.

There is a canonical morphism $A_0 \rightarrow A$ of categories enriched in simplicial sets, and X is a diagram of equivalences if and only if the restriction of X to A_0 is a diagram of equivalences. The condition that X is a diagram of equivalences

is also equivalent to the requirement that the composite square

$$\begin{array}{ccc}
X \times_s \text{Mor}(A_0) & \longrightarrow & \text{Mor}(A_0) \\
\downarrow & & \downarrow \\
X \times_s \text{Mor}(A) & \longrightarrow & \text{Mor}(A) \\
\downarrow \phi & & \downarrow t \\
X & \xrightarrow{\pi} & \text{Ob}(A)
\end{array}$$

is homotopy cartesian.

The homotopy colimit $\text{holim}_A X$ is naturally a bisimplicial set, as is the classifying object BA , and there is a canonical bisimplicial set map $c : \text{holim}_A X \rightarrow BA$. In more detail, BA is the bisimplicial set consisting of the nerves BA_n of the categories A_n , $(\text{holim}_A X)_n$ is the nerve of the translation category for the functor $X_n : A_n \rightarrow \mathbf{Set}$ defined by the n -simplices of X , and the canonical map $(\text{holim}_A X)_n \rightarrow BA_n$ is induced by the canonical functor from the translation category for X_n to A_n . In horizontal degree n , the map c is the projection

$$X_0 \times_{\text{Ob}(A)} BA_n \rightarrow BA_n,$$

where the map $v_0 : BA_n \rightarrow \text{Ob}(A)$ is induced by the ordinal number map $\mathbf{0} \rightarrow \mathbf{n}$ which picks off the vertex 0.

Here is the enriched diagram version of a well known result of Quillen. The first proof of this result was given by Moerdijk [10].

Lemma 2. *Suppose that the enriched A -diagram X is a diagram of equivalences. Then the canonical pullback square*

$$\begin{array}{ccc}
X_0 & \longrightarrow & \text{holim}_A X \\
\pi \downarrow & & \downarrow c \\
\text{Ob}(A) & \longrightarrow & BA
\end{array}$$

is a homotopy cartesian diagram of bisimplicial sets.

Remark 3. The claim in Lemma 2 that the diagram in the statement of the Lemma is homotopy cartesian refers to the closed model structure on the category of bisimplicial sets for which the cofibrations are the inclusions and the weak equivalences are the maps $X \rightarrow Y$ which induce a weak equivalence $d(X) \rightarrow d(Y)$ of the associated diagonal simplicial sets [2], [7]. The left adjoint d^* of the diagonal functor preserves cofibrations and weak equivalences [4, IV.3.12], and it follows that a square diagram of bisimplicial sets is homotopy cartesian if and only if the associated diagram of diagonal simplicial sets is homotopy cartesian.

Proof of Lemma 2. The displayed statement is equivalent to the more common assertion that $X(a) = \pi^{-1}(a)$ is the homotopy fibre of c over $a \in BA_{0,0}$ for each $a \in \text{Ob}(A)$, and this is what we prove. We shall assume that the Lemma holds when A is an ordinary category — this statement has a separate proof [4, IV.5.7], and is Quillen’s original result.

The homotopy colimit functor preserves weak equivalences of enriched A -diagrams. In effect, the induced map $\underline{\text{holim}}_A X \rightarrow \underline{\text{holim}}_A Y$ is the comparison of fibre products

$$X_0 \times_{\text{Ob}(A)} BA_n \rightarrow Y_0 \times_{\text{Ob}(A)} BA_n,$$

in horizontal degree n , which is a weak equivalence since $\text{Ob}(A)$ is discrete (but see also Lemma 12 below). Note also that if $f : X \rightarrow Y$ is a weak equivalence of enriched diagrams, then X is a diagram of equivalences if and only if Y is a diagram of equivalences. It follows that we can assume that X is a fibrant enriched A -diagram.

Form the pullback diagrams of bisimplicial sets

$$\begin{array}{ccc} c^{-1}(\sigma) & \longrightarrow & \underline{\text{holim}}_A X \\ \downarrow & & \downarrow c \\ \Delta^{m,n} & \xrightarrow{\sigma} & BA \end{array}$$

where σ varies over the bisimplices of BA . By standard techniques [4, p.244] (and by using the case of the Lemma where A is an ordinary category) it suffices to show that all maps of bisimplices

$$\begin{array}{ccc} \Delta^{r,s} & \xrightarrow{\theta \times \gamma} & \Delta^{m,n} \\ \tau \searrow & & \swarrow \sigma \\ & BA & \end{array}$$

induce weak equivalences $c^{-1}(\tau) \rightarrow c^{-1}(\sigma)$.

Suppose that $v : \mathbf{0} \rightarrow \mathbf{s}$ is an ordinal number map, and consider the corresponding diagram

$$\begin{array}{ccccc} \Delta^{r,0} & \xrightarrow{1 \times v} & \Delta^{r,s} & & \\ \theta \times 1 \downarrow & & \theta \times \gamma \downarrow & \searrow \tau & \\ \Delta^{m,0} & \xrightarrow{1 \times \gamma(v)} & \Delta^{m,n} & \xrightarrow{\sigma} & BA \end{array}$$

The map $\underline{\text{holim}}_A X \rightarrow BA$ is a fibration in each horizontal degree since X is fibrant, and the map $1 \times v : \Delta^{r,0} \rightarrow \Delta^{r,s}$ is a weak equivalence in each horizontal degree, so that pulling back $\underline{\text{holim}}_A X$ over the maps $1 \times v$ and $1 \times \gamma(v)$ induces weak equivalences of pullbacks. The pullback of $\underline{\text{holim}}_A X$ over the map $\theta \times 1$ is a weak equivalence, by the assumption that X is a diagram of equivalences over A_0 . \square

Suppose that $y \in \text{Ob}(A)$, let $\pi : Z \rightarrow dBA$ be a simplicial set map, and define a simplicial set $\text{pb}(Z)_y$ by the pullback diagram

$$\begin{array}{ccc} \text{pb}(Z)_y & \xrightarrow{c} & Z \\ \downarrow & & \downarrow \pi \\ dB(A/y) & \longrightarrow & dBA \end{array}$$

Here A/y is the simplicial category, which is the slice category A_n/y in each simplicial degree n . The simplicial set $A(y, z)$ of homomorphisms from y to z in A determines a simplicial set map

$$\text{pb}(Z)_y \times A(y, z) \rightarrow \text{pb}(Z)_z,$$

giving $\text{pb}(Z)$ the structure of an enriched A -diagram.

An n -simplex of $\text{pb}(Z)_y$ consists of a triple

$$(x, \sigma : a_0 \rightarrow \cdots \rightarrow a_n, \alpha : a_n \rightarrow y)$$

where $x \in Z_n$, $\pi(x) = \sigma$, σ is a string of arrows of length n in A_n , and α is a morphism of A_n . An n -simplex of the product

$$\text{pb}(Z)_{y_0} \times A(y_0, y_1) \times A(y_1, y_2) \times \cdots \times A(y_{m-1}, y_m)$$

therefore consists of a pair

$$(x, a_0 \rightarrow \cdots \rightarrow a_n \rightarrow y_0 \rightarrow \cdots \rightarrow y_m)$$

where the string of arrows lives in A_n and $\pi(x)$ is the simplex $a_0 \rightarrow \cdots \rightarrow a_n$ of BA_n . The fibre over x of the bisimplicial set map $\text{holim}_A \text{pb}(Z) \rightarrow Z$ can therefore be identified with the simplicial set $B(a_n/A_n)$, which is contractible. We have therefore proved the following result:

Lemma 4. *Suppose that $\pi : Z \rightarrow dBA$ is a simplicial set map, where A is a category enriched in simplicial sets. Then the canonical map $\text{holim}_A \text{pb}(Z) \rightarrow Z$ is a weak equivalence.*

Suppose that X is an enriched A -diagram. There is an identification

$$\text{pb } d(\text{holim}_A X)_a \cong d(\text{holim}_{A/a} X|_{A/a})$$

and each category A_n/a has a terminal object. It follows that there is a weak equivalence

$$\text{pb } d(\text{holim}_A X)_a \simeq X(a).$$

This equivalence is realized by the canonical natural transformation

$$\text{holim}_{A/a} X|_{A/a} \rightarrow X_a$$

which is defined in the various simplicial degrees by the simplicial set maps

$$\bigsqcup_{a_0 \rightarrow \cdots \rightarrow a_n \rightarrow a} (X_{a_0})_n \rightarrow (X_a)_n$$

given in summands by the maps $(X_{a_0})_n \rightarrow (X_a)_n$. The corresponding maps assemble to give a natural weak equivalence

$$\psi : \mathrm{pb} \, d(\mathrm{holim}_A X) \rightarrow X \quad (5)$$

of enriched A -diagrams. This map ψ is split over $\mathrm{Ob}(A)$ by a map $X_0 \rightarrow \mathrm{pb} \, d(\mathrm{holim}_A X)_0$ which is given by the maps $X_a \rightarrow \mathrm{pb} \, d(\mathrm{holim}_A X)_a$ defined by the pullback diagrams

$$\begin{array}{ccc} X_a & \longrightarrow & \mathrm{pb} \, d(\mathrm{holim}_A X)_a \\ \downarrow & & \downarrow \\ * & \xrightarrow{1_a} & dB(A/a) \end{array}$$

2 Diagrams of simplicial presheaves

Suppose now that A is a presheaf of small categories on a small Grothendieck site \mathcal{C} . An enriched A -diagram X taking values in simplicial presheaves consists of a simplicial presheaf map $\pi : X_0 \rightarrow \mathrm{Ob}(A)$ and an action

$$\begin{array}{ccc} X_0 \times_s \mathrm{Mor}(A) & \xrightarrow{\phi} & X_0 \\ \downarrow & & \downarrow \pi \\ \mathrm{Mor}(A) & \xrightarrow{t} & \mathrm{Ob}(A) \end{array}$$

which is associative and respects the composition law and identities of A . Natural transformations $f : X \rightarrow Y$ between enriched A -diagrams in simplicial presheaves are commutative diagrams of simplicial presheaf maps

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & Y_0 \\ \pi \searrow & & \swarrow \pi \\ & \mathrm{Ob}(A) & \end{array}$$

which respect the actions of A on X and Y . The category of enriched A -diagrams with natural transformations between them will be denoted by $s\mathrm{Pre}(\mathcal{C})^A$.

Say that a map $f : X \rightarrow Y$ of enriched A -diagrams is a local weak equivalence (respectively global fibration) if the simplicial set map $f : X_0 \rightarrow Y_0$ is a local weak equivalence (respectively global fibration) of simplicial presheaves over $\mathrm{Ob}(A)$. A cofibration in the category of enriched A -diagrams is a map which has the left lifting property with respect to all trivial fibrations.

Lemma 5. *With these definitions, $s\text{Pre}(\mathcal{C})^A$ has the structure of a proper closed simplicial model category.*

Proof. It is easy to verify the claim when A is discrete. In that case, diagrams are simplicial presheaves defined over $\text{Ob}(A)$ and the model structure of the statement of the Lemma is just the standard model structure for simplicial presheaves over $\text{Ob}(A)$.

Write $j : \text{Ob}(A) \rightarrow A$ for the inclusion of the object set in the simplicial category A , and write j^*Z for the left Kan extension of the functor Z along j . Then j^*Z has the form

$$j^*Z_0 = Z \times_s \text{Mor}(A).$$

over $\text{Ob}(A)$. It follows that the left Kan extension functor preserves cofibrations; it preserves local weak equivalences by a Boolean localization argument. The usual small object arguments give the desired closed model structure. \square

Say that an enriched A -diagram X is a diagram of local equivalences if the square

$$\begin{array}{ccc} X_0 \times_s \text{Mor}(A_0) & \longrightarrow & \text{Mor}(A_0) \\ \phi|_{A_0} \downarrow & & \downarrow t \\ X & \xrightarrow{\pi} & \text{Ob}(A) \end{array} \quad (6)$$

is homotopy cartesian in the category of simplicial presheaves. Then we have the following analogue of Lemma 2:

Lemma 6. *Suppose that the enriched A -diagram X is a diagram of local equivalences. Then the canonical pullback square*

$$\begin{array}{ccc} X_0 & \longrightarrow & \text{holim}_A X \\ \pi \downarrow & & \downarrow c \\ \text{Ob}(A) & \longrightarrow & BA \end{array}$$

is a homotopy cartesian diagram of bisimplicial presheaves.

Remark 7. We use the closed model structure on the category of bisimplicial presheaves for which the cofibrations are the inclusions and the weak equivalences are the maps $X \rightarrow Y$ which induce a local weak equivalence $d(X) \rightarrow d(Y)$ of the associated diagonal simplicial presheaves [7]. The left adjoint d^* of the diagonal functor preserves cofibrations and local weak equivalences (the latter by a Boolean localization argument), and it follows that a square diagram of bisimplicial presheaves is homotopy cartesian if and only if the associated diagram of diagonal simplicial presheaves is homotopy cartesian.

Proof. Lemma 6 follows from Lemma 2, by a Boolean localization argument. \square

Suppose that the enriched A -diagram X is a diagram of local equivalences, and let

$$j : d(\underline{\text{holim}}_A X) \rightarrow Fd(\underline{\text{holim}}_A X)$$

be a choice of fibrant model in objects over dBA . Then the induced map

$$X_0 \rightarrow \text{Ob}(A) \times_{dBA} Fd(\underline{\text{holim}}_A X)$$

is a local weak equivalence — this is just a restatement of Lemma 6. The canonical map $\text{Ob}(A) \rightarrow dBA$ factors as a composite of canonical maps

$$\text{Ob}(A) \xrightarrow{\simeq} \text{pb}(BA)_0 \rightarrow BA$$

and there are corresponding pullback diagrams

$$\begin{array}{ccccc} X_0 & \longrightarrow & \text{pb } d(\underline{\text{holim}}_A X)_0 & \longrightarrow & d(\underline{\text{holim}}_A X) \\ \downarrow \simeq & & \downarrow j_* & & \downarrow \\ \text{Ob}(A) \times_{BA} Fd(\underline{\text{holim}}_A X) & \longrightarrow & \text{pb}(Fd(\underline{\text{holim}}_A X))_0 & \longrightarrow & Fd(\underline{\text{holim}}_A X) \end{array}$$

The map $X_0 \rightarrow \text{pb } d(\underline{\text{holim}}_A X)_0$ is a sectionwise weak equivalence since it splits the map induced by the weak equivalence ψ of (5) over $\text{Ob}(A)$. The map

$$\text{Ob}(A) \times_{BA} F(\underline{\text{holim}}_A X) \rightarrow \text{pb}(F(\underline{\text{holim}}_A X))_0$$

is the pullback of a local weak equivalence along a global fibration, and is therefore a local weak equivalence. It follows that the induced map j_* is a local weak equivalence, so that we have proved the following:

Lemma 8. *Suppose that the enriched A -diagram X is a diagram of local equivalences, and let*

$$j : d(\underline{\text{holim}}_A X) \rightarrow Fd \underline{\text{holim}}_A X$$

be a fibrant model of the corresponding homotopy colimit over dBA . Then the maps

$$X \xleftarrow{\psi} \text{pb } d(\underline{\text{holim}}_A X) \xrightarrow{j_*} \text{pb}(Fd(\underline{\text{holim}}_A X)).$$

are weak equivalences of $s\text{Pre}(\mathcal{C})^A$.

3 Internal localization

Let $f : * \rightarrow I$ be a map of simplicial presheaves on a site \mathcal{C} and form the f -local theory on $s\text{Pre}(\mathcal{C})$ in the style of [3] or [6].

To recall, we say that a simplicial presheaf X is f -injective (or f -fibrant) if X is globally fibrant and if the map $X \rightarrow *$ has the right lifting property with respect to all maps

$$(A \times I) \cup_A B \rightarrow B \times I$$

which are induced by cofibrations $A \rightarrow B$. It suffices to restrict the class of cofibrations to all cofibrations of the form $Y \subset L_U \Delta^n$, where L_U denotes the left adjoint of the U -sections functor $X \mapsto X(U)$. A map $f : X \rightarrow Y$ is an f -equivalence if it induces a weak equivalence of function complexes $\mathbf{hom}(Y, Z) \rightarrow \mathbf{hom}(X, Z)$ for all f -injective Z , and then f -fibrations are defined by a right lifting property with respect to all f -trivial cofibrations. An f -trivial cofibration (respectively fibration) is a cofibration (respectively fibration) which is also an f -equivalence. Then with these definitions and the standard simplicial category structure on $s\text{Pre}(\mathcal{C})$, the category of simplicial presheaves on \mathcal{C} has the structure of a proper closed simplicial model category. The closed simplicial model structure is obtained in [3], and properness is proved in [6].

Suppose that K is a simplicial presheaf on \mathcal{C} . It is essentially obvious that the category $s\text{Pre}(\mathcal{C})/K$ of simplicial presheaf maps $X \rightarrow K$ has a f -local (proper, simplicial) model structure in which a map

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow & \swarrow \\ & K & \end{array}$$

is a f -equivalence (respectively cofibration, f -fibration) if and only if the map $g : X \rightarrow Y$ is an f -equivalence (respectively cofibration, f -equivalence) of $s\text{Pre}(\mathcal{C})$. We shall call this the external f -local structure.

There is also an “internally defined” f -local structure on $s\text{Pre}(\mathcal{C})/K$. Say that a map $g : X \rightarrow Y$ over K (ie. of $s\text{Pre}(\mathcal{C})/K$) is injective if it is a global fibration and has the right lifting property with respect to all maps

$$(A \times_K I) \cup_A B \rightarrow B \times_K I$$

for all cofibrations $A \rightarrow B$ over K . Here the fibre product $A \times_K I$ is the product over K of the object $A \rightarrow K$ and the projection $K \times I \rightarrow K$.

Remark 9. We have tacitly identified I with its pullback $K \times I \rightarrow K$ in $s\text{Pre}(\mathcal{C})/K$. There is an isomorphism $A \times_K I \cong A \times I$ over K which induces the K -structure map

$$A \times I \xrightarrow{pr} A \rightarrow K$$

on $A \times I$. Note that an arbitrary composite $A \times I \rightarrow X \rightarrow K$ may not give the object $A \times I$ the product structure over K , so that maps $A \times_K I \rightarrow X$ over K are somewhat special.

There is a functorial injective replacement functor $j : X \rightarrow \mathcal{L}X$ functor which enjoys the usual properties [7, Lemma 37]. One says that $g : X \rightarrow Y$ is an internal f -equivalence if the induced map $g_* : \mathcal{L}X \rightarrow \mathcal{L}Y$ is a local weak equivalence of $s\text{Pre}(\mathcal{C})/K$. Equivalently g is an internal f -equivalence if and only if g_* is a homotopy equivalence in $s\text{Pre}(\mathcal{C})/K$, or a sectionwise weak equivalence of $s\text{Pre}(\mathcal{C})$. Cofibrations are monomorphisms, and internal f -fibrations are defined by a right lifting property with respect to all internal f -trivial cofibrations.

Proposition 10. *The definitions of internal f -equivalence, cofibration and internal f -fibration give the category $s\text{Pre}(\mathcal{C})/K$ of simplicial presheaves over K the structure of a cofibrantly generated proper closed simplicial model category.*

Proof. Write iK for the category whose objects consist of pairs (x, U) such that U is an object of the underlying site \mathcal{C} and $x : \Delta^n \rightarrow K(U)$ is an n -simplex of $K(U)$. A morphism $(\theta, \phi) : (x, U) \rightarrow (y, V)$ of iK is a pair consisting of a morphism $\phi : V \rightarrow U$ of \mathcal{C} and an ordinal number map $\theta : \mathbf{n} \rightarrow \mathbf{m}$ such that the diagram of simplicial set maps

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta_*} & \Delta^m \\ x \downarrow & & \downarrow y \\ K(U) & \xrightarrow{\phi^*} & K(V) \end{array}$$

commutes. If Y is a presheaf on iK then the family of sets

$$(Y_*)_n(U) = \bigsqcup_{x \in K_n(U)} Y(x, U)$$

defines a simplicial presheaf map $Y_* \rightarrow K$. The assignment $Y \mapsto Y_*$ defines an equivalence of categories between presheaves of sets on the category iK and the category $s\text{Pre}(\mathcal{C})/K$ of simplicial presheaves over K .

The box category \square (see [7]) acts on $s\text{Pre}(\mathcal{C})/K$, by associating to $Y \rightarrow K$ and the product category $\mathbf{1}^{\times n}$ the object

$$Y \otimes \square^n = Y \times (\Delta^1)^{\times n} \xrightarrow{pr} Y \rightarrow K.$$

Let $A_i \rightarrow B_i$ be a set of generating trivial cofibrations for the category of simplicial presheaves on \mathcal{C} , and let S be the set consisting of all maps

$$A_i \rightarrow B_i \rightarrow K,$$

together with the map

$$K \xrightarrow{(1_K, f)} K \times I \rightarrow K$$

Then the (\otimes, S) structure given by Theorem 46 of [7] is the required model structure. The properness of this structure is a consequence of Theorem 47 of [7]. \square

The model structure on $s\text{Pre}(\mathcal{C})/K$ which is given by Proposition 10 will be called the internal f -local structure.

Remark 11. 1) If K is a point (ie. the terminal simplicial presheaf), then the internal f -local structure is the standard f -local structure on $s\text{Pre}(\mathcal{C})$.

2) Every internal f -equivalence over a simplicial presheaf K is an external f -equivalence.

The point of introducing the internal f -local structure over an object K is essentially the following:

Lemma 12. *Suppose that W is a presheaf of sets, identified with a constant simplicial presheaf on the site \mathcal{C} , and suppose that $h : K \rightarrow W$ is a map of simplicial presheaves. Then the pullback functor which sends $X \rightarrow W$ to the projection $K \times_W X \rightarrow K$ preserves internal f -equivalences.*

Proof. Pulling back along h preserves local weak equivalences, by a Boolean localization argument.

Pulling back along h preserves cofibrations, products and all colimits, and therefore sends a map $(W \times_K I) \cup B \rightarrow B \times_K I$ to the map

$$((L \times_K W) \times_L I) \cup (L \times_K B) \rightarrow (L \times_K B) \times_L I.$$

It follows that if $j : X \rightarrow \mathcal{L}X$ is an injective replacement for X over K , then the induced map $j_* : L \times_X K \rightarrow L \times_K \mathcal{L}X$ is an internal f -equivalence over L . If $\mathcal{L}X \rightarrow \mathcal{L}Y$ is a sectionwise weak equivalence of globally fibrant objects over K , then the induced map $L \times_K \mathcal{L}X \rightarrow L \times_K \mathcal{L}Y$ is a sectionwise weak equivalence of globally fibrant objects over L . \square

Remark 13. Lemma 43 of [7] asserts in the case at hand that if a map $p : X \rightarrow Y$ is of simplicial presheaves over K is injective and the object $Y \rightarrow K$ is injective, then p is an f -fibration. It follows that all injective objects are f -fibrant in the internal f -local structure of Proposition 10.

Example 14. Suppose that U is an object of \mathcal{C} , and consider the forgetful map $j_U : \mathcal{C}/U \rightarrow \mathcal{C}$. If X is a simplicial presheaf on \mathcal{C} , write $X|_U$ for the composition

$$(\mathcal{C}/U)^{op} \xrightarrow{j_U} \mathcal{C}^{op} \xrightarrow{X} \mathbf{S}$$

Recall from [8] that the category $s\text{Pre}(\mathcal{C})/U$ is equivalent to $s\text{Pre}(\mathcal{C}/U)$. In effect, a functor $F : s\text{Pre}(\mathcal{C})/U \rightarrow s\text{Pre}(\mathcal{C}/U)$ is defined for an object $\pi : X \rightarrow U$ by setting

$$F(\pi)(\phi : V \rightarrow U) = \pi^{-1}(\phi) \subset X(V),$$

while a functor $G : s\text{Pre}(\mathcal{C}/U) \rightarrow s\text{Pre}(\mathcal{C})/U$ is defined by setting

$$G(X)(V) = \bigsqcup_{\phi : V \rightarrow U} X(\phi),$$

and the functors F and G are inverse to each other up to natural isomorphism. If I is a simplicial presheaf on \mathcal{C} , then there is a natural isomorphism $G(I|_U) \cong I \times U$ of objects over U . One can check that the equivalence of categories

$$F : s\text{Pre}(\mathcal{C})/U \simeq s\text{Pre}(\mathcal{C}/U) : G$$

induces an equivalence of the internal f -local structure on $s\text{Pre}(\mathcal{C})/U$ with the $f|_U$ -local structure on $s\text{Pre}(\mathcal{C}/U)$.

Say that a map $g : X \rightarrow Y$ of enriched A -diagrams in $s\text{Pre}(\mathcal{C})$ is an f -equivalence (respectively f -fibration) if the induced map

$$\begin{array}{ccc} X_0 & \xrightarrow{g} & Y_0 \\ \pi_X \searrow & & \swarrow \pi_Y \\ & \text{Ob}(A) & \end{array}$$

is an internal f -equivalence (respectively internal f -fibration) of simplicial presheaves over $\text{Ob}(A)$. An f -cofibration is a map which has the left lifting property with respect to all maps which are simultaneously f -fibrations and f -equivalences. We have the following analogue of Lemma 5:

Lemma 15. *With the definitions of f -equivalence, f -fibration and f -cofibration given above, the category $s\text{Pre}(\mathcal{C})^A$ of enriched A -diagrams has the structure of a proper closed simplicial model category.*

Proof. Write $j : \text{Ob}(A) \rightarrow A$ for the inclusion of the discrete subcategory of objects of A , and observe that the left Kan extension j^*Z of an object $Z \rightarrow \text{Ob}(A)$ (ie. an enriched functor $\text{Ob}(A) \rightarrow s\text{Pre}(\mathcal{C})$) has objects defined by the map $Z \times_s \text{Mor}(A) \rightarrow \text{Ob}(A)$ arising from the pullback diagram

$$\begin{array}{ccc} Z \times_s \text{Mor}(A) & \longrightarrow & \text{Mor}(A) \\ \downarrow & \searrow \text{dotted} & \downarrow s \\ Z & \longrightarrow & \text{Ob}(A) \end{array}$$

It follows from Lemma 12 that the left Kan extension functor $Z \mapsto j^*Z$ preserves internal f -equivalences; this functor also preserves cofibrations. The functor $X \mapsto X_0$ preserves colimits. It follows that applying the left Kan extension functor j^* to the generating sets of cofibrations and trivial cofibrations for the internal f -local structure on $s\text{Pre}(\mathcal{C})/\text{Ob}(A)$ gives corresponding sets of generators for $s\text{Pre}(\mathcal{C})^A$. \square

Say that an enriched A -diagram X is a diagram of f -equivalences if there is an f -fibrant replacement $j : X \rightarrow Z$ of enriched A -diagrams such that the diagram

$$\begin{array}{ccc} Z_0 \times_{\text{Ob}(A), s} \text{Mor}(A_0) & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \pi_Z \\ \text{Mor}(A_0) & \xrightarrow{t} & \text{Ob}(A) \end{array}$$

is a homotopy cartesian diagram of simplicial presheaves. Note that any two (internal) f -fibrant replacements of X are globally fibrant and sectionwise equivalent over $\text{Ob}(A)$, so that the choice of f -fibrant replacement doesn't matter.

Lemma 16. *Suppose that X is a diagram of local equivalences in the category of enriched A -diagrams. Then X is a diagram of f -equivalences.*

Proof. We can suppose that $X_0 \rightarrow \text{Ob}(A)$ is a global fibration. We are presuming that the diagram

$$\begin{array}{ccc} X_0 \times_{\text{Ob}(A),s} \text{Mor}(A_0) & \longrightarrow & X_0 \\ \downarrow & & \downarrow \pi_X \\ \text{Mor}(A_0) & \xrightarrow{t} & \text{Ob}(A) \end{array}$$

is homotopy cartesian in the category of simplicial presheaves. In particular, the induced map of global fibrations

$$\begin{array}{ccc} X_0 \times_{\text{Ob}(A),s} \text{Mor}(A_0) & \xrightarrow{\cong} & \text{Mor}(A_0) \times_{\text{Ob}(A),t} X_0 \\ & \searrow & \swarrow \\ & \text{Mor}(A_0) & \end{array}$$

is a weak equivalence of globally fibrant objects over $\text{Mor}(A)$. If $X \rightarrow Z$ is an f -fibrant replacement for X , then the induced map

$$\begin{array}{ccc} Z_0 \times_{\text{Ob}(A),s} \text{Mor}(A_0) & \longrightarrow & \text{Mor}(A_0) \times_{\text{Ob}(A),t} Z_0 \\ & \searrow & \swarrow \\ & \text{Mor}(A_0) & \end{array}$$

is a comparison of internal f -fibrant replacements for the diagram above, by Lemma 12, and is therefore a weak equivalence of globally fibrant objects over $\text{Mor}(A_0)$. In particular, Z is a diagram of local equivalences. \square

If X is an A -diagram then the simplicial presheaf of n -simplices $(\underline{\text{holim}}_A X)_n$ of the homotopy colimit sits in a pullback diagram

$$\begin{array}{ccc} (\underline{\text{holim}}_A X)_n & \longrightarrow & BA_n \\ \downarrow & & \downarrow v_0 \\ X_0 & \longrightarrow & \text{Ob}(A) \end{array}$$

It follows from Remark 11 and Lemma 12 that the assignment

$$X \mapsto (\underline{\text{holim}}_A X)_n$$

preserves f -equivalences (which are defined internally over $\text{Ob}(A)$.) Any f -equivalence of enriched A -diagrams $X \rightarrow Y$ therefore induces an f -equivalence

of simplicial presheaves $d(\underline{\text{holim}}_A X) \rightarrow d(\underline{\text{holim}}_A Y)$, by properness of the f -local model structure.

Suppose that X is a diagram of f -equivalences with f -fibrant replacement $j : X \rightarrow Z$, and let $d(\underline{\text{holim}}_A X) \rightarrow Fd(\underline{\text{holim}}_A X)$ be a natural choice of external f -fibrant replacement over dBA . Then in the diagram

$$\begin{array}{ccccc} X & \xleftarrow{\simeq} & \text{pb } d(\underline{\text{holim}}_A X) & \longrightarrow & \text{pb } Fd(\underline{\text{holim}}_A X) \\ j \downarrow & & \downarrow & & \simeq \downarrow j_* \\ Z & \xleftarrow{\simeq} & \text{pb } d(\underline{\text{holim}}_A Z) & \xrightarrow{\simeq} & \text{pb } Fd(\underline{\text{holim}}_A Z) \end{array}$$

the indicated maps are local weak equivalences of enriched A -diagrams. This follows partly from Lemma 8, which gives the equivalences on the bottom row. It follows that the natural maps

$$X \leftarrow \text{pb } d(\underline{\text{holim}}_A X) \rightarrow \text{pb } Fd(\underline{\text{holim}}_A X) \quad (7)$$

are f -equivalences of enriched A -diagrams if X is a diagram of f -equivalences.

If $X \rightarrow dBA$ is an object of $s\text{Pre}(\mathcal{C})/dBA$ with external f -fibrant model $j : X \rightarrow FX$ over dBA , then $\text{pb}(FX)$ is a diagram of local equivalences and hence a diagram of f -equivalences by Lemma 16. There is, furthermore, a natural diagram

$$\begin{array}{ccc} d(\underline{\text{holim}}_A \text{pb}(X)) & \xrightarrow{\simeq} & X \\ j_* \downarrow & & \downarrow j \\ d(\underline{\text{holim}}_A \text{pb}(FX)) & \xrightarrow{\simeq} & FX \end{array} \quad (8)$$

in which the indicated maps are local weak equivalences, and of course j is an external f -equivalence.

Write $\text{Ho}(s\text{Pre}(\mathcal{C})^A)_{f,e}$ for the full subcategory on diagrams of f -equivalences in the homotopy category $\text{Ho}(s\text{Pre}(\mathcal{C})^A)_f$ associated to the f -local structure on the category of enriched A -diagrams of simplicial presheaves. Write

$$\text{Ho}(s\text{Pre}(\mathcal{C})/dBA)_f^{ext}$$

for the homotopy category associated to the external f -local structure on the category of simplicial presheaves over dBA . We have seen that the homotopy colimit functor $X \mapsto d(\underline{\text{holim}}_A X)$ takes f -equivalences of enriched A -diagrams to external f -equivalences; it therefore induces a functor

$$\underline{\text{holim}} : \text{Ho}(s\text{Pre}(\mathcal{C})^A)_{f,e} \rightarrow \text{Ho}(s\text{Pre}(\mathcal{C})/dBA)_f^{ext}.$$

On account of Lemma 16, the assignment $Z \mapsto \text{pb } F(Z)$ defines a functor

$$R\text{pb} : \text{Ho}(s\text{Pre}(\mathcal{C})/dBA)_f^{ext} \rightarrow \text{Ho}(s\text{Pre}(\mathcal{C})^A)_{f,e},$$

which, as the notation suggests, is a homotopy derived functor associated to the functor pb .

The natural equivalences displayed in (7) and (8) are used to prove the following result:

Theorem 17. *The functors pb and $\underline{\text{holim}}_A$ determine an equivalence of categories*

$$R\text{pb} : \text{Ho}(s\text{Pre}(\mathcal{C})/dBA)_f^{ext} \xrightarrow{\sim} \text{Ho}(s\text{Pre}(\mathcal{C})^A)_{f,e} : \underline{\text{holim}}_A.$$

An f -local A -torsor is defined to be an diagram of f -equivalences X on A such that the canonical map $d(\underline{\text{holim}}_A X) \rightarrow *$ is an f -equivalence. Write $f - \mathbf{Tors}_A$ for the full subcategory of $s\text{Pre}(\mathcal{C})^A$ on the class of A -torsors.

The homotopy colimit functor (by definition) induces a functor

$$\underline{\text{holim}} : f - \mathbf{Tors}_A \rightarrow \mathbf{Triv}/dBA.$$

where \mathbf{Triv}/dBA is the category of objects $Z \rightarrow dBA$ such that the map $Z \rightarrow *$ is an f -equivalence of simplicial presheaves. Recall that the functor $R\text{pb} : s\text{Pre}(\mathcal{C})/dBA \rightarrow s\text{Pre}(\mathcal{C})^A$ is defined by $R\text{pb}(Z) = \text{pb } F(Z)$, where $j : Z \rightarrow FZ$ is a natural f -fibrant replacement. Then $R\text{pb}(Z)$ is a diagram of f -equivalences, and $R\text{pb}(Z)$ is an A -torsor if $Z \rightarrow dBA$ is a member of \mathbf{Triv}/BA . It follows that the functor $R\text{pb}$ induces a functor

$$R\text{pb} : \mathbf{Triv}/BA \rightarrow f - \mathbf{Tors}_A.$$

Then Lemma 4 and Lemma 8 together imply the following:

Theorem 18. *The functors $\underline{\text{holim}}$ and $R\text{pb}$ induce a bijection*

$$\pi_0(f - \mathbf{Tors}_A) \cong \pi_0(\mathbf{Triv}/dBA),$$

and there is a bijection

$$\pi_0(\mathbf{Triv}/dBA) \xrightarrow{\cong} [*, dBA].$$

In the statement of Theorem 18, $[*, dBA]$ denotes morphisms in the f -local homotopy category for simplicial presheaves on \mathcal{C} . The displayed map

$$\pi_0(\mathbf{Triv}/dBA) \rightarrow [*, dBA].$$

is induced by associating to an element

$$* \xleftarrow{\sim} Z \rightarrow dBA$$

the corresponding map $* \rightarrow dBA$ in the f -local homotopy category. Lemma 8 of [9] (the proof of which is essentially elementary and will not be reproduced here) asserts that this map is a bijection, since the f -local structure on $s\text{Pre}(\mathcal{C})$ is right proper.

Example 19. Suppose that S is a scheme of finite dimension, and let $(Sm|_S)_{Nis}$ be the Grothendieck site of smooth schemes over S endowed with the Nisnevich topology, as in [6]. The motivic homotopy category is obtained by localizing the category $s\text{Pre}(Sm|_S)_{Nis}$ of simplicial presheaves on the smooth Nisnevich site at a rational point $* \rightarrow \mathbb{A}^1$ of the affine line over S . If A is a presheaf of categories enriched in simplicial sets then the f -local A -torsors are more properly called motivic A -torsors, and Theorem 18 identifies the path components of the category of motivic A -torsors with the maps $[*, dBA]$ in the motivic homotopy category.

The internal motivic model structure arising from Proposition 10 for objects $X \rightarrow K$ over a fixed simplicial presheaf K is of interest in its own right. It coincides with the external motivic structure on simplicial presheaves over K in the case that K is a “discrete” object which is represented by an S -scheme U .

There is nothing special about the localized model structures appearing in Theorem 17 and Theorem 18, outside of the critical underlying properness assumption for the model structures. For example, localization at the map $f : * \rightarrow I$ could be replaced by localization at a set of maps $f_j : * \rightarrow I_j$, and this set could be empty.

In particular, the derived pullback functor Rpb and the homotopy colimit functors make sense for diagrams of local equivalences. Write $\text{Ho}(s\text{Pre}(\mathcal{C})^A)_e$ for the full subcategory of the homotopy category of enriched A -diagrams on the diagrams of local equivalences. Then we have the following:

Theorem 20. *The functors pb and $\underline{\text{holim}}_A$ determine an equivalence of categories*

$$Rpb : \text{Ho}(s\text{Pre}(\mathcal{C})/dBA) \xrightarrow{\sim} \text{Ho}(s\text{Pre}(\mathcal{C})^A)_e : \underline{\text{holim}}_A.$$

Say that an enriched A -diagram X is an A -torsor if X is a diagram of equivalences and the canonical map $d(\underline{\text{holim}}_A X) \rightarrow *$ is a local weak equivalence. In other words the map $d(\underline{\text{holim}}_A X) \rightarrow dBA$ lands in the category \mathbf{Triv}/dBA consisting of objects $Z \rightarrow dBA$ such that the canonical map $Z \rightarrow *$ is a local equivalence.

Theorem 21. *The functors $\underline{\text{holim}}_A$ and Rpb induce a bijection*

$$\pi_0(\mathbf{Tors}_A) \cong \pi_0(\mathbf{Triv}/dBA),$$

and there is a bijection

$$\pi_0(\mathbf{Triv}/dBA) \cong [*, dBA].$$

Remark 22. Theorem 17 of [9] is the special case of Theorem 21 corresponding to enriched diagrams on presheaves of groupoids enriched in simplicial sets. All such diagrams are diagrams of equivalences.

Remark 23. Theorem 20 and Theorem 21 concern the case of “localization” at an empty set of maps, in which the potential distinction between external and internal model structures for simplicial presheaves over a fixed object K disappears.

References

- [1] A.K. Bousfield and D.M. Kan, *Homotopy limits completions and localizations*, Springer Lecture Notes in Math. **304** (2nd corrected printing), Springer-Verlag, Berlin-Heidelberg-New York (1987).
- [2] D-C. Cisinski, *Les préfaisceaux comme modèles des types d'homotopie*, Thèse de doctorat de l'Université Paris VII (2002).
- [3] P.G. Goerss and J.F. Jardine, *Localization theories for simplicial presheaves*, Canad. J. Math. **50**(5) (1998), 1048–1089.
- [4] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Math. **174**, Birkhäuser, Basel-Boston-Berlin (1999).
- [5] J.F. Jardine, *Boolean localization, in practice*, Doc. Math. **1** (1996), 245–275.
- [6] J.F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552.
- [7] J.F. Jardine, *Categorical homotopy theory*, Preprint (2003), to appear in Hmology Homotopy Appl.
- [8] J.F. Jardine, *Fibred sites and stack cohomology*, Preprint (2004).
- [9] J.F. Jardine and Z. Luo, *Higher principal bundles*, Preprint (2004), to appear in Math. Proc. Cambridge Philos. Soc.
- [10] I. Moerdijk, *Bisimplicial sets and the group completion theorem*, “Algebraic K-Theory: Connections with Geometry and Topology”, NATO ASI Series C, **279**, Kluwer (1989), 225–240.
- [11] D. Quillen, *Higher algebraic K-theory, I*, Springer Lecture Notes in Math. **341** (1973), 85–147.