DELETING AND INSERTING FIXED-POINT SETS ON DISKS UNDER THE STRONG GAP CONDITION

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Abstract. It is a basic problem to determine manifolds which can occur as the fixed-point sets of smooth actions on spheres of a given finite group. This article answers to the problem in the case where the finite group is a perfect group or a nilpotent Oliver group. We obtain the answer as an application of a new deleting and inserting theorem which is formulated to delete (or insert) fixed-point sets from (or to) disks with smooth actions of finite groups. One of keys to the proof is an equivariant interpretation of the surgery theory developed by S. E. Cappell and J. L. Shaneson to get homology equivalences.

1. Introduction

Throughout this paper, let $G$ be a finite group. It is a basic problem to describe necessary and sufficient conditions of manifolds $F$ which are realizable as the fixed-point sets of smooth actions on specific manifolds $M$, namely $F = M^G$. The study of this problem goes back to a famous result by P. A. Smith [27]. He showed that if $G$ is a $p$-group for a prime $p$, namely $|G| = p^m$ with a nonnegative integer $m$, then the fixed-point set of any $Z_p$-acyclic $G$-CW complex of finite dimension is itself $Z_p$-acyclic, where $Z_p = \mathbb{Z}/p\mathbb{Z}$. A. Edmonds and R. Lee [7, Proposition (3.2)] showed that if $G$ is a finite group with a normal 2-Sylow subgroup then the fixed-point set of any $Z_2$-acyclic compact smooth $G$-manifold is stably complex. What these two results and the converse showed by L. Jones [8, Theorems 1.1 and 2.1] together amount to is that if $G$ is a nontrivial $p$-group then a compact smooth manifold can be the fixed-point set of a smooth $G$-action on a disk if and only if it is stably complex and $Z_p$-acyclic. For a finite group $G$ not of prime-power order, B. Oliver [22] solved the problem in the case where ambient manifolds $M$ are disks.

It is interesting to study the problem in the case where ambient manifolds are spheres. Concise surveys of relevant results obtained so far are described in [20] and [25, §8]. We studied the problem in [20] under the condition that each connected component of $F$ is simply connected or stably parallelizable. In the study, it turned out important to develop methods to delete (or insert) some connected components of the fixed-point set from (or to) a given smooth $G$-manifold: namely, deleting and inserting theorems. This paper


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provides a new deleting and inserting theorem on disks as Theorem 1.1 and determines
the manifolds which are realizable as the fixed-point sets of smooth actions on spheres of
nontrivial perfect groups in Theorem 1.3 as well as nilpotent Oliver groups in Theorem 1.4
without assuming the condition mentioned above. Keys to the proof of Theorem 1.1 are
the equivariant interpretation (Lemmas 2.3 and 2.5) of Cappell-Shaneson’s surgery theory
[3] for homology equivalences, the observation (Proposition 2.2 and Theorem 2.6) of the
canonical homomorphism
\[ \Gamma^h_{2k}(\mathbb{Z}[G \rtimes \pi] \to \mathbb{Z}(p)[G]) \rightarrow L^h_{2k}(\mathbb{Z}(p)[G]), \]
the induction theory [5], [6], [16], [9], and the equivariant connected-sum operation associated
with an element of the Burnside ring \( \Omega(G) \). Here \( p \) is a prime, \( \pi \) is a finite group
of order prime to \( p \), \( \mathbb{Z} \) denotes the ring of integers, and \( \mathbb{Z}(p) \) denotes the localization of \( \mathbb{Z} \)
at \( p \); namely,
\[ \mathbb{Z}(p) = \{ x/y \in \mathbb{Q} \mid x, y \in \mathbb{Z}, (y, p) = 1 \}. \]

In order to state our results, we introduce several families of finite groups which have
already appeared in [23], [22], [10], [20], and so on. Let \( p \) and \( q \) be primes. We say
that \( G \) is a mod-\( p \) cyclic (resp. mod-\( p \) q-hyperelementary, mod-\( p \) hyperelementary) group
if there is a normal subgroup \( P \) of \( G \) such that \( P \) is a p-group, where \( P \) may be the
trivial group, and \( G/P \) is cyclic (resp. q-hyperelementary, hyperelementary). Thus, if
\( G \) is a mod-\( p \) q-hyperelementary group, then there is a normal series \( P \triangleleft H \triangleleft G \)
such that \( P \) is a \( p \)-group, \( H/P \) is cyclic, and \( G/H \) is a \( q \)-group. If \( G \) is mod-\( p \) cyclic (resp.
mod-\( p \) q-hyperelementary, mod-\( p \) hyperelementary) for some prime \( p \), then we say that
\( G \) is mod-\( \mathcal{P} \) cyclic (resp. mod-\( \mathcal{P} \) q-hyperelementary, mod-\( \mathcal{P} \) hyperelementary). A finite
group \( G \) is called an Oliver group if \( G \) is not a mod-\( \mathcal{P} \) hyperelementary group. By [23], \( G \)
is an Oliver group if and only if \( G \) can smoothly act on a disk without fixed points. By
[10], it is also equivalent to the case where \( G \) can smoothly act on a sphere with exactly
one fixed point. The subgroup \( G^{(q)} \) of \( G \) is defined as the smallest normal subgroup of
\( G \) such that \( G/G^{(q)} \) is a q-group, and \( G^{(q)} \) is called the Dress subgroup of type \( q \) of \( G \).
Concerning subgroups of \( G \), we use the notation:
\[
\begin{align*}
\mathcal{S}(G) & \quad \text{the set of all subgroups of } G \\
\mathcal{P}(G) & \quad \text{the set of all subgroups of } G \text{ of prime-power order} \\
\mathcal{G}^1(G) & \quad \text{the set of all mod-}\mathcal{P}\text{ cyclic subgroups of } G \\
\mathcal{G}(G) & \quad \text{the set of all mod-}\mathcal{P}\text{ hyperelementary subgroups of } G \\
\mathcal{L}(G) & \quad \text{the set of all subgroups of } G \text{ containing some Dress subgroups}
\end{align*}
\]
These sets are regarded as \( G \)-sets by conjugations.

Let \( X \) and \( Y \) be compact smooth \( G \)-manifolds. For a subgroup \( H \) of \( G \), let \( Y^H \) denote
the \( H \)-fixed-point set of \( Y \) and \( Y^H = H \) the subset
\[ \{ y \in Y^H \mid G_y = H \}, \]
where $G_y$ is the isotropy subgroup of $G$ at $y$. For $\alpha \in \pi_0(Y^H)$, let $Y^H_\alpha$ denote the underlying space of $\alpha$.

**Theorem 1.1.** Let $G$ be a finite Oliver group and $Y$ a smooth $G$-manifold such that the underlying manifold of $Y$ is diffeomorphic to the disk of dimension $n \geq 5$ and $Y^G \neq \emptyset$. Let $F_1, \ldots, F_t$ denote the connected components of $Y^G$, and let $n_1, \ldots, n_t$ be nonnegative integers. Suppose the following:

(1.1.1) $\dim Y^P > 2(\dim Y^H + 1)$ for any $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P \subset H$.
(1.1.2) $\dim Y^{-H} \geq 3$ for any $H \in \mathcal{G}^1(G)$.
(1.1.3) $\dim Y^P \geq 5$ for any $P \in \mathcal{P}(G)$.
(1.1.4) $\pi_1(Y^P)$ is finite and of order prime to $|P|$ for any $P \in \mathcal{P}(G)$.
(1.1.5) For $1 \leq i, j \leq t$, $n_i$ coincides with $n_j$, if some connected component $Y^H_\alpha$ of $Y^H$, $H \in \mathcal{L}(G)$, contains both $F_i$ and $F_j$.
(1.1.6) For $1 \leq i \leq t$, $n_i$ is equal to 1 if some connected component $Y^H_\alpha$ of $Y^H$, $H \in \mathcal{L}(G)$, contains $F_i$ and $\partial Y^H_\alpha \neq \emptyset$.

Then there exists a smooth $G$-action on the disk $D$ of dimension $n$ such that

(i) $\partial D$ is $G$-diffeomorphic to $\partial Y$,
(ii) $D^G$ has the form of the disjoint union of copies of $F_i$’s:

\[ D^G = \bigsqcup_{i=1}^{t} \prod_{j=1}^{n_i} F_{i,j} \quad (each \ F_{i,j} \ is \ diffeomorphic \ to \ F_i), \]

(iii) the normal bundle $\nu(F_{i,j}, D^n)$ is $G$-isomorphic to $\nu(F_i, Y)$.

Furthermore if $Y^H$ (resp. $Y^P$) is connected (resp. simply connected) for an element $H \in \mathcal{G}^1(G)$ (resp. $P \in \mathcal{P}(G)$), then one can choose the $G$-action so that $D^H$ (resp. $D^P$) is connected (resp. simply connected) for the subgroup $H$ (resp. $P$).

If $n_i, 1 \leq i \leq t$, in the theorem are all equal to 1 then the conclusion of the theorem is evidently valid by taking $Y$ as $D$. If $n_i \neq 1$ holds for an $i$, where $1 \leq i \leq t$, then $F_i \cap \partial Y = \emptyset$ holds for the $i$ by (1.1.6).

**Remark 1.2.** Let $Y$ be as in Theorem 1.1. Then $Y^P$ is connected and orientable for all $P \in \mathcal{P}(G)$. So, it follows that

\[ \dim Y^H_\alpha \equiv \dim Y^H_\beta \mod 2 \]

for all $H \in \mathcal{G}^1(G)$ and $\alpha, \beta \in \pi_0(Y^H)$.

We will see significance of Theorem 1.1 in the study of smooth actions of gap Oliver groups on spheres. A finite group $G$ is called a gap group if there exists a real $G$-representation space $V$ of finite dimension such that $\dim V^P > 2 \dim V^H$ for any $P \in \mathcal{P}(G)$ and any $H \in \mathcal{S}(G)$ with $P \subset H$, and $V^K = 0$ for any $K \in \mathcal{L}(G)$. If $G$ is a nontrivial perfect group or a nilpotent group whose order is divisible by at least 3 distinct primes then $G$ is a gap group. For further information of gap groups, the reader can refer to [21] and [28]. After P. Smith, various authors studied the problem: which closed manifolds
$F$ can occur as the $G$-fixed-point sets of smooth actions on spheres. Here we mean by a closed manifold a compact (smooth) manifold without boundary. An answer for $G$ in a large family of gap Oliver groups is given in [20] under the assumption that each connected component of $F$ is simply connected or stably parallelizable. Making use of Theorem 1.1 together with results in [22], [18], [19] and [20], one can remove this assumption and obtain a satisfactory answer to the problem in the case where $G$ is a nontrivial perfect group or a nilpotent Oliver group.

The following classification of finite groups not of prime-power order was introduced by B. Oliver [22] in order to determine the $G$-fixed-point sets of smooth $G$-actions on disks.

Type (A): $G$ has a subquotient group isomorphic to a dihedral group of order $2pq$ with distinct primes $p$ and $q$.

Type (B): $G$ contains an element $g$ not of prime-power order that is conjugate to $g^{-1}$, and $G$ is not of Type (A).

Type (C): $G$ contains an element $g$ not of prime-power order, $G$ is not of Types (A)−(B), and $G_2 \nsubseteq G$.

Type (D): $G$ contains an element $g$ not of prime-power order, $G$ is not of Types (A)−(B), and $G_2 \nsubseteq G$.

Type (E): $G$ does not contain an element not of prime-power order and $G_2 \nsubseteq G$.

Type (F): $G$ does not contain an element not of prime-power order and $G_2 \leq G$.

Here $G_2$ is a 2-Sylow subgroup of $G$. For a smooth manifold $F$, let $T(F)$ denote the tangent bundle of $F$, $c : \widetilde{KSp}(F) \to \tilde{K}(F)$ the complexification and $r : \tilde{K}(F) \to \tilde{KO}(F)$ the realification.

If $G$ is a nontrivial perfect group, then $G$ is not a mod-$\mathcal{P}$ hyperelementary group, hence $G$ is an Oliver group, $G$ is of even order, and any 2-Sylow subgroup of $G$ is not normal in $G$. Thus neither Type (D) nor Type (F) occurs for a nontrivial perfect group $G$.

In the case where $G$ is a nilpotent group, $G$ is an Oliver group if and only if $G$ contains at least 3 noncyclic Sylow subgroups. A quotient group $G/N$ of $G$ is called a cyclic $pqr$-quotient if $G/N$ is a cyclic group of order $pqr$ for some distinct primes $p$, $q$, and $r$. A nilpotent Oliver group is necessarily a group of Type (D) with a cyclic $pqr$-quotient.

The following two theorems are improvements of Theorems 4 and 5 in [20] (cf. [25, Theorem 9.2]).

**Theorem 1.3.** Let $G$ be a nontrivial perfect group and $F$ a closed smooth manifold. Then there exists a smooth $G$-action on a sphere $\Sigma$ such that $\Sigma^G = F$ and $\Sigma^F \neq \Sigma^G$ for any Sylow subgroup $P$ of $G$ if and only if the following condition is fulfilled respectively to the type:

Type (A): No restriction on $F$.

Type (B): $c([T(F)]) \in c(\widetilde{KSp}(F)) + \text{tor}(\tilde{K}(F))$.

Type (C): $[T(F)] \in r(\tilde{K}(F)) + \text{tor}(\tilde{KO}(F))$. 


Type (E): $[T(F)] \in \text{tor}(\tilde{K}_O(F))$.

**Theorem 1.4.** Let $G$ be an Oliver group of Type (D) with a cyclic $pqr$-quotient and $F$ a closed smooth manifold. Then there exists a smooth $G$-action on a sphere $\Sigma$ such that $\Sigma^G = F$ and $\Sigma^F \neq \Sigma^G$ for any Sylow subgroup $P$ of $G$ if and only if $F$ is stably complex.

The rest of the present paper is organized as follows. In Section 2, we interpret Cappell-Shaneson’s surgery theory in equivariant settings as Lemmas 2.3 and 2.5. In the even dimensional case, the surgery-obstruction group is not necessarily Wall’s group. Proposition 2.2 and Theorem 2.6 provide situation where the surgery obstruction essentially lies in Wall’s group. This enables us to apply Dress’ induction theorem to (Cappell-Shaneson’s) equivariant surgery obstructions. Section 3 is devoted to explanation of equivariant connected sum of $G$-framed maps. In Section 4 we construct a $G$-framed map $(f,b)$ from a disk $Y$ specified with a smooth $G$-action. Performing equivariant surgery of $(f,b)$, we prove Theorem 1.1 in Section 5. Finally, Section 6 is devoted to the proofs of Theorems 1.3 and 1.4.

**2. G-SURGERY OBSTRUCTION FOR HOMOLOGY EQUIVALENCES**

A ring with unit is a ring with a unique multiplicative identity element $1$. Unless otherwise stated, a homomorphism of rings with unit is a ring homomorphism preserving the unit $1$. Let $A$ be a ring with unit. We mean by an involution — on $A$ a map $A \rightarrow A$ satisfying $\overline{a} = a$, $\overline{a + b} = \overline{a} + \overline{b}$, and $\overline{ab} = \overline{b} \overline{a}$ for all $a, b \in A$. Let $\Lambda$ be also a ring with unit and involution. Let $\mathcal{F} : A \rightarrow \Lambda$ be an involution-preserving, locally epic homomorphism of rings with unit: namely, for (finitely many) arbitrary elements $b_1, \ldots, b_k \in \Lambda$ there exists a unit $u$ of $\Lambda$ such that $ub_1, \ldots, ub_k$ lie in $\mathcal{F}(A)$ (the notion is defined in [3, p.288]).

Let $\lambda$ stand for the symmetry $1$ or $-1$. In this paper we give the definition of the abelian group $\Gamma^H_4(\mathcal{F})$ by using left modules. Readers who need the definition of the group by using right modules can refer to [3, §1]. Let $\text{min}_\lambda(A)$ denote the minimal (quadratic) form parameter: namely,

$$\text{min}_\lambda(A) = \{a - \lambda \overline{a} \mid a \in A\}.$$ 

We mean by a $\lambda$-form over $\mathcal{F}$ a triple $\alpha = (H, \varphi, \mu)$, where $H$ is a finitely generated left $A$-module, $\varphi : H \times H \rightarrow A$ is a biadditive map and $\mu : H \rightarrow A/\text{min}_\lambda(A)$ is a map satisfying the following properties (compare them with [3, p.286, (Q1)–(Q6)]):

(Q1') $\varphi(ax, by) = b\varphi(x, y)\overline{a}$,
(Q2') $\varphi(x, y) = \lambda \varphi(y, x)$,
(Q3') $\varphi(x, x) = \mu(x) + \lambda \mu(x)$,
(Q4') $\mu(x + y) - \mu(x) - \mu(y) = \varphi(x, y)$ in $A/\text{min}_\lambda(A)$,
(Q5') $\mu(ax) = a\mu(x)\overline{a}$,
(Q6') $H_\lambda := \Lambda \otimes_A H$ is a stably free $\Lambda$-module (without specifying any stable basis), and the map

$$\text{Ad}_\varphi : H_\lambda \rightarrow \text{Hom}_\lambda(H_\lambda, \Lambda)$$
given by $\text{Ad} \varphi_\Lambda(u)(v) = \varphi_\Lambda(u, v)$ is a bijection,
where $a, b \in A$, $x, y \in H$, $u, v \in H_\Lambda$, $\mu(x) \in A$ is a lifting of $\mu(x)$, and $\varphi_\Lambda : H_\Lambda \times H_\Lambda \to \Lambda$ is the map induced from $\varphi$.

The $\alpha$ above induces a (nonsingular) $\lambda$-quadratic module $\alpha_\Lambda = (H_\Lambda, \varphi_\Lambda, \mu_\Lambda)$ over $\Lambda$, where $\mu_\Lambda : H_\Lambda \to \Lambda/\min_\Lambda(\Lambda)$. Let $-\alpha$ denote the $\lambda$-form $(H, -\varphi, -\mu)$ over $\mathcal{F}$.

We say that the $\lambda$-form $\alpha$ over $\mathcal{F}$ is strongly equivalent to zero, and write $\alpha \approx 0$, if there exists a presubkernel $K$: namely, $K$ is an $\Lambda$-submodule of $H$ such that $\varphi(K, K) = \{0\}$, $\mu(K) = \{0\}$, the image $K'$ of $K_\Lambda$ in $H_\Lambda$ is a stably $\Lambda$-free, $\Lambda$-direct summand of $H_\Lambda$, $K'$ coincides with the submodule

$$K'^\perp = \{x \in H_\Lambda \mid \varphi_\Lambda(y, x) = 0 \text{ for all } y \in K'\},$$

and the map $H_\Lambda / K' \to \text{Hom}_\Lambda(K', \Lambda)$ induced from $\text{Ad} \varphi_\Lambda$ is an isomorphism. It is easy to show that

$$\alpha \perp -\alpha \approx 0$$

for any $\lambda$-form $\alpha$ over $\mathcal{F}$, where $\perp$ stands for the orthogonal sum. Let $H(A^s)$ denote the $\lambda$-hyperbolic $\Lambda$-module of rank $2s$ (cf. [12, p.468], [2, p.280], [1, p.7]). It can be regarded as a $\lambda$-form over $\mathcal{F}$, and is strongly equivalent to zero. We say that two $\lambda$-forms $\alpha$ and $\beta$ over $\mathcal{F}$ are stably equivalent and write $\alpha \sim \beta$ if $\alpha \perp (-\beta) \perp \gamma \approx 0$ for some $\gamma \approx 0$. The following lemma is a key to understanding Cappell-Shaneson’s surgery theory.

**Lemma 2.1** ([3, Lemma 1.3]). Let $\alpha$ be a $\lambda$-form over $\mathcal{F}$. If $\alpha \approx 0$, then $\alpha \perp H(A^s) \approx 0$ for some nonnegative integer $s$.

Let $\Gamma^h(\mathcal{F})$ denote the set of all stable equivalence classes of $\lambda$-forms over $\mathcal{F}$. We can see that $\Gamma^h(\mathcal{F})$ is an abelian group under orthogonal sum (cf. [3, p.287]). For an even integer $n = 2k$, let $\Gamma^h_n(\mathcal{F})$ denote the group $\Gamma^h_{(-1)^n}(\mathcal{F})$.

We say that a homomorphism $f : A \to A'$ of rings with unit is strongly locally epic if for (finitely many) arbitrary elements $b_1, \ldots, b_k \in A'$ there exists a unit $u$ of $A'$ such that all $u, ub_1, \ldots, ub_k$ belong to $f(A)$.

**Proposition 2.2.** Let $A'$ be a ring with unit and involution, and let $f : A \to A'$ and $\mathcal{F}' : A' \to \Lambda$ be involution-preserving, locally epic homomorphisms of rings with unit such that $\mathcal{F} = \mathcal{F}' \circ f$. If $f : A \to A'$ is a monomorphism (resp. strongly locally epic monomorphism) such that $f(\min_\Lambda(A)) = \min_\Lambda(A') \cap f(A)$, then the canonically induced map $\Gamma^h_\Lambda(\mathcal{F}) \to \Gamma^h_\Lambda(\mathcal{F}')$: namely, the map of changing rings, is a monomorphism (resp. isomorphism).

**Proof.** We regard $A$ as a subring of $A'$ via $f$.

**Injectivity:** It suffices to show $\alpha \approx 0$ for an arbitrary $\lambda$-form $\alpha = (H, \varphi, \mu)$ over $\mathcal{F}$ such that $\alpha_\Lambda \approx 0$. By [3, Lemma 1.2], we may suppose that $H$ is a free module over $A$. By [3, Lemma 1.3], there exists a nonnegative integer $s$ such that $\alpha_\Lambda \perp H(A^s) \approx 0$. Hence $\alpha_\Lambda \perp H(A'^s)$ has a presubkernel $L$. There exists a subset $\{x_1, \ldots, x_m\}$ of $L$ whose image is a $\Lambda$-basis of the image of $L_\Lambda$ in $H_\Lambda \oplus \Lambda^s \oplus \Lambda^s$. We regard $A^s \subseteq A'^s$ and $H \subseteq H_{A'}$. Since
$f$ is locally epic, there exists an unit $u$ of $A'$ such that \{ux_1, \ldots, ux_m\} \subset H \oplus A^s \oplus A^s$. Define $U$ to be the $A$-submodule of $H \oplus A^s \oplus A^s$ generated by \{ux_1, \ldots, ux_m\}. Since \(\min(A) = \min(A') \cap A\), $U$ is a presubkernel of $\alpha \perp H(A^s)$. Thus we conclude $\alpha \sim 0$.

**Surjectivity:** Here we suppose $f : A \to A'$ is strongly locally epic. Let $\beta = (K, \gamma, \omega)$ be a $\lambda$-form over $\mathcal{F}'$ such that $K$ is a free $A'$-module. Let \{y_1, \ldots, y_m\} be an $A'$-basis of $K$. Consider the elements $\gamma(y_i)$ and $\omega(y_i)$ in $A'$, where $1 \leq i, j \leq m$ and $\omega(y_i)$ are liftings of $\omega(y_i)$, respectively. Then there exists an element $u \in A$ such that $u$ is invertible in $A'$ and all $u\gamma(y_i), u\omega(y_i)$ belong to $A$. Set $x_1 = uy_1$, $\ldots$, $x_m = uy_m$, and let $H$ be the $A$-submodule of $K$ generated by $x_1$, $\ldots$, $x_m$. A $\lambda$-Hermitian map $\varphi : H \times H \to A$ and a quadratic map $\mu : H \to A/\text{min}(A)$ are obtained by restricting $\gamma$ and $\omega$ to $H \times H$ and $H$, respectively. Then, the equivalence class $[K, \gamma, \omega] \in \Gamma^h(\mathcal{F}')$ is the image of $[H, \varphi, \mu] \in \Gamma^h(\mathcal{F})$ by the map of changing rings. \hfill \Box

Let $G$ be a finite group and $R$ a commutative ring with unit such that the canonical homomorphism $\mathbb{Z} \to R$ is locally epic. Let $Y$ be a compact, connected, smooth $G$-manifold. Then the sequence

\begin{equation}
\{e\} \to \pi_1(Y) \longrightarrow \pi_1(EG \times_G Y) \xrightarrow{\zeta} G \to \{e\}
\end{equation}

is exact, where $EG$ is a contractible CW-complex with free $G$-action. Let $\hat{G}$ denote the fundamental group $\pi_1(EG \times_G Y)$ and $\tilde{Y}$ the universal covering space of $Y$. As is explained in [17, §1], we can regard $\tilde{Y}$ as a $\hat{G}$-manifold. We note that the canonical homomorphism

$$
\mathcal{F} : \mathbb{Z}[\hat{G}] \to R[G]
$$

is locally epic. Let $X$ be a compact, connected, smooth $G$-manifold and $f : X \to Y$ a $G$-map. As is well known, the covering space $f^*\tilde{Y}$ over $X$ induced from $\tilde{Y}$ is a $\hat{G}$-space, and $f^*\tilde{Y}$ is the universal covering space of $X$ in the case where $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

In the remainder of the current section, we assume that $X$ and $Y$ are oriented. The homomorphism

$$
w_Y : G \to \{1, -1\}
$$

is defined by setting $w_Y(g) = 1$ for $g \in G$ that preserves the orientation of $Y$ and $w_Y(g) = -1$ for $g \in G$ that reverses the orientation of $Y$. Since $Y$ is oriented, $\pi_1(Y)$ is contained in $\text{Ker}(w_Y)$: in other words, each element of $\pi_1(Y)$ acts on $\tilde{Y}$ as an orientation-preserving transformation. The involution $- \circ R[G]$ is defined as the additive map such that $rg = rw_Y(g)g^{-1}$ for $r \in R$ and $g \in G$. Hence, $\mathcal{F}$ is involution preserving.

We mean by the **singular set** of $X$ the subset

$$
X_{\text{sing}} := \bigcup_{g \in G \setminus \{e\}} X^g.
$$
For a subgroup $H$ of $G$, we say that a $G$-manifold $X$ satisfies the **strong gap condition for $H$** if

$$\dim X^H_\alpha > 2(\dim X^K + 1) \quad \text{for all } \alpha \in \pi_0(X^H) \text{ and } K \in \mathcal{S}(G) \text{ with } H \subseteq K.$$ 

Note that we presented Theorem 1.1 for a smooth $G$-action on a disk satisfying the strong gap condition for all $P$ in $\mathcal{P}(G)$.

Let $f : (X, \partial X) \to (Y, \partial Y)$ be a $G$-map and $b : T(X) \oplus f^* \eta \to f^* \xi$ a $G$-vector bundle isomorphism (covering the identity map on $X$), where $\eta$ and $\xi$ are real $G$-vector bundles over $Y$. We refer to such a pair $(f, b)$ as a $G$-framed map. When we say that $(f, b)$ is an $R$-homology equivalence, $m$-connected, or of degree one, we mean that $f$ is an $R$-homology equivalence, $m$-connected, or of degree one, respectively.

The following two lemmas are equivariant interpretations of Cappell-Shaneson’s surgery theory.

**Lemma 2.3** (Even Dimensional Case). Let $p$ be a prime, $R = \mathbb{Z}_p$, and let $(f, b)$, $f : (X, \partial X) \to (Y, \partial Y)$, be a degree-one $G$-framed map with $\dim X = n = 2k$ (even) $\geq 6$. Suppose

1. $\dim X > 2(\dim X^K + 1)$ for all $K \in \mathcal{S}(G)$ with $K \neq \{e\}$,
2. $\partial f := f|_{\partial X} : \partial X \to \partial Y$ is an $R$-homology equivalence,
3. $f^P : X^P \to Y^P$ is an $R$-homology equivalence for any $p$-group $P \neq \{e\}$,
4. $\chi(X^g) = \chi(Y^g)$ for any $g \in G \setminus \{e\}$.

Then the $G$-framed map $(f, b)$ determines an element $\sigma(f, b)$ of $\Gamma^h_n(\mathcal{F})$ and the following three statements are equivalent:

1. $\sigma(f, b)$ vanishes.
2. $(f, b)$ is $G$-framed cobordant, relatively to the boundary and the singular set, to a degree-one $R$-homology equivalence.
3. $(f, b)$ is $G$-framed cobordant, relatively to the boundary and the singular set, to a degree-one, $(k - 1)$-connected $R$-homology equivalence.

**Proof.** This follows from [3, Theorem 1.7 and Addendum to 1.7] and [22, Lemma A.10] (or [24, Lemmas 2.1 and 2.4]), cf. [12, §3]. □

**Remark 2.4.** Let $\zeta : \hat{G} \to G$ be as above, and let $\mathcal{F} : \mathbb{Z}[\hat{G}] \to \mathbb{Z}_p[\hat{G}]$ and $\mathcal{F}_p : \mathbb{Z}_p[\hat{G}] \to \mathbb{Z}_p[G]$ be the canonical homomorphisms. Since the homomorphism $\Gamma^h_n(\mathcal{F}) \to \Gamma^h_n(\mathcal{F}_p)$ is an isomorphism by Proposition 2.2, it is better to regard $\sigma(f, b)$ of Lemma 2.3 as an element in $\Gamma^h_n(\mathcal{F}_p)$.

**Lemma 2.5** (Odd Dimensional Case). Let $p$ be a prime, $R = \mathbb{Z}_p$, and let $(f, b)$, $f : (X, \partial X) \to (Y, \partial Y)$, be a degree-one $G$-framed map with $\dim X = n = 2k + 1$ (odd) $\geq 5$. Suppose

1. $\dim X > 2(\dim X^K + 1)$ for all $K \in \mathcal{S}(G)$ with $K \neq \{e\}$,
2. $\partial f := f|_{\partial X} : \partial X \to \partial Y$ is an $R$-homology equivalence,
(2.5.3) \( f^P : X^P \to Y^P \) is an \( R \)-homology equivalence for any \( p \)-group \( P \neq \{e\} \),
(2.5.4) \( \chi(X^g) = \chi(Y^g) \) for any \( g \in G \setminus \{e\} \).

Then the \( G \)-framed map \((f, b)\) determines an element \( \sigma(f, b) \) of \( L^h_n(R[G], \omega_Y) \) and the following three statements are equivalent:

1. \( \sigma(f, b) \) vanishes.
2. \( (f, b) \) is \( G \)-framed cobordant, relatively to the boundary and the singular set, to a degree-one \( R \)-homology equivalence.
3. \( (f, b) \) is \( G \)-framed cobordant, relatively to the boundary and the singular set, to a degree-one, \((k - 1)\)-connected \( R \)-homology equivalence.

**Proof.** This follows from [3, Proposition 2.1] and [22, Lemma A.10] (or [24, Lemmas 2.1 and 2.4]), cf. [12, §4].

Note that if \( Y^G \neq \emptyset \), then the exact sequence (2.1) splits. Hereafter, we assume the sequence (2.1) splits: namely, \( \hat{G} \) is a semi-direct product \( G \rtimes Q(\hat{G}) \).

**Theorem 2.6.** Let \( p \) be a prime, \( R = \mathbb{Z}(p) \) and \( Y \) a compact, connected, oriented \( G \)-manifold of dimension \( n = 2k \) (even) such that \( Y^G \neq \emptyset \). If \( Q(\hat{G}) \) is a finite group of order prime to \( p \), then the canonical homomorphism \( j_* : \Gamma_n^h(\mathcal{F}(p)) \to L^h_n(R[G], \omega_Y) \) is an isomorphism, where \( \mathcal{F}(p) : R[\hat{G}] \to R[G] \) is the canonical homomorphism.

**Proof.** Since \( j_* \) is surjective ([3, p. 288, lines 7–8 from the bottom]), it suffices to prove the injectivity of \( j_* \). Let \( \alpha = (H, \phi, \mu) \) be a \((-1)^k\)-form over \( \mathcal{F}(p) \). By adding a hyperbolic module to \( \alpha \) if necessary, we suppose \( H_{R[G]} = R[G] \otimes_{R[\hat{G}]} H \) is a free \( R[G] \)-module. Let \( \{x_1, \ldots, x_m\} \) be a basis of \( H_{R[G]} \) over \( R[G] \). Take a lift \( \tilde{x}_i \) in \( H \) for each \( x_i \) and define an \( R[G] \)-homomorphism \( \tau : H_{R[G]} \to H \) by

\[
\tau(x_i) = \frac{\Sigma_{\pi_1(Y)}(\tilde{x}_i)}{\pi_1(Y)} \tilde{x}_i,
\]

where

\[
\Sigma_{\pi_1(Y)} = \sum_{c \in \pi_1(Y)} c.
\]

Later we use the property

\[
(2.3) \quad \tau(H_{R[G]}) \subset H^{\pi_1(Y)}.
\]

We will prove in Step 1 that \( \phi_{R[G]}(x, y) = 0 \) implies \( \phi(\tau(x), \tau(y)) = 0 \), and in Step 2 that \( \phi_{R[G]}(x, x) = 0 \) and \( \mu_{R[G]}(x) = 0 \) imply \( \mu(\tau(x)) = 0 \). Once these were shown, it follows that if \( \phi_{R[G]} \) has a subkernel \( K' \), then \( \tau(K') \) is a presubkernel of \( \alpha \); namely, \( \alpha_{R[G]} \sim 0 \) implies \( \alpha \sim 0 \).

**Step 1.** Let \( \phi_c : H \times H \to R \) denote the \( c \)-component of \( \phi : H \times H \to R[G] \): namely,

\[
\phi_c(\tilde{x}, \tilde{y}) = \sum_{a \in \hat{G}} \phi_c(\tilde{x}, a^{-1}\tilde{y})a
\]
for \( \hat{x}, \hat{y} \in H \), where \( e \) is the identity element of \( \hat{G} \). Let \( \zeta : R[\hat{G}] \to R[G] \) denote the homomorphism induced from \( \zeta : \hat{G} \to G \). By definition, we have

\[
\varphi(\tau(x), \tau(y)) = \sum_{a \in \hat{G}} \varphi_e(\tau(x), a^{-1}\tau(y)) a
\]

\[
= \sum_{g \in G} \left( \sum_{a \in \zeta^{-1}(g)} \varphi_e(\tau(x), g^{-1}\tau(y)) a \right)
\]

\[
= \sum_{g \in G} \varphi_e(\tau(x), g^{-1}\tau(y)) \sum_{\pi_1(Y)} g,
\]

and

\[
\varphi_{R[\mathcal{C}]}(x, y) = \zeta_* (\varphi(\tau(x), \tau(y)))
\]

\[
= \zeta_* \left( \sum_{g \in G} \varphi_e(\tau(x), g^{-1}\tau(y)) \sum_{\pi_1(Y)} g \right)
\]

\[
= |\pi_1(Y)| \sum_{g \in G} \varphi_e(\tau(x), g^{-1}\tau(y)) g.
\]

Thus, the equality

\[
\varphi(\tau(x), \tau(y)) = \frac{\sum_{\pi_1(Y)} \varphi_{R[\mathcal{C}]}(x, y)}{|\pi_1(Y)|}
\]

holds for any \( x, y \in H_{R[\mathcal{C}]} \). Now it is clear that \( \varphi(\tau(x), \tau(y)) \) vanishes if and only if \( \varphi_{R[\mathcal{C}]}(x, y) \) does.

**Step 2.** Suppose \( \varphi_{R[\mathcal{C}]}(x, x) = 0 \) and \( \mu_{R[\mathcal{C}]}(x) = 0 \). By Step 1, we have \( \varphi(\tau(x), \tau(x)) = 0 \). For a subset \( S \) of \( G \), we adopt the notation

\[
S(2) = \{ g \in S \mid g^2 = e, g \neq e \},
\]

\[
S_q = \{ g \in S \mid g^2 = e, w_Y(g) = (-1)^{k+1} \},
\]

\[
S_s = S \setminus S_q.
\]

Decompose \( G \) to a disjoint union of the form

\[
G = \{ e \} \amalg G(2)_q \amalg G(2)_s \amalg C \amalg C'
\]

such that \( C' = \{ g^{-1} \mid g \in C \} \). Set \( \mathcal{R} = \{ e \} \cup G(2) \cup C \), and define the module \( Q_g, g \in G \), by

\[
Q_g = \begin{cases} R & (g \in G_s), \\ R/2R & (g \in G_q). \end{cases}
\]

Then we can regard \( \mu_{R[\mathcal{C}]}(x) \) as the formal sum

\[
\mu_{R[\mathcal{C}]}(x) = \sum_{g \in \mathcal{R}} (\mu_{R[\mathcal{C}]})_g(x) g
\]
with \((\mu_{R[G]})_g(x) \in Q_g\). Similarly choosing a subset \(\widehat{\mathcal{R}}\) of \(\widehat{G}\) so that \(\zeta(\widehat{\mathcal{R}}) = \mathcal{R}\), we can write \(\mu(\widehat{x})\) in the form
\[
\mu(\widehat{x}) = \sum_{a \in \widehat{\mathcal{R}}} \mu_a(\widehat{x}) a
\]
with \(\mu_a(\widehat{x}) \in Q_a\).

Note that for \(a \in \widehat{G}_s \cap \widehat{\mathcal{R}}, \mu_a(\widehat{x}) = 0\) follows from \(\varphi_c(\widehat{x}, a^{-1} \widehat{x}) = 0\).

Next, note that if \(p\) is an odd integer and \(a \in \widehat{G}_q \cap \widehat{\mathcal{R}}\), then \(Q_a = R/2R = 0\), and hence \(\mu_a(\widehat{x}) = 0\).

It remains to consider \(\mu_a(\widehat{x})\) in the case where \(p = 2\) and \(a \in \widehat{G}_q \cap \widehat{\mathcal{R}}\). Let \(g \in G_q\). By definition, we have
\[
(\mu_{R[G]})_g(x) = \sum_{b \in \zeta^{-1}(g) \cap \widehat{\mathcal{R}}} \mu_b(\tau(x)) \quad \text{in } R/2R
\]
\[
= \sum_{b \in \zeta^{-1}(g) \cap \widehat{G}_q} \mu_b(\tau(x)) \quad \text{(because } \varphi(\tau(x), \tau(x)) = 0).\]

In particular, if the identity element \(e\) belongs to \(G_q\), then
\[
(\mu_{R[G]})_g(e) = \mu_e(\tau(x)),
\]
since \(\pi_1(Y)\) is of odd order. We conclude \(\mu_e(\tau(x)) = 0\). So, let \(g \in G(2)_q\). For \(b \in \zeta^{-1}(g) \cap \widehat{G}(2)_q\), we can write \(b\) in the form \(b = cg\) with \(c \in \pi_1(Y)\). Note that \(g^2 = e, g \neq e\) and \(g = c^{-1}\). If \(c \neq e\), then the group \(\langle c, g \rangle\) generated by \(c\) and \(g\) is a dihedral group. Since \(c(e^2g)c^{-1} = c^{u+2}g\) and
\[
\mu(\tau(x)) = \mu(c\tau(x)) = c\mu(\tau(x))c^{-1} \quad \text{(cf. (2.3))},
\]
we obtain \(\mu_{c^{u+2}g}(\tau(x)) = \mu_{c^ug}(\tau(x))\), and hence \(\mu_b(\tau(x)) = \mu_g(\tau(x))\) for all \(b \in \zeta^{-1}(g) \cap \widehat{G}(2)_q\). Noting that \(|\zeta^{-1}(g) \cap \widehat{G}_q|\) is an odd integer, we get
\[
(\mu_{R[G]})_g(x) = \mu_g(\tau(x)).
\]
Thus, \(\mu_g(\tau(x)) = 0\) as well as \(\mu_b(\tau(x)) = 0\) for \(b \in \zeta^{-1}(g) \cap \widehat{G}(2)_q\).

Putting all this together, we have showed \(\mu(\tau(x)) = 0\). \(\square\)

3. Equivariant connected sum of \(G\)-framed maps

Let \(G\) be a finite group and \(n = 2k, 2k + 1 \geq 5\) with an integer \(k\). Let \(X\) and \(Y\) be compact connected oriented smooth \(G\)-manifolds of dimension \(n\). Suppose \(\partial Y \neq \emptyset\) and \((\text{Int}(Y))^G \neq \emptyset\). Let \(y_0\) be a base point of \(Y\) located in \((\text{Int}(Y))^G\).

Let \(f = (f, b) : (X, \partial X, T(X)) \to (Y, \partial Y, f^*T(Y))\) be a \(G\)-framed map. Here \(b\) is a real \(G\)-vector bundle isomorphism
\[
\varepsilon_X(\mathbb{R}) \oplus T(X) \oplus f^*\eta \to f^*\tau \quad \text{(where } \tau = \varepsilon_Y(\mathbb{R}) \oplus T(Y) \oplus \eta)\]
for some real \(G\)-vector bundle \(\eta\) over \(Y\).

In this section we invoke
Assumption 3.1 (Boundary Assumption). The restriction $\partial f : \partial X \to \partial Y$ of $f$ is a orientation preserving diffeomorphism and the restriction

$$\partial b : \varepsilon_{\partial X}(\mathbb{R}) \oplus T(X)|_{\partial X} \oplus (\partial f)^* \eta|_{\partial Y} \to (\partial f)^* \tau|_{\partial Y}$$

of $b$ coincides with

$$id_{\varepsilon_{\partial X}(\mathbb{R})} \oplus df|_{\partial X} \oplus id_{(\partial f)^* \eta}|_{\partial Y}.$$ 

By this assumption, we can regard $\partial X = \partial Y$, $\partial f = id_{\partial Y}$ and $\partial b = id_{r|_{\partial Y}}$.

Define $\Delta X = (-Y) \cup_{\partial Y} X$. Particularly if $X = Y$ then $\Delta X$ is the double of $Y$. Let $Z$ be the smooth $G$-manifold obtained by smoothing the corner of $[-\delta, \delta] \times Y$ with small $\delta > 0$, where $Z \subset [-\delta, \delta] \times Y$.

Then the boundary $\partial Z$ of $Z$ is $\Delta Y = (-Y) \cup_{\partial Y} Y$.

Let $\nu(\partial Z, Z)$ denote the normal bundle of $\partial Z$ in $Z$. Since $Z$ is orientable, $\nu(\partial Z, Z)$ is isomorphic to the product bundle $\varepsilon_{\partial Z}(\mathbb{R})$ as real $G$-vector bundles. We obtain the isomorphism

$$\psi : \varepsilon_{\partial Z}(\mathbb{R}) \oplus T(\partial Z) \to \nu(\partial Z, Z) \oplus T(\partial Z)$$

$$\quad \to T(Z)|_{\partial Z}$$

$$\quad \to T([-\delta, \delta] \times Y)|_{\partial Z}$$

$$\quad \to (\varepsilon_{[-\delta, \delta]}(\mathbb{R}) \times T(Y))|_{\partial Z}$$

$$\quad \to \varepsilon_{\partial Z}(\mathbb{R}) \oplus h^*T(Y),$$

where $h : \partial Z \to Z \to [-\delta, \delta] \times Y \to Y$ is the composition of canonical maps. We may regard that $h$ is the identity map on $-Y$ except a thin collar neighborhood of $\partial(-Y)$ and so on $Y$ except a thin collar neighborhood of $\partial Y$. Thus, on the fibers over most part of $-Y$, $A$ say, $\psi(u, v) = (-u, v)$ where $(u, v) \in \varepsilon_A(\mathbb{R}) \oplus T(\partial(-Y))|_A$; and on the fibers over most part of $Y$, $B$ say, $\psi(u, v) = (u, v)$ where $(u, v) \in \varepsilon_B(\mathbb{R}) \oplus T(\partial Y)|_B$. Define $\Delta f : \Delta X \to Y$ as the composition

$$\Delta X \xrightarrow{id_{\Delta Y} \cup f} \Delta Y \xrightarrow{h} Y.$$
Define $\Delta b : \varepsilon_{\Delta X}(\mathbb{R}) \oplus T(\Delta X) \oplus (\Delta f)^* \eta \to (\Delta f)^* \tau$ to be the induced map from the composition

$$
\varepsilon_{\Delta X}(\mathbb{R}) \oplus T(\Delta X) \oplus (\Delta f)^* \eta \xrightarrow{id \cup b} \varepsilon_Z(\mathbb{R}) \oplus T(\partial Z) \oplus h^* \eta \\
\downarrow_{\psi \oplus id} \\
\varepsilon_Z(\mathbb{R}) \oplus h^* T(Y) \oplus h^* \eta
$$

covering $id_Y \cup f$. Let $\Delta f$ denote the $G$-framed map $(\Delta f, \Delta b)$ from the closed manifold $\Delta X$ to the compact manifold $Y$ with boundary.

Define $-\Delta f$ to be the $G$-framed map $(-\Delta f, \Delta b)$, where $-\Delta f : -\Delta X \to Y$ is (if we forget the orientation) the same as $\Delta f$ and $\Delta b$ denotes the composition of

$$
\varepsilon_{\Delta X}(\mathbb{R}) \oplus T(\Delta X) \oplus (\Delta f)^* \eta \xrightarrow{(id_{\Delta X} \times -id_{\eta}) \oplus id \oplus id} \varepsilon_{\Delta X}(\mathbb{R}) \oplus T(\Delta X) \oplus (\Delta f)^* \eta
$$

and $\Delta b$.

Let $V_{y_0}$ be a (linear) slice $G$-neighborhood of $y_0$ in the interior of $Y$ and $D_{y_0}$ the (closed) unit $G$-disk of $V_{y_0}$ centered at $y_0$ with respect to some $G$-invariant inner product.

**Definition 3.2.** Let $x_1 \in f^{-1}(y_0)$ with $Gx_1 = H$, $V_{x_1}$ a (linear) slice $G$-neighborhood of $x_1$ in the interior of $X$. The point $x_1$ is called an $(H, +)$- (resp. $(H, -)$) point on $f$ if the following conditions are satisfied.

(3.2.1) The restriction of $f$ to $V = V_{x_1}$ is a linear isomorphism $V \to V_{y_0}$ preserving inner product.

(3.2.2) The restriction of $b$ over $V$ coincides with

$$
b_{V,+} = (id_V \times id_{\mathbb{R}}) \oplus (id_V \times df_{x_1}) \oplus id_{f^* \eta|_V}
$$

(resp. $b_{V,-} = (id_V \times -id_{\mathbb{R}}) \oplus (id_V \times df_{x_1}) \oplus id_{f^* \eta|_V}$).

An $(H, +)$- (resp. $(H, -)$) point on $\Delta f$ or $-\Delta f$ is similarly defined.

For an $(H, +)$- or $(H, -)$-point $x_1$ on $f$, we have the $G$-diffeomorphism

$$G \cdot V_{x_1} (\cong G \times_H V_{x_1}) \to G \times_H V_{y_0}; \ gx (= [g, x]) \mapsto [g, f(x)]$$

and

$$\varphi_{x_1} : G \cdot D_{x_1} (\cong G \times_H D_{x_1}) \to G \times_H D_{y_0}$$

denotes the restriction of the $G$-diffeomorphism, where $D_{x_1}$ is the (closed) unit disk of $V_{x_1}$. 
Let $f' = (f', b') : (X', \partial X', T(X')) \to (Y, \partial Y, f'^* T(Y))$, where

$$b' : \varepsilon_{X'}(\mathbb{R}) \oplus T(X') \oplus f'^* \eta \to f'^* \tau \quad \text{(with } \tau = \varepsilon_Y(\mathbb{R}) \oplus T(Y) \oplus \eta),$$

be a $G$-framed map satisfying Assumption 3.1 for $f$ replaced by $f'$.

Let $\zeta : \mathbb{R} \to \mathbb{R}^2$, $\zeta(t) = (\zeta_1(t), \zeta_2(t))$, be a smooth curve such that

$$\zeta_1(t) = t, \quad \zeta_2(t) = \frac{1}{2} \quad (\text{for } 0 \le t \le \frac{1}{2}),$$

$$\zeta_1(t) > 0, \quad \zeta_2(t) > 0 \quad (\text{for } \frac{1}{2} < t < 1),$$

$$\zeta_1(t) = 1, \quad \zeta_2(t) = t \quad (\text{for } t \ge 1),$$

$$\zeta_1(-t) = -\zeta_1(t), \quad \zeta_2(-t) = \zeta_2(t) \quad (\text{for } t \ge 0).$$

If $V$ is a finite dimensional real vector space with inner product, define the submanifold $\text{Connector}(V)$ of $\mathbb{R} \times V$ as the union of the three subspaces

$$\{(1, v) \mid v \in V \text{ and } ||v|| \ge 1\},$$

$$C_{(-1,1)} := \{((\zeta_1(t), v) \mid -1 < t < 1 \text{ and } ||v|| = \zeta_2(t)\},$$

and

$$\{(1, v) \mid v \in V \text{ and } ||v|| \ge 1\}.$$

If $x_1$ is an $(H, +)$-point on $f$ then the smooth $G$-manifold

$$X \#_{G, (x_1, y_0)} G \times_H \Delta X' \quad (= X'' \text{ say})$$

is obtained from the three manifolds

$$X \setminus G \cdot \text{Interior}(D_{x_1}), \quad G \times_H \text{Connector}(V_{x_1}),$$

and

$$G \times_H (\Delta X' \setminus \text{Interior}(D_{y_0}))$$

by identifying $gv \in G \cdot (V_{x_1} \setminus \text{Interior}(D_{x_1})) \subset X$ with $[g, (-1, v)] \in G \times_H \text{Connector}(V_{x_1})$, and $[g, (1, v)] \in G \times_H \text{Connector}(V_{x_1})$ with $[g, f(v)] \in G \times_H (V_{y_0} \setminus \text{Interior}(D_{y_0})) \subset \Delta X'$.

Thus, $X \#_{G, (x_1, y_0)} G \times_H \Delta X'$ is $G$-homeomorphic to the attaching space

$$(X \setminus \text{Interior}(G \cdot D_{x_1})) \cup_{\partial x_1} G \times_H (\Delta X' \setminus \text{Interior}(D_{y_0})).$$

Define the $G$-map

$$f'' : X \#_{G, (x_1, y_0)} G \times_H \Delta X' \to Y \quad (= f'' \text{ say})$$

by

$$f''(x) = f(x) \quad \text{for } x \in X \setminus G \cdot \text{Interior}(D_{x_1}),$$

$$f''([g, (t, v)]) = gf(v) \quad \text{for } g \in G, \ (t, v) \in \text{Connector}(V) \text{ with } -1 < t < 1,$$

and

$$f''([g, x']) = g(\Delta f')(x') \quad \text{for } g \in G, \ x' \in \Delta X' \setminus \text{Interior}(D_{y_0}).$$

The tangent bundle $T(X'')$ is the union of

$$T(X \setminus G \cdot \text{Interior}(D_{x_1})), \quad T(G \times_H C_{(-1,1)}) \quad \text{and} \quad G \times_H T(\Delta X' \setminus \text{Interior}(D_{y_0})).$$
Define
\[ \text{Band}(V) = \mathbb{R} \times V, \text{ where } V = V_{x_1}. \]
The total space of \( T(\text{Band}(V)) \) is the cartesian product of the base space \( \text{Band}(V) \) and the fiber \( \mathbb{R} \times T_{x_1}(X) \). We have the canonical isomorphism \( \rho \):
\[
\varepsilon_{\text{Connector}(V)}(\mathbb{R}) \oplus T(\text{Connector}(V)) \xrightarrow{\rho} T(\text{Band}(V))|_{\text{Connector}(V)}
\]
which satisfies
\[
\rho(((1, x), t), ((1, x), v)) = (((1, x), t), ((1, x), v))
\]
and
\[
\rho(((1, x), t), ((1, x), v)) = (((1, x), -t), ((1, x), v))
\]
for all \( x \in V \setminus \text{Interior}(D_{x_1}), t \in \mathbb{R}, \text{ and } v \in T_{x_1}(X) \). The map
\[
b\#_{G, (x_1, y_0)} G \times_H \Delta b' : \varepsilon_{X''}(\mathbb{R}) \oplus T(X'') \oplus f'' \eta \to f'' \tau \quad (= b'' \text{ say})
\]
is defined by
\[
b''|_{X \setminus G \cdot \text{Interior}(D_{x_1})} = b|_{X \setminus G \cdot \text{Interior}(D_{x_1})} \ni (pr^* b) \circ (\rho \oplus id),
\]
where \( pr : \text{Connector}(V) \to \mathbb{R} \times V \to V \) is the composition of canonical maps, and
\[
b''|_{G \times_H (\Delta X' \setminus \text{Interior}(D_{y_0}))} = G \times_H (\Delta b'|_{\Delta X' \setminus \text{Interior}(D_{y_0})}).
\]
Putting all this together, we define the \( G \)-connected sum
\[
f\#_{G, (x_1, y_0)} G \times_H \Delta f' = (f \#_{G, (x_1, y_0)} G \times_H \Delta f', b\#_{G, (x_1, y_0)} G \times_H \Delta b')
\]
of \( f \) with \( G \times_H \Delta f' \) of type \((H, +)\).

If \( x_1 \) is an \((H, -)\)-point on \( f \) then define the \( G \)-connected sum
\[
f\#_{G, (x_1, y_0)} G \times_H (\Delta f') = (f \#_{G, (x_1, y_0)} G \times_H (\Delta f'), b\#_{G, (x_1, y_0)} G \times_H (\Delta b'))
\]
of \( f \) with \( G \times_H (\Delta f') \) of type \((H, -)\), where the source manifold of the \( G \)-framed map is \( X \#_{G, (x_1, y_0)} G \times_H (\Delta X') \). If we forget the orientation of the source manifold then \( f\#_{G, (x_1, y_0)} G \times_H (\Delta f') \) is the same as \( f\#_{G, (x_1, y_0)} G \times_H \Delta f' \). Clearly, \( X \#_{G, (x_1, y_0)} G \times_H (\Delta X') \) is \( G \)-homeomorphic to the attaching space
\[
(X \setminus \text{Interior}(G \cdot D_{x_1})) \cup a_{x_1} G \times_H (\Delta X' \setminus \text{Interior}(D_{y_0})).
\]

**Proposition 3.3.** If we forget the \( G \)-action, \( X \#_{G, (x_1, +)} G \times_H \Delta X' \) and \( X \#_{G, (x_1, -)} G \times_H (\Delta X) \) coincide with the ordinary connected sums
\[
\underbrace{X \# \Delta X' \# \cdots \# \Delta X'}_{|G/H|-fold}
\]
and
\[
X \# (-\Delta X') \# \cdots \# (-\Delta X'),
\]
respectively, as oriented manifolds.

If \( \chi(Y) = 1 \) and \( \chi(\partial Y) = 1 + (-1)^{n-1} \) then
\[
\chi(X \# G, (x_1, \pm) G \times_H \pm \Delta X') = \chi(X) + |G/H| (\chi(X') - 1).
\]

Proof. The first claim is evident. For the second claim, observe
\[
\chi(\Delta X') = \chi(X') + \chi(Y) - \chi(\partial Y) = \chi(X') + (-1)^n.
\]
Letting \( X'' = X \# G, (x_1, \pm) G \times_H \pm \Delta X' \),
\[
\chi(X'') = \chi(X) + |G/H| \chi(\Delta X') - 2|G/H|(-1)^n - |G/H| \chi(S^{n-1})
= \chi(X) + |G/H| (\chi(X') - 1).
\]

In the following we invoke
\textbf{Assumption 3.4 (Gap Condition).} For all \( g \in G \setminus \{e\} \),
\[
\dim X^g \leq \begin{cases} 
  k - 1 & \text{if } n = 2k \\
  k & \text{if } n = 2k + 1.
\end{cases}
\]

In addition, we assume that \( X' \) also satisfies Gap Condition above. Define
\[G(2) = \{ g \in G \mid g^2 = e \text{ and } g \neq e\}\]
and
\[Q(G, X) = \begin{cases} 
  \{ g \in G(2) \mid \dim X^g = k - 1 \} & \text{if } n = 2k, \\
  \{ g \in G(2) \mid \dim X^g = k \} & \text{if } n = 2k + 1.
\end{cases}\]

Let \( R \) be a subring of \( \mathbb{Q} \) containing \( \mathbb{Z} \) and let \( w = w_Y : G \to \{ \pm 1 \} \) denote the orientation homomorphism associated with \( Y \). Then the group ring \( R[G] \) has the involution satisfying \( g \mapsto g^{-1} \) for \( g \in G \). Set \( \lambda = (-1)^k \). Let \( (Q(G, X))_R \) denote the form parameter on \( R[G] \) generated by \( Q(G, X) \), namely
\[(Q(G, X))_R = \{ a - \lambda \pi \mid a \in R[G] \} + R[Q(G, X)].\]

So, the data \((R[G], (-, \lambda), (Q(G, X))_R)\) is a form ring of Bak [1]. Let
\[W_n(R[G], (Q(G, X))_R; w_Y)\]
denote the \( G \)-surgery obstruction group defined in [12], and \( \mathcal{A} \) the category of abelian groups. By [16, Proposition 12.7] and [9, Theorem 1.1], the correspondence
\[S(G) \to \mathcal{A}; H \mapsto W_n(R[H], (Q(H, \text{Res}_H^G X))_R; w_{\text{Res}_H^G Y})\]
is a \( w \)-Mackey functor (cf. [9, Section 2]). Thus the Burnside ring acts on the \( G \)-surgery obstruction group \( W_n(R[G], (Q(G, X))_R; w_Y) \).

\textbf{Theorem 3.5.} Let \( f \) and \( f' \) be as above. Suppose
(1) $Y$ is homeomorphic to the standard disk of dimension $n$,
(2) $f : X \to Y$ and $f' : X' \to Y$ are $k$-connected,
(3) the surgery kernel

$$K(X; R) = \begin{cases} 
\text{Ker}[f_* : H_k(X; R) \to H_k(Y; R)] & \text{if } n = 2k, \\
\text{Ker}[f_* : H_k(X_0; R) \to H_k(Y_0; R)] \text{ (see [12, §4])} & \text{if } n = 2k + 1,
\end{cases}$$

and $K(X'; R)$ are $R[G]$-projective.

Then

$$[K(X \#_{G, (x_1, y_0)} G \times_H (+\Delta f')); R)] = [K(X; R)] + [G/H][K(X'; R)]$$

in the reduced projective class group $\tilde{K}_0(R[G])$. If moreover $K(X; R)$ and $K(X'; R)$ are stably $R[G]$-free and $Q(G, X) = Q(G, X') \supset Q(G, Y)$, then the $G$-surgery obstructions $\sigma(-)$ satisfy the formula

$$\sigma(f \#_{G, (x_1, y_0)} G \times_H (\epsilon \Delta f')) = \sigma(f) + \epsilon[G/H] \sigma(f') \ (\epsilon = +, -).$$

Here $f \#_{G, (x_1, y_0)} G \times_H (+\Delta f')$ and $\sigma(f \#_{G, (x_1, y_0)} G \times_H (+\Delta f'))$ denote $f \#_{G, (x_1, y_0)} G \times_H \Delta f'$ and $\sigma(f \#_{G, (x_1, y_0)} G \times_H \Delta f')$ respectively, and $[G/H]$ stands for the element in the Burnside ring.

The proof is straightforward.

The $G$-connected sum $f \#_{G, (x_1, y_0)} G \times_H (\pm \Delta f')$ is usually abbreviated as

$$f \#_G G \times_H (\pm \Delta f') \text{ and } f \vdash (\pm[G/H]f').$$

4. Construction of a $G$-framed map

Let a smooth $G$-manifold $Y$, connected components $F_1, \ldots, F_t$ of $Y^G$, and nonnegative integers $n_1, \ldots, n_t$ be as in Theorem 1.1. In this section, we suppose $n_1 \neq 1$. Since $F_1 \cap \partial Y = \emptyset$, we can take a base point $y_0$ of $Y$ in $F_1 \cap \text{Int}(Y)$. Let $\nu(F_i, Y)$ denote the $G$-tubular neighborhood of $F_i$ in $Y$. For each conjugacy class $(H) \in \mathcal{G}(G)/G$, take a point $y_H$ in a highest-dimensional connected component of $Y^H \setminus \partial Y$, and set

$$Z = \{ y \in Y \mid G_y \in \mathcal{L}(G) \} \cup \partial Y \cup \bigcup_{(H) \in \mathcal{G}(G)/G} G y_H.$$ 

Then, let $A_1, \ldots, A_t$ denote the connected components of $(Z/\partial Y)/G$ and set $Z_i = \pi^{-1}(A_i)$, where $\pi : Z \to (Z/\partial Y)/G$ is the canonical projection. Reordering $Z_i$’s if necessary, we may assume

(1) $Z_t \supset \partial Y$,
(2) for $1 \leq i \leq u$, $Z_i$ contains $F_j$,
(3) for all $u < i < t$, $Z_i \cap Y^G = \emptyset$. 

Apply Theorems 3.6 and 4.4 of [13] for the elements \( \gamma_i \) of the Burnside ring \( \Omega(G) \) (see [4], [15]) with the formal expression
\[
\gamma_i = \begin{cases} 
(n_i - 0)[G/G] + \sum_{(H) \neq (G)} (1 - 1)[G/H] & (1 \leq i \leq u), \\
(1 - 0)[G/G] & (u < i \leq \ell),
\end{cases}
\]
and obtain a \( G \)-framed map \((f, b)\) consisting of a degree-one \( G \)-map
\[
f : (X, \partial X) \to (Y, \partial Y)
\]
and a \( G \)-vector bundle isomorphism
\[
b : T(X) \oplus \varepsilon_X(\mathbb{R}^{2m+1}) \to f^*(T(Y) \oplus \varepsilon_Y(\mathbb{R}^{2m+1})).
\]
By construction, we can arrange \((f, b)\) as follows:

**Property 4.1.** The degree-one \( G \)-framed map \((f, b)\) satisfies the properties below.

1. **X contains a \( G \)-invariant neighborhood** \( U \) of \( \partial Y \cup \bigcup_{(H) \in G\backslash G} G y_H \) in \( Y \) as a \( G \)-subset and \( \partial X = \partial Y \).
2. **The restrictions** \( f|_U : U \to U \) and \( b|_U : T(X)|_U \oplus \varepsilon_U(\mathbb{R}^{2m+1}) \to f|_U^*(T(Y) \oplus \varepsilon_Y(\mathbb{R}^{2m+1})) \) of \( f \) and \( b \) are the identity maps, respectively.
3. **For each** \( 1 \leq i \leq t \), \( f^{-1}(F_i) = \bigsqcup_{j=1}^{m_i} F_{i,j} \) (connected components).
4. **For each** \( 1 \leq i \leq t \), \( f^{-1}(F_i)^G = \bigsqcup_{j=1}^{m_i} F_{i,j} \).
5. **For each** \( 1 \leq i \leq t \) and \( 1 \leq j \leq m_i \), the isotropy subgroups (with respect to the \( G \)-action) appearing all over \( F_{i,j} \) coincide with one another. Let \( H_{i,j} \) denote the isotropy subgroup, i.e. \( \text{Iso}(G, F_{i,j}) = \{ H_{i,j} \} \).
6. **For each** \( 1 \leq i \leq t \) and \( 1 \leq j \leq m_i \), \( F_{i,j} \cap gF_{i,j} = \emptyset \) whenever \( g \in G \setminus H_{i,j} \).
7. **For each** \( 1 \leq i \leq t \) and \( 1 \leq j \leq m_i \), \( f|_{F_{i,j}} : F_{i,j} \to F_i \) is a diffeomorphism.
8. **For each** \( 1 \leq i \leq t \) and \( 1 \leq j \leq m_i \), \( f\nu(i,j) := f|_{F_{i,j}} : \nu(i,j) \to \nu(F_i, Y) \) can be regarded as an \( H_{i,j} \)-vector bundle map covering \( f_{i,j} \), where \( \nu(i,j) := \nu(F_{i,j}, X) \) is some \( H_{i,j} \)-tubular neighborhood of \( F_{i,j} \) in \( X \).
9. **For each** \( H \in \mathcal{S}(G) \setminus \{ G \} \), there exist points \( a(H,+) \in F_{1,j_H,+} \) and \( a(H,-) \in F_{1,j_H,-} \) such that \( H_{1,j_H,+} = H = H_{1,j_H,-} \), \( f(a(H,+)) = y_0 = f(a(H,-)) \), and the restrictions
\[
b_{a(H,+)} : T_{a(H,+)}(X) \oplus \mathbb{R} \oplus \mathbb{R}^{2m} \to T_{y_0}(Y) \oplus \mathbb{R} \oplus \mathbb{R}^{2m}, \\
b_{a(H,-)} : T_{a(H,-)}(X) \oplus \mathbb{R} \oplus \mathbb{R}^{2m} \to T_{y_0}(Y) \oplus \mathbb{R} \oplus \mathbb{R}^{2m}
\]
of \( b \) coincide with
\[
df_{a(H,+)} \oplus id_{\mathbb{R}} \oplus id_{\mathbb{R}^{2m}},
\]
\[
df_{a(H,-)} \oplus (-id_{\mathbb{R}}) \oplus id_{\mathbb{R}^{2m}},
\]
respectively.
10. **For** \( 1 \leq i \leq t \), \( 1 \leq j \leq n_i \), the restriction \( b_{\nu(i,j)} \) of \( b \) to \( \nu(i,j) \) coincides with the map induced from the stabilization of \( df_{\nu(i,j)} \).
(4.1.11) For each \( H \in S(G) \setminus \mathcal{L}(G) \), there exists an \( H \)-framed cobordism from \( \text{Res}_H^G(f, b) \) to the identity framed map \( \text{Res}_H^G(id_Y, id_{id_{\mathbb{R}^{2m+1}}}) \).

The properties (4.1.1) and (4.1.2) follow from the fact that \( \gamma_i, u < i \leq \ell \), has the form \((1 - 0)[G/G]\) (see [13, Theorem 4.4 (4.4.1)])

5. Proof of Theorem 1.1

For the proof of Theorem 1.1, we can suppose \( n_1 \neq 1 \) without loss of generality. We will alter the \( G \)-framed map \((f, b)\) obtained in Section 4 to \((f', b')\) such that \( X'^G = X^G \), \( \partial X' = \partial X (= \partial Y)\), \( \partial f' = id_{\partial Y} \), and \( X' \) is contractible. We will use \( G \)-surgery and \( G \)-connected sum for this alteration. The procedure of the alteration is an induction on elements \( H \) of \( \mathcal{G}^1(G) \).

We begin with the \( G \)-framed map constructed in Section 4. Recall that by Oliver [23], \( \chi(Y^H) = 1 \) holds for all \( H \in \mathcal{G}^1(G) \), and by the Smith theory \( Y^P \) is \( \mathbb{Z}_p \)-acyclic for all \( P \in \mathcal{P}(G) \), where \( p \) is the prime dividing \( |P| \). Let \( \mathcal{G}^1(G, X) \) denote the (unique) maximal subset of \( \mathcal{G}^1(G) \) satisfying the following conditions:

1. If \( H \leq K \in \mathcal{G}^1(G) \) and \( H \in \mathcal{G}^1(G, X) \), then \( K \) belongs to \( \mathcal{G}^1(G, X) \).
2. If \( H \in \mathcal{G}^1(G, X) \) and \( Y^H \) is connected, then \( X^H \) is connected.
3. If \( H \in \mathcal{G}^1(G, X) \), then \( \chi(X^H) = 1 \).
4. If \( P \in \mathcal{P}(G) \cap \mathcal{G}^1(G, X) \) and \( Y^P \) is simply connected, then \( X^P \) is simply connected.
5. If \( P \in \mathcal{P}(G) \cap \mathcal{G}^1(G, X) \) and \( P \neq \{e\} \), then \( X^P \) is \( \mathbb{Z}_p \)-acyclic, where \( p \) is the prime dividing \( |P| \).
6. If \( \{e\} \in \mathcal{G}^1(G, X) \), then \( X \) is contractible.

If \( \mathcal{G}^1(G, X) = \mathcal{G}^1(G) \), then the proof of the theorem is already completed. Thus, supposing that \( H \) is a maximal element of \( \mathcal{G}^1(G) \setminus \mathcal{G}^1(G, X) \), we will show that there exists a degree-one \( G \)-framed map \((f', b')\) satisfying the properties required for \((f, b)\) in Section 4 and

\[
\mathcal{G}^1(G, X) \cup \{H\} \subset \mathcal{G}^1(G, X').
\]

If it is done, the proof of the theorem is completed because we can do the procedure until \( \mathcal{G}^1(G, X') = \mathcal{G}^1(G) \) becomes true. The proof of the existence of \((f', b')\) above is divided into the three cases: the case \( H \notin \mathcal{P}(G) \), the case \( H \in \mathcal{P}(G) \) with \( H \neq \{e\} \), and the case \( H = \{e\} \). The equivariant connected sum operation in Section 3 plays a key role in our proof. Under certain assumptions on \( M \) and \( X \) (cf. Assumption 3.1 and Definition 3.2), the \( G \)-connected sums

\[
M \cup [G/K]X = M \#_G(G \times_K \Delta X)
\]

and

\[
M \cup (-[G/K]X) = M \#_G(G \times_K (-\Delta X))
\]

are defined. For an element \( \omega \) of the Burnside ring with the expression

\[
\omega = \sum_{i=1}^{s} a_i[G/K_i] \ (a_i \in \mathbb{Z}),
\]
$M \setminus \omega X$ stands for the $G$-manifold $M_s$ resulted by a sequence of $G$-connected sums:

\[ M_0 := M, \quad M_i := M_{i-1} \#_G (G \times K_i, \epsilon (a_i) \Delta X) \# \cdots \#_G (G \times K_i, \epsilon (a_i) \Delta X) \]

for $1 \leq i \leq s$, where $\epsilon (a_i)$ is the sign of $a_i$. If $M = X$ then $M \setminus \omega X$ is denoted by $(1 - \omega)X$. Similarly define $(1 - \omega)f$, $(1 - \omega)b$ and $(1 - \omega)f = ((1 - \omega)f, (1 - \omega)b)$.

For each subgroup $L$ of $G$, there is a canonical homomorphism $\chi_L : \Omega (G) \to \mathbb{Z}$ defined by $\chi_L ([A]) = \chi (A^L)$, where $A$ is a finite $G$-CW complex. Choose and fix an element $\beta \in \Omega (G)$ such that

\[ \chi_G (\beta) = 0, \quad \chi_H (\beta) = 1 \text{ for all } H \in \mathcal{S} (G) \setminus \mathcal{L} (G) \]

(see [10, Theorem 1.3], [15, Theorem 2.3]). Write $\beta$ in the form

\[ \beta = \sum_{(H) \in \mathcal{S} (G) / G} b_H [G / H] \]

with $b_H \in \mathbb{Z}$ and $b_G = 0$, and obtain $(- \beta)^G \in \Omega (G)$ such that

\[ (- \beta)^G = \sum_{(H) \in \mathcal{S} (G) / G} c_H [G / H] \]

with $c_H$ satisfying $c_G = 0$, $c_H \geq 0$, and

\[ c_H \equiv -b_H \mod 2 |G| |\tilde{K}_0 (\mathbb{Z} [G])| \]

for all $H \neq G$. By Proposition 3.3,

\[ \chi ((1 - \pm [G / K]) X^L) - 1 = \chi_L (1 + [G / K]) (\chi (X^L) - 1). \]

This provides the formula

\[ \chi ((1 - (- \beta)^G) X^L) - 1 = \chi_L (1 + (- \beta)^G) (\chi (X^L) - 1). \]

Case 1: $H \in \mathcal{G}^1 (G) \setminus \mathcal{P} (G)$. First, perform $G$-surgery of $(f, b)$ of isotropy type $(H)$ so that modified $X^H_\alpha$ is connected for any $\alpha \in \pi_0 (Y^H)$ with dim $X^H_\alpha \geq 3$.

By Remark 1.2, all dim $Y^H_\alpha$, $\alpha \in \pi_0 (Y^H)$, have the same parity: namely, they are simultaneously even or odd.

If dim $Y^H_\alpha$ is odd, then

\[ \chi (X^H) = \chi ((X^H) - \chi (Y^H) + \chi (Y^H) \]

\[ = \chi ((X^H) - \chi (Y^H) + \chi (Y^H) \]

\[ = \chi (Y^H) \]

\[ = 1. \]

Thus, the requirement for the Euler characteristic is automatically fulfilled.

So, suppose dim $Y^H_\alpha$ is even. Consider a $G$-connected sum $X'$ of the form

\[ X' = (1 - (- \beta)^G) X. \]
If \( L \in G^1(G) \) and \( \chi(X^L) = 1 \), then \( \chi(X^{iL}) = 1 \) follows from (5.1). Since

\[
\chi_H((-\beta)^\%e) \equiv \chi_H(-\beta) = -1 \quad \text{mod } 2|G|,
\]

we obtain

\[
\chi(X^{iH}) \equiv 1 \quad \text{mod } 2|G|.
\]

Thus, we can perform \( G \)-surgery of dimension 0 or 1, on a \( G \)-neighborhood of \( G X^i = H \) of isotropy type \( (H) \), where \( X^i = H \) is a highest dimensional connected component of \( X^i = H \), so that resulting \( X'' \) fulfills \( \chi(X''^H) = 1 \).

In result, if \( Y^H \) is connected, then \( X''^H \) is also connected.

**Case 2**: \( H \in \mathcal{P}(G) \) and \( H \neq \{e\} \). Let \( p \) be the prime dividing \( |H| \). Performing \( G \)-surgery of \((f, b)\) of isotropy type \((H)\) if necessary, we may assume that \( f^H : X^H \to Y^H \) is \([d(H)/2]-\)connected, where \( d(H) = \dim X^H \) and \([d(H)/2]\) is the largest integer not exceeding \( d(H)/2 \).

The obstruction \( \sigma(f^H, b^H) \) to convert \((f, b)\) so that modified \( f^H : X^H \to Y^H \) is a \(([d(H)/2] - 1)-\)connected \( \mathbb{Z}_p(\mathbb{Z}[W], w_{Y^H}) \) according as \( d(H) \) is even or odd, where

\[
W = N_G(H)/H
\]

and

\[
\mathcal{F}(p) : \mathbb{Z}_p[W \ltimes \pi_1(Y^H)] \to \mathbb{Z}_p[W].
\]

However, by Theorem 2.6 one can regard \( \sigma(f^H, b^H) \in L^h_{d(H)}(\mathbb{Z}_p[W], w_{Y^H}) \) even if \( d(H) \) is even.

Let \( \gamma \) denote the element in \( \Omega(G) \) such that \( 1 + \gamma = (1 - \beta)^{|G|} \). We consider a \( G \)-connected sum \( X' = (1 + \gamma)X \) and a \( G \)-framed map \((f', b')\) of the form

\[
(f', b') = ((1 + \gamma)f, (1 + \gamma)b).
\]

If \( H' \) is conjugate to \( H \), then the reduced homology groups \( \overline{H}_i(X^H; \mathbb{Z}_p) \) and \( \overline{H}_i(Y^H; \mathbb{Z}_p) \) are trivial for all \( i < [d(H)/2] \) and \( \overline{H}_i(\partial X^H; \mathbb{Z}_p) \) are also trivial for all \( i < d(H) - 1 \).

It follows that \( \overline{H}_i(X^H; \mathbb{Z}_p) = 0 \) for all \( i < [d(H)/2] \), although \( f' \) may not be \([d(H)/2]-\)connected.

Suppose \( d(H) = \dim Y^H \) is even, \( d(H) = 2k \) say. Consider the \((-1)^k\)-form

\[
\alpha = (K_k(X^H; \mathbb{Z}_p[\pi_1(Y^H)]), \varphi_{X^H}, \mu_{X^H})
\]

over \( \mathcal{F}(p) \) that determines \( \sigma(f^H, b^H) \), where

\[
K_k(X^H; \mathbb{Z}_p[\pi_1(Y^H)]) = \ker [\widetilde{H}_k : H_k(X^H; \mathbb{Z}_p) \to H_k(Y^H; \mathbb{Z}_p)],
\]

\( \varphi_{X^H} \) and \( \mu_{X^H} \) are the equivariant intersection and self-intersection forms, respectively. In this situation, the Hurewicz homomorphism \( \mathbb{Z}_p \otimes \pi_{k+1}(\overline{f^H}) \to K_k(X^H; \mathbb{Z}_p[\pi_1(Y^H)]) \) is
an isomorphism, and $K_k(X^H; \mathbb{Z}_p[\pi_1(Y^H)])$ is a $\mathbb{Z}_p[\widehat{W}]$-direct summand of $H_k(\widehat{X}^H; \mathbb{Z}_p)$, where

$$\widehat{W} = \pi_1(EW \times_W Y^H).$$

Let $\widehat{X}^H$ denote the covering space $f'^H \cdot \widehat{Y}^H$ over $X^H$ induced from $\widehat{Y}^H$ by $f'^H : X'^H \to Y^H$, and $f'^H : \widehat{X}^H \to \widehat{Y}^H$ the canonical map covering $f'^H$. The canonical inclusion

$$X_0 \to X' = X \rightharpoonup Z \quad \text{(to the first factor),}$$

where $X_0$ is a $G$-subset obtained by removing finitely many small open disks from $X$, and $Z = \gamma X$, induces the canonical monomorphisms

$$\pi_1(X^H) \to \pi_1(X'^H)$$

and

$$\widehat{W} \to \widehat{W},$$

where $\widehat{W} = \pi_1(EW \times_W X'^H)$. Note that forgetting $G$-actions, $X'$ is a connected sum of copies of $X$, moreover $X'^H$ is a connected sum of copies of certain $X'^1Hg_j$'s, say

$$X'^H = \bigoplus_{j=1}^m (g_j, X_j^{g_j^{-1}Hg_j}).$$

Thus, the $\mathbb{Z}_p[\widehat{W}]$-module

$$K_k(X'^H; \mathbb{Z}_p[\pi_1(Y^H)]) = \text{Ker}[f'^H_* : H_k(\widehat{X}^H; \mathbb{Z}_p) \to H_k(\widehat{Y}^H; \mathbb{Z}_p)]$$

includes the $\mathbb{Z}_p[\widehat{W}]$-direct summand

$$\widehat{M} = \bigoplus_{j=1}^m K_k(X_j^{g_j^{-1}Hg_j}; \mathbb{Z}_p[\pi_1(Y^H)]).$$

Moreover $\mathbb{Z}_p \otimes \pi_{k+1}(f'^H)$ contains the $\mathbb{Z}_p$-submodule

$$\bigoplus_{j=1}^m \mathbb{Z}_p \otimes \pi_{k+1}(f_j^{g_j^{-1}Hg_j}) \quad (= \widehat{M})$$

and this module has a natural $\widehat{W}$-action. Let $\widehat{X}^H$ denote the universal covering space of $X'^H$ and $f'^H$ the canonical map $X'^H \to Y^H$. Then the diagram

$$\begin{array}{ccc}
X'^H & \xrightarrow{f'^H} & Y^H \\
\downarrow & & \downarrow \\
\widehat{X}^H & \xrightarrow{f'^H} & \widehat{Y}^H
\end{array}$$

commutes. Then $H_{k+1}(f'^H; \mathbb{Z}_p)$ contains the $\mathbb{Z}_p[\widehat{W}]$-submodule

$$\overline{M} = \bigoplus_{j=1}^m \mathbb{Z}_p[\pi_1(X'^H)] \otimes_{\mathbb{Z}_p[\pi_1(Y^H)]} K_k(X_j^{g_j^{-1}Hg_j}; \mathbb{Z}_p[\pi_1(Y^H)]).$$
and $\mathbb{Z}_p(\otimes \pi_{k+1}(f^H))$ does the $\mathbb{Z}_p[\hat{W}]$-submodule
\[
\bigoplus_{j=1}^m \mathbb{Z}_p[\pi_1(X^H)] \otimes_{\mathbb{Z}_p[\pi_1(X^H)]} (\mathbb{Z}_p(\otimes \pi_{k+1}(f_j^{-1}H))) = \hat{M}.
\]

In addition, one has the canonical $\mathbb{Z}_p[\hat{W}]$-splitting
\[
\psi : \hat{M} \to \hat{M} \quad (\subseteq \mathbb{Z}_p(\otimes \pi_{k+1}(f^H))).
\]

Each element of $\hat{M}$ can be represented by a commutative diagram
\[
\begin{array}{ccc}
S^k & \xrightarrow{h} & X^H \\
\downarrow & & \downarrow f^H \\
D^{k+1} & \longrightarrow & \hat{Y}^H
\end{array}
\]
such that $h$ is an immersion with trivial normal bundle. Thus, we have the equivariant intersection form
\[
\tilde{\varphi} : \hat{M} \times \hat{M} \to \mathbb{Z}_p[\hat{W}]
\]
and the equivariant self-intersection form
\[
\tilde{\mu} : \hat{M} \to \mathbb{Z}_p[\hat{W}] / \min(-1)^k(\mathbb{Z}_p[\hat{W}]).
\]

Then, $\tilde{\alpha} = (\hat{M}, \tilde{\varphi}, \tilde{\mu})$ is a $(-1)^k$-form over $\mathcal{F}_p : \mathbb{Z}_p[\hat{W}] \to \mathbb{Z}_p[\hat{W}]$. Since $\hat{M}_{\mathbb{Z}_p[\hat{W}]} = \hat{M}$, we can define $\hat{\alpha} = (\hat{M}, \tilde{\varphi}, \tilde{\mu})$ to be the induced $(-1)^k$-form
\[
\tilde{\alpha}_{\mathbb{Z}_p[\hat{W}]} = (\hat{M}_{\mathbb{Z}_p[\hat{W}]} \tilde{\varphi}_{\mathbb{Z}_p[\hat{W}]} \tilde{\mu}_{\mathbb{Z}_p[\hat{W}]}).
\]

over $\mathcal{F}_p$. By construction, $\tilde{\varphi}(\psi(x), \psi(y))$ vanishes if $\tilde{\varphi}(x, y) = 0$; and $\tilde{\mu}(\psi(x))$ vanishes if $\tilde{\mu}(x) = 0$, where $x, y \in \hat{M}$. It implies the next claim.

**Claim 5.1.** If $\hat{\alpha}$ is strongly equivalent to zero, then so is $\tilde{\alpha}$.

Furthermore, if $[\hat{\alpha}] = 0$ in $\Gamma_{\mathcal{F}_p}(\mathcal{F}_p)$, then we can perform $(k - 1, k)$-dimensional $W$-surgery of $(f^H, b^H)$, relatively to the boundary and the singular set, so that the resulting map is a $\mathbb{Z}_p$-homology equivalence. For the goal, we will show that $[\hat{\alpha}]$ vanishes. Since the natural homomorphism $\Gamma_{\mathcal{F}_p}(\mathcal{F}_p) \to L_{2k}^h(\mathbb{Z}_p[\hat{W}], w_{Y^n})$ is an isomorphism by Theorem 2.6, it suffices to show $[\hat{\alpha}] = 0$ in $L_{2k}^h(\mathbb{Z}_p[\hat{W}], w_{Y^n})$.

Let $\text{Fix}_H^G : \Omega(G) \to \Omega(W)$ denote the homomorphism $[A] \mapsto [A^H]$ for a finite $G$-CW complex $A$. The bifunctor
\[
T \mapsto L_{2k}^h(\mathbb{Z}_p[T], w_{Y^n})
\]
with canonical correspondence on morphisms from the category of subgroups of $W$ to the category of abelian groups is not necessarily a Mackey functor, but is a $w_{Y^n}$-Mackey functor (see [16, Section 2] or [9, Section 2]). So, it is a module over the Burnside ring functor $T \mapsto \Omega(T)$, where $T \in S(W)$. Furthermore, by Theorem 12.10 of [16]
(cf. [30, Theorem 1]), the bifunctor is a module over the Grothendieck-Witt ring functor $T \mapsto GW_0(\mathbb{Z}, T)$, where $T \in \mathcal{S}(W)$. By the construction of $\widehat{\alpha}$, we have

\begin{equation}
\widehat{\alpha} = \text{Fix}^G_H(1 + \gamma) \sigma(f^H, b^H) \quad \text{in } L^h_{2k}(\mathbb{Z}(p)[W], w_{Y^H}).
\end{equation}

Since $\text{Res}^G_K(1 - \beta) = 0$ for all $K \in \mathcal{G}_1(G)$, it follows that $\text{Res}^W_K \text{Fix}^G_H(1 - \beta) = 0$ for all $H \subset K \subset N_G(H)$ such that $K/H$ is cyclic. Thus, by Dress' induction theorem [5, Theorem 3] $(\text{Fix}^G_H(1 - \beta)) \cdot 1_{GW_0(\mathbb{Z}, W)}$ is annihilated by 4. Furthermore, Proposition 6.8 of [10] implies

\[(\text{Fix}^G_H(1 + \gamma)) \cdot 1_{GW_0(\mathbb{Z}, W)} = 0.
\]

(Note $|G| \geq 2\ell + 3$ for the integer $\ell$ such that $|G| = 2^\ell m$ with $m$ odd.) By (5.2), we obtain $[\widehat{\alpha}] = 0$ in $L^h_{2k}(\mathbb{Z}(p)[W], w_{Y^H})$.

Thus, one can perform $W$-surgery on $(f^H, b^H)$ relatively to the singular set and the boundary so that the resulting map is a $\mathbb{Z}(p)$-homology equivalence. Furthermore, the $W$-surgery of $(f^H, b^H)$ can be thickened to a $G$-surgery of $(f', b')$ of isotropy type $(H)$ relative to the boundary so that $f'^H : X'^H \to Y^H$ is a $\mathbb{Z}(p)$-homology equivalence for resulting $(f'', b'')$.

Next, suppose $d(H) = \dim Y^H$ is odd, $d(H) = 2k + 1$ say. We can construct the same butterfly diagram as [12, Diagram 4.2] (cf. [3, p.295, *(B)]) for $X$ and $G$ replaced by $X^H$ and $W = N_G(H)/H$ respectively, and obtain the nonsingular $(-1)^k$-formation (cf. [26])

$$\alpha = (K_k(\partial WU; \mathbb{Z}(p)), K_{k+1}(WU, \partial WU; \mathbb{Z}(p)), K_{k+1}(X^H_0, \partial WU; \mathbb{Z}(p)), \varphi_{\partial WU}, \mu_{\partial WU})$$

over $\mathbb{Z}(p)[W]$. Here we can choose $U \subset \text{Int}(X^H)$ such that $U \cap gU = \emptyset$ for all $g \in G \setminus H$. The equivalence class of $\alpha$ in $L^h_{2k+1}(\mathbb{Z}(p)[W], w_{Y^H})$ is the surgery obstruction $\sigma(f^H, b^H)$. By Theorems 1.1 and 1.2 of [9], the bifunctor

$$T \mapsto L^h_{2k+1}(\mathbb{Z}(p)[T], w_{Y^H}|_T)$$

with canonical correspondence on morphisms from the category of subgroups of $W$ to the category of abelian groups is a module over the Burnside ring functor $T \mapsto \Omega(T)$, where $T \in \mathcal{S}(W)$, and moreover a module over the Grothendieck-Witt ring functor $T \mapsto GW_0(\mathbb{Z}, T)$, where $T \in \mathcal{S}(W)$.

For each $G$-component $G \times_{L_j} X$ of $\gamma X$ with $H \subset L_j$ up to conjugation, consider the subset $S_j := G \times_{L_j} (GU)$, where $GU = \{gu \in X | g \in G, \ u \in U\}$. Let $WU'$ denote the subset of $X'^H$ defined by

$$WU' = WU \cup \bigcup_j S^H_j,$$

where the union is taken over $j$ for all $G$-components $G \times_{L_j} X$ of $\gamma X$ with $H \subset L_j$ up to conjugation, and set $X'^H_0 = X'^H \setminus \text{Int}(WU')$. Then we obtain the $(-1)^k$-formation

$$\alpha' = (K_k(\partial WU'; \mathbb{Z}(p)), K_{k+1}(WU', \partial WU'; \mathbb{Z}(p)), K_{k+1}(X'^H_0, \partial WU'; \mathbb{Z}(p)), \varphi_{\partial WU'}, \mu_{\partial WU'})$$
over $\mathbb{Z}_p[W]$, and $\alpha'$ determines an element $\overline{\sigma}(f^H, b^H)$ in $L^h_{d(H)}(\mathbb{Z}_p[W], w_Y)$. It follows from the construction that

$$\overline{\sigma}(f^H, b^H) = \text{Fix}_H^G(1 + \gamma)\sigma(f^H, b^H).$$

Thus, by Dress’ induction argument as in the case where $d(H)$ is even, we have $\overline{\sigma}(f^H, b^H) = 0$. By using the same arguments as [12, §4], we can alter $(f', b')$, by $(k-1, k$-dimensional) $G$-surgery of isotropy type $(H)$ relative to the boundary, so that $f''^H : X''^H \to Y^H$ is a $\mathbb{Z}_p$-homology equivalence for resulting $(f'', b'')$.

In result, if $Y^H$ is simply connected, then $X''^H$ is also simply connected.

**Case 3** : $H = \{e\}$. First, convert $f : X \to Y$ to a $k$-connected one, where $k = \lfloor n/2 \rfloor$ and $n = \dim Y$, by $G$-surgery on $(f, b)$ of isotropy type $\{e\}$ relative to the boundary. According as $n$ is even or odd, $K_k(X; \mathbb{Z})$ or $K_k(X_0; \mathbb{Z})$ is projective over $\mathbb{Z}[G]$.

We claim that $(1 - \beta)\tilde{K}_0(\mathbb{Z}[G]) = 0$ and hence

$$(1 + (-\beta)^\%\tilde{K}_0(\mathbb{Z}[G]) = 0.$$ 

By Swan [29, Corollary 9.1], $\tilde{K}_0(\mathbb{Z}[G])$ is a module over the rational representation ring $R(G, \mathbb{Q})$. The module $\mathbb{Q}$ with trivial $G$-action represents the identity element of $R(G, \mathbb{Q})$. Since Res$_C^G(1 - \beta) = 0$ for all cyclic subgroups $C$ of $G$, $(1 - \beta)[\mathbb{Q}]$ vanishes in $R(G, \mathbb{Q})$. Thus we get

$$(1 - \beta)\tilde{K}_0(\mathbb{Z}[G]) = ((1 - \beta)[\mathbb{Q}])\tilde{K}_0(\mathbb{Z}[G]) = 0.$$ 

Consider a $G$-connected sum $X'$ of the form $X' = (1 + (-\beta)^\%X$ and a $G$-framed map $(f', b') = ((1 + (-\beta)^\%f, (1 + (-\beta)^\%b).$

Then $K_k(X'; \mathbb{Z})$ or $K_k(X_0; \mathbb{Z})$ is stably free over $\mathbb{Z}[G]$. Thus the obstruction $\sigma(f', b')$ for converting it to a homotopy equivalence by $G$-surgery of isotropy type $\{e\}$ relative to the boundary, lies in $L^h_{d(H)}(\mathbb{Z}[G], w_Y)$. By Dress’ induction theorem (or [10, Proposition 6.8]), for a $G$-framed map $(f'', b'')$ of the form

$$(f'', b'') = ((1 + \gamma)f', (1 + \gamma)b'),$$

the $G$-surgery obstruction $\sigma(f'', b'')$ vanishes. Thus we can perform $G$-surgery of $(f'', b'')$, of isotropy type $\{e\}$ relative to the boundary, so that resulting $f^{(3)} : X^{(3)} \to Y$ is a homotopy equivalence.

We have completed the proof of Theorem 1.1.

6. **Applications of Theorem 1.1**

In the present section we prove Theorems 1.3 and 1.4, which determine the diffeomorphism types of closed manifolds which can be obtained as the fixed-point sets of smooth $G$-actions on spheres.

**Proof of Theorem 1.3.** Let $G$ be a nontrivial perfect group and $F$ a closed manifold. In this case, $\mathcal{L}(G) = \{G\}$ holds.
The ‘only if part’ follows from [20, Theorem 0.1]. But for reader’s convenience, we give
the proof here. Suppose there exists a smooth $G$-action on a sphere $\Sigma$ such that $\Sigma^G = F$
and $\Sigma^P \neq \Sigma^G$ for all Sylow subgroups $P$ of $G$. Let $\eta$ denote the $G$-vector bundle $T(\Sigma)|_F$,
the restriction of $T(\Sigma)$ over $F$. For a Sylow subgroup $P$ of $G$, take a point $x_P \in \Sigma^P \setminus F$
and a $P$-invariant open disk-neighborhood $U_P$ of $x_P$ in $\Sigma \setminus F$, and set $X_P = \Sigma \setminus U_P$.
Clearly $X_P$ is contractible, which implies $[\text{Res}_P^G \eta] = 0$ in $KO(F)$. Since $F \subset X_P^P$, by
the Smith theory we get $[\text{Res}_P^G \eta] = 0$ in $KO_P(F)(p)$, where $p$ is the prime dividing $|P|$.
In addition, we have $\eta^G = T(F)$. Thus, by [22, Theorem 0.2] the condition on $T(F)$
described in Theorem 1.3 is necessary.

Next, we prove the ‘if part’. Suppose the condition on $T(F)$ described in Theorem 1.3
is fulfilled. Since $n_G = 1$, by [22, Theorem 0.2] there exists a smooth $G$-action on a disk
$D$ such that $F = \Delta^G$. Moreover, we can choose the $G$-action so that both $\Delta^P$ and $\partial \Delta^P$
are connected, the canonical homomorphism $\pi_1(\partial \Delta^P) \to \pi_1(\Delta^P)$ is an isomorphism and
$\pi_1(\Delta^P)$ is finite abelian and of order prime to $|P|$ for any $P \in \mathcal{P}(G)$. (This follows from
[18, Theorem 1.4] for $T = \emptyset = \mathcal{F}$ and $A = F$ with the equivariant thickening theorem,
alternatively from [19, Theorem 0.3 and Remark 3.8].) Let $V(G)$ denote the orthogonal
complement of $\mathbb{R}[G]^G$ in $\mathbb{R}[G]$ with respect to a $G$-invariant inner product: namely,
$$\mathbb{R}[G] = \mathbb{R}[G]^G \oplus V(G).$$
In order to use Theorem 1.1, here we adopt $n_i = 0$ for each connected component $F_i$
of $F$, and $Y = \Delta \times D(V(G))^m$ for an arbitrary integer $m$ with $m \geq \dim \Delta + 3$. Then,
it is straightforward to check that $Y$ satisfies Conditions (1.1.1)–(1.1.6) in Theorem 1.1.
Thus, there exists a smooth $G$-action on the disk $D$ of the dimension same as $Y$ with the
properties (i)–(iii) described in the theorem. Particularly, $D^G = \emptyset$ holds. Now set
$$\Sigma = (-Y) \cup_{\partial} D.$$ Clearly $\Sigma$ is a homotopy sphere, $\Sigma^G = F$ and $\dim \Sigma^P > \dim \Sigma^G$ for any $P \in \mathcal{P}(G)$. By
[11, Proposition 1.3], namely taking a $G$-connected sum of $\Sigma$ and $G \times P$ $\Sigma$’s for Sylow
subgroups $P$ of $G$, we can modify $\Sigma$ to the standard sphere of the same dimension with
a desired smooth $G$-action.

$$\text{Proof of Theorem 1.4.}$ Let $G$ be an Oliver group of Type (D) with a cyclic $pqr$-quotient.
The ‘only if part’ follows from [20, Theorem 0.1]. Hereafter, we prove the ‘if part’. Let
$F$ be a stably complex, closed, smooth manifold. Since $G$ possesses a cyclic $pqr$-quotient,
there exists a pair $(V, W)$ consisting of complex $G$-modules $V$ and $W$ such that
$$\text{Res}_P^G V \cong \text{Res}_P^G W$$
for all Sylow subgroups $P$ of $G$, $\dim_{\mathbb{C}} V^H = 1$ and $W^H = \{0\}$ for all $H \in \mathcal{L}(G)$ (see
[19, Example 1.5] or [20, Example 32]). Since $F$ is stably complex, the vector bundle
$T(F) \oplus \epsilon_F(\mathbb{R}^s)$ has a complex structure for some nonnegative integer $s$. Let $\nu_0$ be a
complex normal bundle of $F$, more precisely of $T(F) \oplus \epsilon_F(\mathbb{R}^s)$, with trivial $G$-action.
Consider the $G$-vector bundles
\[
\eta = \left( (T(F) \oplus \varepsilon_F(\mathbb{R}^n)) \otimes_{\mathbb{C}} V \right) \oplus (\nu_0 \otimes_{\mathbb{C}} W)
\]
and
\[
\nu = \eta - \eta^G \quad (= \eta - (T(F) \oplus \varepsilon_F(\mathbb{R}^n))).
\]
By Theorem 0.3 and Remark 3.8 of [19], we can obtain a smooth $G$-action on a disk $\Delta$ satisfying the properties:

1. $\Delta^G = F$,
2. $\Delta^H \subseteq \Delta^G$ or $\Delta^H \cap \Delta^G = \emptyset$ for any connected component $\Delta^H$ of $\Delta^G$ with $H \in \mathcal{L}(G)$,
3. $\partial \Delta^P$ and $\partial \Delta^P$ are connected,
4. the canonical homomorphism $\pi_1(\partial \Delta^P) \to \pi_1(\Delta^P)$ is an isomorphism,
5. $\pi_1(\Delta^P)$ is a finite abelian group of order prime to $|P|$, for any $P \in \mathcal{P}(G)$. Let $V(G)$ denote the $G$-module
\[
(\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \mid |G|} (\mathbb{R}[G/G^p] - \mathbb{R})
\]
defined in [10, §2]. Using Theorem 2.3 of [10], we can check that the disk
\[
Y = \Delta \times D(V(G)^m)
\]
satisfies Conditions (1.1.1)–(1.1.3), where $m$ is an arbitrary integer with $m \geq \dim \Delta + 3$. (Note that $V(G)$ is a gap module in the sense of [21].) Thus, Theorem 1.1 provides a smooth $G$-action on a disk $D$ such that $\partial D = \partial Y$ (as smooth $G$-manifolds), $D^G = \emptyset$ and $\dim D^H = \dim Y^H$ for any subgroup $H \in S(G) \setminus \mathcal{L}(G)$. Then the homotopy sphere $\Sigma = (-Y) \cup_0 D$ has the property that $\Sigma^G = F$ and $\dim \Sigma^P > \dim \Sigma^G$ for any subgroup $P \in \mathcal{P}(G)$. By [11, Proposition 1.3], there exists a smooth $G$-action on the standard sphere $S$ such that $\dim S = \dim \Sigma$, $S^G = F$ and $\dim S^P > \dim S^G$ for all $P \in \mathcal{P}(G)$. □

References


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