THE LOWER ALGEBRAIC $K$-THEORY OF $\Gamma_3$

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Abstract. We explicitly compute the lower algebraic $K$-theory of $\Gamma_3$ a discrete subgroup of the group of isometries of hyperbolic 3-space.

1. Introduction

In this paper we prove the following theorem:

Main Theorem. Let $\Gamma_3 = O^+(3,1) \cap GL(4,\mathbb{Z})$. Then the lower algebraic $K$-theory of the integral group ring of $\Gamma_3$ is given as follows:

$\mathrm{Wh}(\Gamma_3) = 0$,

$K_0(\mathbb{Z}[\Gamma_3]) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$,

$K_{-1}(\mathbb{Z}[\Gamma_3]) \cong \mathbb{Z} \oplus \mathbb{Z}$, and

$K_n(\mathbb{Z}[\Gamma_3]) = 0$, for $n < -1$.

For an arbitrary discrete group $\Gamma$, it has been conjecture that the algebraic $K$-theory of the integral group ring $\mathbb{Z}[\Gamma]$ may be computed from the corresponding $K$-groups of certain subgroups of $\Gamma$. More precisely, the Farrell and Jones Isomorphism Conjecture [FJ93] states that the algebraic $K$-theory of $\mathbb{Z}[\Gamma]$ may be computed from the algebraic $K$-theory of the virtually cyclic subgroups of $\Gamma$ (where a group is called virtually cyclic if it has a cyclic subgroup of finite index) via an appropriate “assembly map” (see Section 2 for a precise statement and definitions). In [FJ93] Farrell and Jones proved the Isomorphism Conjecture in lower algebraic $K$-theory for cocompact discrete subgroups of a virtually connected Lie group, in particular for discrete groups acting properly discontinuously and cocompactly by isometries on a simply connected symmetric Riemannian manifold $M$ with sectional curvature non-positive everywhere. In [BFPP00] Berkove, Farrell, Pineda and Pearson extend this result to discrete groups acting properly discontinuously on hyperbolic $n$-space via isometries whose orbit space has finite volume (but non necessarily compact).

Let $\Gamma_n = O^+(n,1) \cap GL(n+1,\mathbb{Z})$, where $O^+(n,1)$ denotes the group of isometries of the Riemannian manifold $\mathbb{H}^n$. The group $\Gamma_n$ is a discrete subgroup of $O^+(n,1)$, since $\Gamma_n$ is a subgroup of the discrete group $GL(n+1,\mathbb{Z})$. The groups $\Gamma_n$ are hyperbolic, non-cocompact, $n$-simplex, reflection groups for $n = 3, \ldots, 9$, (see Section 3). The groups $\Gamma_n$ form a nice family of infinite groups with torsion for which the Isomorphism Conjecture in algebraic $K$-theory holds. In this paper we compute the lower algebraic $K$-theory of the integral group ring $\mathbb{Z}[\Gamma_3]$. We accomplish our computations using the fundamental results of Farrell and Jones [FJ93] and the algebraic techniques of Davis and Lück [DL98].
2. Review of the Isomorphism Conjecture in $K$-Theory

In this section we introduce some notation that is used throughout this paper. The definitions and results provided here are brief, and the interested reader should refer to [DL98].

Let $\Gamma$ be a discrete group and $\mathcal{F}$ be a family of subgroups of $\Gamma$ closed under inclusion and conjugation, i.e. if $H \in \mathcal{F}$ then $g^{-1}H'g \in \mathcal{F}$ for all $H' \subseteq H$ and all $g \in \Gamma$. Some examples for $\mathcal{F}$ are $\mathcal{TR}$, $\mathcal{ELN}$, $\mathcal{VC}$, and $\mathcal{ACL}$, which are the families consisting of the trivial group, finite subgroups, virtually cyclic groups, and all subgroups respectively.

The orbit category $\text{Or}(\Gamma)$ is the category whose objects are homogeneous $\Gamma$-spaces $\Gamma/H$, considered as left $\Gamma$-sets, and whose morphisms are $\Gamma$-maps. More generally, for a family of subgroups $\mathcal{F}$, defined the restricted orbit category $\text{Or}(\Gamma, \mathcal{F})$ to be the category whose objects are homogeneous spaces $\Gamma/H$ where $H \in \mathcal{F}$. If $\mathcal{F}$ is the family $\mathcal{ACL}$, we abbreviate $\text{Or}(\Gamma, \mathcal{ACL})$ by $\text{Or}(\Gamma)$.

A covariant (contravariant) $\text{Or}(\Gamma)$-space $X$ is a covariant (contravariant) functor $X : \text{Or}(\Gamma) \rightarrow \text{SPACES}.$

from $\text{Or}(\Gamma)$ to the category of compactly generated spaces. A map between $\text{Or}(\Gamma)$-spaces is a natural transformation of such functors. A covariant (contravariant) $\text{Or}(\Gamma, \mathcal{TR})$-space is the same as a left (right) $\Gamma$-space. Maps of $\text{Or}(\Gamma, \mathcal{TR})$-spaces correspond to $\Gamma$-maps.

A spectrum $E = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of based spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps $\sigma(n) : (E(n) \wedge S^1) \rightarrow E(n+1)$, called structure maps. A spectrum $E$ is called a $\Omega$-spectrum if for each structure map, its adjoint $E_n \rightarrow \Omega E_{n+1} = \text{map}(S^1, E_{n+1})$ is a weak homotopy equivalence of spaces. We denote by $\text{SPECTRA}$ the corresponding full subcategory of $\text{SPACES}$. A map of spectra is a sequence of maps which strictly commute with the structure maps in an obvious sense.

The homotopy groups of a spectrum $E$ are defined by

$$\pi_q(E) := \lim_{n \rightarrow \infty} \pi_{q+n}(E(n)),$$

where the system $\pi_{q+n}(E(n))$ is given by the composition

$$\pi_{q+n}(E(n)) \xrightarrow{\Sigma} \pi_{q+n+1}(E(n) \wedge S^1) \xrightarrow{\sigma(n)} \pi_{q+n+1}(E(n+1))$$

of the suspension homomorphism and the homomorphism induced by the structure maps.

More generally, one can also take the homology of a space with coefficients in a spectrum by

$$H_q(Y; E) := \lim_{n \rightarrow \infty} \pi_{q+n}(Y_n \wedge E(n)),$$

letting $Y$ be a point recovers the coefficient groups. Homology with coefficients in a specified spectrum is a generalized homology theory; furthermore any generalized homology theory has a spectrum giving rise to it.

Associated to each covariant $\text{Or}(\Gamma)$-spectrum $E(\cdot)$ (we use a question mark to indicate the place where objects are plugged into the functor) is a generalized equivariant homology theory satisfying the WHE-axiom and the disjoint union axiom, for example, there is a long exact Mayer-Vietories sequence, and the WHE-axiom requires that weak homotopy equivalence of covariant $\text{Or}(\Gamma)$-spaces induces an isomorphism on homology groups.
This generalized homology theory is constructed as follows: every \( \Gamma \)-space \( X \) gives rise to a contravariant \( \text{Or}(\Gamma) \)-space \( X^\gamma = \text{map}_\Gamma(\Gamma/\gamma, X) \) and we can form the balance smash product over the orbit category between a contravariant \( \text{Or}(\Gamma) \)-space and a covariant \( \text{Or}(\Gamma) \)-spectrum to obtain an ordinary spectrum, e.g.,

\[
X^\gamma \wedge_{\text{Or}(\Gamma)} \mathbb{K}(?) = \bigvee_{\Gamma/H \in \text{Or}(\Gamma)} X^H_+ \wedge \mathbb{K}(H) / \sim.
\]

Compare [DL98, pg.237]. This construction is functorial in \( X \) and satisfies the properties listed above. To stress the homological behavior we write the homotopy groups of the spectrum as

\[
H^\text{Or}(\Gamma)_n(X; \mathbb{K}) = \pi_n(X^\gamma_+ \wedge_{\text{Or}(\Gamma)} \mathbb{K}(?)�).
\]

Note that if \( \mathcal{C} \) is a category with a single object, all whose morphisms are isomorphism (e.g. \( \text{Or}(\Gamma, \mathcal{T} \mathcal{R}) \)), this generalized homology theory reduces to Borel homology.

Let \( \mathcal{C} \) be a (small) additive category. In [PedW85] the authors construct a non-connective spectrum whose homotopy groups are the algebraic \( K \)-groups of \( \mathcal{C} \) (including the negative groups). We denote this spectrum by \( \mathbb{K}^{-\infty}(\mathcal{C}) \). In fact \( \mathbb{K}^{-\infty} \) is a functor from additive categories to \( \text{SPECTRA} \), i.e. an additive and hence exact functor induces a map of spectra.

Let us now recall the construction of the algebraic \( K \)-theory \( \text{Or}(\Gamma, \mathcal{F}) \)-spectrum defined by Davis and Lück in [DL98]. We will denote it by \( \mathbb{K}R^{-\infty} \) with \( R \) an arbitrary associative ring with unit. Given any \( \Gamma \)-set \( S \) define the transport category \( \mathcal{S} \) the category whose objects are the elements of \( S \) and morphisms \( s \mapsto \{g \in \Gamma \mid gs = t \} \). The transport category is a groupoid, i.e. every morphism is an isomorphism. Given any small category \( \mathcal{C} \), we can form the the associated \( R \)-linear category \( R \mathcal{C} \) with the same objects and new morphism set \( \text{mor}_{R \mathcal{C}}(c, d) = R \text{mor}_{\mathcal{C}}(c, d) \) (the free \( R \)-module generated by the old morphism set). Finally we turn \( R \mathcal{C} \) into an additive category, i.e. we artificially introduce finite sums (or products). The resulting category is denoted by \( R \mathcal{C}_\oplus \). The Davis-Lück functor is now given as

\[
\mathbb{K}R^{-\infty} : \text{Or}(\Gamma, \mathcal{F}) \to \text{SPECTRA}
\]

\[
\Gamma/H \mapsto \mathbb{K}^{-\infty}(R \Gamma \big/ \big/ H_\oplus).
\]

Note that \( \mathbb{K}R^{-\infty} \) and \( \mathbb{K}^{-\infty}(R) \) are different objects.

For any quotient, the category \( \Gamma \big/ \big/ H \) is equivalent to the category \( \text{Or}(\Gamma, \mathcal{T} \mathcal{R}) \), as a consequence the \( \text{Or}(\Gamma) \)-spectrum defined above has the key property that \( \pi_n(\mathbb{K}R^{-\infty} (\Gamma/H)) = K_n(RH) \).

For a family \( \mathcal{F} \) of subgroups of \( \Gamma \) we denote by \( E(\Gamma, \mathcal{F}) \) the universal space among \( \Gamma \)-spaces with isotropy in \( \mathcal{F} \). It is characterized by the universal property that for every \( \Gamma \)-CW complex \( X \) whose isotropy groups are all in \( \mathcal{F} \) one can find an equivariant continuous map \( X \to E(\Gamma, \mathcal{F}) \) which is unique up to equivariant homotopy. A \( \Gamma \)-CW-complex \( E \) is a model \( E(\Gamma, \mathcal{F}) \) if the \( H \)-fixed point sets \( E^H \) are contractible for all \( H \in \mathcal{F} \) and empty otherwise. The two extreme cases are \( \mathcal{F} = \mathcal{ALL} \), where \( E(\Gamma, \mathcal{F}) \) can be taken to be a point, and \( \mathcal{F} = \mathcal{T} \mathcal{R} \), where \( E(\Gamma, \mathcal{F}) \) is a model for \( ET \).

The projection map \( \text{pr} : E(\Gamma, \mathcal{F}) \to E(\Gamma, \mathcal{ALL}) = \{pt\} \) induces a map

\[
H^\text{Or}(\Gamma)_n(E(\Gamma, \mathcal{F}); \mathbb{K}R^{-\infty}) \to H^\text{Or}(\Gamma)_n(\{pt\}; \mathbb{K}R^{-\infty}) = K_n(R\Gamma),
\]
which is called the assembly map.

The Isomorphism Conjecture for a discrete group $\Gamma$ and a family $\mathcal{F}$ of subgroups is that the assembly map

$$H_n^{\text{Or}(\Gamma)}(E(\Gamma, \mathcal{F}); \mathbb{K}R^{-\infty}) \rightarrow H_n^{\text{Or}(\Gamma)}(\{pt\}; \mathbb{K}R^{-\infty}) = K_n(\mathbb{R} \Gamma),$$

is an isomorphism for all $n \in \mathbb{Z}$.

It is clear that for an arbitrary $\Gamma$ the Isomorphism Conjecture need not be valid. However, the Isomorphism Conjecture is always true (and therefore pointless!) when $\mathcal{F}$ is the family of all subgroups. The philosophy is that the smaller the family, the easier it is to compute $H_n^{\text{Or}(\Gamma)}(E(\Gamma, \mathcal{F}); \mathbb{K}R^{-\infty})$. The larger is the family, the closer the end result is to $K$-theory.

The Isomorphism Conjecture of Farrell and Jones for $\mathbb{R} \Gamma$ states that the assembly map

$$H_n^{\text{Or}(\Gamma)}(E(\Gamma, \mathcal{V} \mathcal{C}); \mathbb{K}R^{-\infty}) \rightarrow H_n^{\text{Or}(\Gamma)}(\{pt\}; \mathbb{K}R^{-\infty}) = K_n(\mathbb{R} \Gamma),$$

is an isomorphism for all $n \in \mathbb{Z}$.

The point of this conjecture is that they express the target, which is the group one wants to compute, by the source, which only involves the $K$-theory of the virtually cyclic subgroups and it is much easier to compute.

The following theorem establishes this result ([FJ93, Theorem 2.1]).

**Theorem 2.1** (Farrell, F.T. and Jones, L.E). Let $\Gamma$ be a cocompact discrete subgroup of a virtually connected Lie group. Then the assembly map

$$H_n^{\text{Or}(\Gamma)}(E(\Gamma, \mathcal{V} \mathcal{C}); \mathbb{K}Z^{-\infty}) \rightarrow K_n(\mathbb{Z} \Gamma)$$

is an isomorphism for $n \leq 1$ and a surjection for $n = 2$.

Farrell and Jones also proved Theorem 2.1 for discrete cocompact groups acting properly discontinuously by isometries on a simply connected Riemannian manifold $M$ with everywhere non-positive curvature ([FJ93, Proposition 2.3]). Berkov, Farrell, Pineda, and Pearson extended this result to discrete groups acting properly discontinuously on hyperbolic space $n$-space via isometries whose orbit space has finite volume (but not necessarily compact), (see [BFPP00, Theorem A]). In particular this result is valid for $\Gamma$ a hyperbolic, non-cocompact, $n$-simplex reflection group.

Sometimes we can use smaller families than $\mathcal{V} \mathcal{C}$ such as the family $\mathcal{F} \mathcal{L} \mathcal{N}$ of finite subgroups as explained in the next result which appeared in [LS00, Theorem 2.3] and for $n = \infty$ in [FJ93, Theorem A.10].

**Theorem 2.2**. Let $\mathcal{F} \subset \mathcal{F}'$ be two families of subgroups of $\Gamma$. For each $Q \in \mathcal{F}' - \mathcal{F}$, define the induced family of subgroups $\mathcal{F}_Q$ of $Q$ as $\mathcal{F}_Q = \{G \cap Q | G \in \mathcal{F}\}$. Suppose that for all $Q \in \mathcal{F}' - \mathcal{F}$ the assembly map

$$H_q^{\text{Or}(\Gamma, \mathcal{F}_Q)}(E(Q, \mathcal{F}_Q); \mathbb{K}Z^{-\infty}) \rightarrow K_q(\mathbb{Z}Q)$$

is an isomorphism for all $q \leq n$. Then the relative assembly map

$$H_q^{\text{Or}(\Gamma, \mathcal{F})}(E(\Gamma, \mathcal{F}); \mathbb{K}Z^{-\infty}) \rightarrow H_q^{\text{Or}(\Gamma, \mathcal{F}' \mathcal{F}(\Gamma, \mathcal{F}'; \mathbb{K}Z^{-\infty})$$

is an isomorphism for all $q \leq n$.

Our intention is to use Theorem 2.1 to compute the lower algebraic $K$-theory of the integral group ring $\mathbb{Z} \Gamma$. We would like to reduce this problem even further by applying Theorem 2.2 to the case $\mathcal{F} = \mathcal{F} \mathcal{L} \mathcal{N}$, $\mathcal{F}' = \mathcal{V} \mathcal{C}$ and $n < 2$. Here our first
task is to determined up to isomorphism all finite subgroups and all infinite virtually cyclic subgroups of $\Gamma_3$. Once this is established, we must check the assembly map condition given in Theorem 2.2 for each infinite virtually cyclic subgroup. If we can accomplished this, then we have shown that the relative assembly map

$$H^n_{\text{Or}}(\Gamma_3; E(\Gamma_3, F\mathcal{L}N); \mathbb{Z}^\infty) \rightarrow H^n_{\text{Or}}(\Gamma_3; V(C); \mathbb{Z}^\infty)$$

is an isomorphism for $n < 2$. By combining this assembly map with the assembly map in Theorem 2.1, we have shown that the assembly map

$$H^n_{\text{Or}}(\Gamma_3; E(\Gamma_3, F\mathcal{L}N); \mathbb{Z}^\infty) \rightarrow K_n(\mathbb{Z}\Gamma_3)$$

is an isomorphism for all $n < 2$.

Thus to compute the lower algebraic $K$-theory of the integral group ring $\mathbb{Z}\Gamma_3$ it suffices to compute for $n < 2$ the homotopy groups

$$H^n_{\text{Or}}(\Gamma_3; E(\Gamma_3, F\mathcal{L}N); \mathbb{Z}^\infty).$$

These computations are feasible using spectral sequences described by Davis and Lück in [DL98].

**Theorem 2.3** (Davis, J. and Lück W.). *There exists a spectral sequence*

$$E^2_{p,q} = H_p(E(G, F) / G; \{K_q(\mathbb{Z}G, \mathbb{Z})\}) \Rightarrow H_{p+q}^{\text{Or}}(\Gamma; \mathbb{Z}^\infty).$$

**3. The groups $\Gamma_n$**

Let $\mathbb{R}^{n}$ denote Minkowski space, that is, an $(n+1)$-dimensional real vector space with coordinates $x = (x_1, x_2, \ldots, x_{n+1})$, equipped with the bilinear form defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1}.$$ 

Hyperbolic $n$-space $\mathbb{H}^n$ can be defined as one sheet of the hyperboloid $\langle x, x \rangle = -1$, defined by $x_{n+1} > 0$. Let $O(n, 1)$ denote the isometry group of the bilinear form and let $O^+(n, 1)$ be the subgroup of index 2 that preserves the sheets of the hyperboloid. Then $O^+(n, 1)$ is the isometry group of the Riemannian manifold $\mathbb{H}^n$.

In the conformal ball model of hyperbolic $n$-space, we can identify boundary at infinity of hyperbolic $n$-space $\partial\mathbb{H}^n$, with the boundary of the ball, i.e., the sphere at infinity $\mathbb{S}^{n-1}$.

A *hyperbolic Coxeter $n$-simplex* is an $n$-dimensional simplex in $\mathbb{H}^n$, all of whose dihedral angles are submultiples of $\pi$ or zero. We allow a simplex in $\mathbb{H}^n$ to be unbounded with ideal vertices on the sphere at infinity of $\mathbb{H}^n$. Let $S$ be a side of a Coxeter $n$-simplex $\Delta$ in $\mathbb{H}^n$. The reflection of $\mathbb{H}^n$ in the side $S$ of $\Delta$ is the reflection of $\mathbb{H}^n$ in the hyperplane $\langle S \rangle$ spanned by $S$.

**Definition 3.1.** A *hyperbolic Coxeter $n$-simplex reflection group* is the group generated by reflections in the sides of a Coxeter $n$-simplex in $\mathbb{H}^n$.

A *hyperbolic $n$-simplex group* is a hyperbolic Coxeter $n$-simplex reflection group, with fundamental domain its defining Coxeter $n$-simplex $\Delta$. Hyperbolic $n$-simplex groups arise naturally in geometry as groups of symmetries of regular tessellations of $\mathbb{H}^n$. The hyperbolic Coxeter simplices were classified by H. S. M. Coxeter and G. J. Whidrow [CxW50], F. Lannér [L50], J.-L. Koszul [K68] and M. Chein [Ch69]. There are only finitely many hyperbolic Coxeter simplices for each dimension $n \geq 3$, and such simplices exist only in dimensions $n = 2, 3, \ldots, 9$.

A *Coxeter group* $W$ is an abstract group defined by a group presentation of the form $\langle S_1 \mid (S_i S_j)^{m_{ij}} \rangle$, where
(1) the indices \( i, j \) vary over some countable index set \( I \);
(2) \( m_{ij} \) is either a positive integer or \( \infty \) for each \( i, j \);
(3) \( m_{ij} = m_{ji} \);
(4) \( m_{ii} = 1 \) for each \( i \);
(5) \( m_{ij} > 1 \) if \( i \neq j \); and
(6) if \( m_{ij} = \infty \), then the relator \((S_i S_j)^{m_{ij}}\) is deleted.

Note that if \( i \neq j \), then we can obtain the relator \((S_i S_j)^{m_{ij}}\) from the relators \(S_i^2\), \(S_j^2\), and \((S_i S_j)^{m_{ij}}\); and therefore only one of the relators \((S_i S_j)^{m_{ij}}\) and \((S_j S_i)^{m_{ij}}\) is required and the other one may be deleted.

Let \( W = \langle S_i \mid (S_i S_j)^{m_{ij}} \rangle \) be a Coxeter group. The Coxeter graph of \( W \) is the labeled graph with vertices \( I \) and edges

\[
\{(i, j) : m_{ij} > 2\}.
\]

Each edge \((i, j)\) is labeled by \( m_{ij} \). For simplicity, the edges with \( m_{ij} = 3 \) are usually not labeled in a representation of a Coxeter graph.

Let \( W = \langle S_i \mid (S_i S_j)^{m_{ij}} \rangle \) be a Coxeter group. For a subset \( T \subset \{S_i\}_{i \in I} \) of the generating set, \( W_T \) is defined as the subgroup of \( W \) generated by \( T \), and is called a parabolic subgroup. It is known that \( W_T \) is also a Coxeter group (see [V72]). If \( T \) is the empty set, then \( W_T \) is defined to be the trivial group. It is well known that any finite subgroup of a Coxeter group \( W \) is conjugate to a subgroup of a parabolic group \( W_T \) for some \( T \) subset of the generating set (see [Da87]).

Let \( \Gamma \) be the group generated by the reflections in the sides on a Coxeter \( n \)-simplex \( \Delta \). Let \( \{S_i\} \) be the set of sides of \( \Delta \), and for each pair of indices \( i, j \), let \( m_{ij} = \pi/\theta(S_i, S_j) \), where \( \theta(S_i, S_j) \) is the dihedral angle between \( S_i \) and \( S_j \). Then the Coxeter group \( W = \langle S_i \mid (S_i S_j)^{m_{ij}} \rangle \) is isomorphic to \( \Gamma \), ([R94, Theorem 7.1.4]), e.g., \( \Gamma \) is a Coxeter group.

Let \( \Gamma_n \) be the subgroup of \( O^+(n, 1) \) that preserves the standard integer lattice \( \mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1} \), that is, \( \Gamma_n = O^+(n, 1) \cap GL(n+1, \mathbb{Z}) \).

Since \( \Gamma_n \) is a subgroup of the discrete group \( GL(n+1, \mathbb{Z}) \), it is also a discrete group of \( O^+(n, 1) \). The group \( \Gamma_n \) is a hyperbolic, noncompact, \( n \)-simplex, reflection group for \( n = 3, \ldots, 9 \) (see [R94, pg. 301]). The Coxeter graphs of these groups are listed in Figure 1.

![Figure 1](image-url)

**Figure 1.** The Coxeter graphs of the groups \( \Gamma_n \) for \( n = 3, \ldots, 9 \).
The groups $\Gamma_n$ form a nice family of discrete subgroups of isometries of hyperbolic $n$-space for which the Farrell and Jones Isomorphism Conjecture in algebraic $K$-theory holds. In this paper we use this result to explicitly compute the lower algebraic $K$-theory of the integral group ring $\mathbb{Z}\Gamma_3$.

In order to use Theorem 2.1 to compute the lower algebraic $K$-theory of $\mathbb{Z}\Gamma_3$, we must first classify up to isomorphism the family $\mathcal{V}_C$ of all virtually cyclic subgroups of $\Gamma_3$, where $\Gamma_3$ is a hyperbolic, tetrahedron, reflection group with group presentation

$$\Gamma_3 = \langle S_1, S_2, S_3, S_4 \mid S_1^2 = (S_1S_2)^2 = (S_1S_4)^2 = (S_1S_2S_3)^3 = (S_2S_3)^4 = (S_3S_4)^4 = 1 \rangle,$$

and Coxeter graph

![Coxeter graph of $\Gamma_3 = [3, 4, 4]$](image)

**Figure 2.** The Coxeter graph of $\Gamma_3 = [3, 4, 4]$

Recall that a group $G$ is **virtually cyclic** if $G$ is either finite or contains $\mathbb{Z}$ as a subgroup of finite index. We start with the classification of the finite subgroups of $\Gamma_3$. By looking at subsets of the generating set $\{S_1, S_2, S_3, S_4\}$, the following are all parabolic subgroups of $\Gamma_3$:

- $\langle S_i \mid S_i^3 = 1 \rangle \cong \mathbb{Z}/2$
- $\langle S_1, S_2 \mid S_1^2 = (S_1S_2)^3 = 1 \rangle \cong D_3$,
- $\langle S_1, S_3 \mid S_1^2 = (S_1S_3)^2 = 1 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$,
- $\langle S_1, S_4 \mid S_1^2 = (S_1S_4)^2 = 1 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$,
- $\langle S_2, S_3 \mid S_2^3 = (S_2S_3)^4 = 1 \rangle \cong D_4$,
- $\langle S_2, S_4 \mid S_2^3 = (S_2S_4)^4 = 1 \rangle \cong D_4$,
- $\langle S_1, S_2, S_3 \mid S_1^2 = (S_1S_2)^3 = (S_1S_3)^3 = (S_2S_3)^4 = 1 \rangle \cong [3, 4] \cong \mathbb{Z}/2 \times S_4$,
- $\langle S_1, S_2, S_4 \mid S_1^2 = (S_1S_2)^3 = (S_1S_4)^3 = (S_2S_4)^4 = 1 \rangle \cong \mathbb{Z}/2 \times D_3 \cong D_6$,
- $\langle S_1, S_3, S_4 \mid S_1^2 = (S_1S_3)^2 = (S_1S_4)^2 = (S_3S_4)^4 = 1 \rangle \cong \mathbb{Z}/2 \times D_4$,
- $\langle S_2, S_3, S_4 \mid S_2^2 = (S_2S_3)^4 = (S_3S_4)^4 = (S_2S_4)^2 = 1 \rangle \cong [4, 4] \cong P4m$.

Here $\mathbb{Z}/n$ denotes the cyclic group of order $n$, $D_n$ the dihedral group of order $2n$, $S_n$ the symmetric group of order $n!$, and $P4m$ is a two-dimensional crystallographic group (see [Pe98]).
Any finite subgroup of $\Gamma_3$ is conjugate to a subgroup of a parabolic group (see [Da87, Lemma 1.3]), thus the finite subgroups of $\Gamma_3$ up to isomorphism are: $\mathbb{Z}/2$, $\mathbb{Z}/3, \mathbb{Z}/4, D_2, \mathbb{Z}/6, D_3 \equiv S_3, \mathbb{Z}/2 \times \mathbb{Z}/4, (\mathbb{Z}/2)^3$, $D_4$, $D_6$, $A_4$, $\mathbb{Z}/2 \times D_4$, $S_4$, $\mathbb{Z}/2 \times A_4$, $\mathbb{Z}/2 \times S_4$.

To classify the infinite virtually cyclic subgroups of $\Gamma_3$, we use the following well known result (see [FJ95, Lemma 2.5]).

**Lemma 3.2.** Any infinite virtually cyclic group is either of type $F \rtimes_{\alpha} \mathbb{Z}$, where $F$ is a finite group, or it maps onto $D_\infty$ with finite kernel.

Here $F \rtimes_{\alpha} \mathbb{Z}$ denotes the semi-direct product of $F$ and $\mathbb{Z}$ where $\mathbb{Z}$ acts on $F$ by an automorphism $\alpha$ of $F$, and $D_\infty$ denotes the infinite dihedral group.

First, to identify the groups of type $F \rtimes_{\alpha} \mathbb{Z}$ which occurs in $\Gamma_3$ we observe that since $F$ is a finite group then $F \rtimes_{\alpha} \mathbb{Z}$ contains $1 \times |F|! \mathbb{Z}$. Since $1 \times |F|! \mathbb{Z}$ is contained in the centralizer of $F$ in $\Gamma_3$, then $F \rtimes_{\alpha} \mathbb{Z}$ can occur in $\Gamma_3$ only if $C_{\Gamma_3}(F)$ contains an element of infinite order.

There is some literature concerning centralizers of subgroups of Coxeter groups. In [Br96], Brink describes $C(W_T)$ in case that $T$ consists of a single generator. Using [Br96, Theorem in Section 2], we can show that the centralizers $C_{\Gamma_3}(S_i)$ for $i = 1,\ldots, 4$ are the Coxeter groups given in Figure 3.

![Coxeter graphs](image)

**Figure 3.** The Coxeter graphs of the centralizers $C_{\Gamma_3}(S_i)$ for $i = 1,\ldots, 4$

In [BM], Mihalik and Bahl give a complete description of the centralizer of an arbitrary parabolic subgroup of an even Coxeter group in terms of the generators. We thank M. Mihalik who kindly informed us of these results and made them accessible to us. He point out that the techniques used to prove [BM, Theorem
1.1] also apply to the groups \( \Gamma_n \). Using this information, the following are the centralizers of the remaining parabolic subgroups of \( \Gamma_3 \):

\[
C_{\Gamma_3}(\langle S_1, S_2 \rangle) = \langle S_4 \rangle \cong \mathbb{Z}/2,
\]

\[
C_{\Gamma_3}(\langle S_1, S_3 \rangle) = \langle S_1, S_3, S_4 S_2 S_1 \rangle \cong (\mathbb{Z}/2)^3,
\]

\[
C_{\Gamma_3}(\langle S_1, S_4 \rangle) = \langle S_1, S_4, S_2 S_3 S_2 \rangle \cong (\mathbb{Z}/2)^3,
\]

\[
C_{\Gamma_3}(\langle S_2, S_3 \rangle) \cong \langle 1 \rangle,
\]

\[
C_{\Gamma_3}(\langle S_2, S_4 \rangle) = \langle S_2, S_4 \rangle \cong (\mathbb{Z}/2)^2,
\]

\[
C_{\Gamma_3}(\langle S_3, S_4 \rangle) = \langle S_1, (S_4 S_3)^2 \rangle \cong (\mathbb{Z}/2)^2,
\]

\[
C_{\Gamma_3}(\langle S_1, S_2, S_3 \rangle) \cong \langle 1 \rangle,
\]

\[
C_{\Gamma_3}(\langle S_1, S_2, S_4 \rangle) = \langle S_4 \rangle \cong \mathbb{Z}/2,
\]

\[
C_{\Gamma_3}(\langle S_1, S_3, S_4 \rangle) = \langle S_1, (S_4 S_3)^2 \rangle \cong (\mathbb{Z}/2)^2.
\]

With this information we can exclude all parabolic subgroups except for those isomorphic to \( \mathbb{Z}/2 \).

**Proposition 3.3.** Let \( G \) be a subgroup of \( \Gamma_3 \) isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \cong D_2 \), then \( C_{\Gamma_3}(G) \) is finite.

**Proof.** Let \( X_3 \) be the fundamental 3-simplex of \( \Gamma_3 \) and let \( V \) be the set of vertices of \( X_3 \cap \mathbb{H}^3 \). Note that the fixed point set of \( G \) in \( \mathbb{H}^3 \) is either

\[
X^G = (\mathbb{H}^3)^G = \begin{cases} 
* & \text{a point in } \mathbb{H}^3 \\
L & \text{a line in } \mathbb{H}^3 \\
P & \text{a plane in } \mathbb{H}^3 
\end{cases}
\]

and \( C_{\Gamma_3}(G) \) leaves invariant \( X^G \). If \( X^G = * \), then \( C_{\Gamma_3}(G) \) is finite (\( C_{\Gamma_3}(G) \) acts properly discontinuously on \( \mathbb{H}^3 \) since \( \Gamma_3 \) does, and fixes a point). Hence we need only to consider the case where a line \( L \subset X^G \). Since \( \bigcup_{\gamma \in \Gamma_3} \langle \Gamma_3 V \rangle \) is countable, then \( \gamma \cdot x \in X_3 - V \) for some point \( x \in L \) and \( \gamma \in \Gamma_3 \). Therefore \( G \) is conjugate to a subgroup of a parabolic subgroup of \( \Gamma_3 \) which is different from

\[
\langle S_1, S_2, S_3 \rangle, \quad \langle S_1, S_2, S_4 \rangle, \quad \langle S_1, S_3, S_4 \rangle, \quad \langle S_2, S_3, S_4 \rangle.
\]

In fact it must be either

\[
P_1 = \langle S_1, S_3 \rangle, \quad P_2 = \langle S_1, S_4 \rangle,
\]

\[
P_3 = \langle S_2, S_3 \rangle, \quad P_4 = \langle S_2, S_4 \rangle \quad \text{or} \quad P_5 = \langle S_3, S_4 \rangle.
\]

If \( G = \gamma P_i \gamma^{-1} \) with \( i = 1, 2 \) or \( 4 \), then \( C_{\Gamma_3}(G) \) is finite by the previous calculations. Therefore we only need to consider the two cases: \( P_3 = \langle S_2, S_3 \rangle \) and \( P_5 = \langle S_3, S_4 \rangle \). In both of this two cases \( G \) fixes a line \( L \) such that \( X_3 \cap L \) is a half line. Hence \( C_{\Gamma_3}(G) \) can not contain an element of infinite order because if it does, i.e. if there exist \( \gamma \in C_{\Gamma_3}(G) \) of infinite order, then \( \gamma \) leaves \( L \) invariant since \( C_{\Gamma_3}(G) \) does and acts on \( L \) by isometries of \( L \); but since \( \text{Isom}(L) = \mathbb{R} \times \mathbb{Z}/2 \), then \( \gamma \) acts on \( L \) by a non-trivial translation (recall that that \( \Gamma_3 \) acts properly discontinuously on \( \mathbb{H}^3 \)), therefore \( \gamma \) maps some point of \( X_3 \cap L \) into a different point, but this is impossible since \( X_3 \) is the fundamental domain for \( \Gamma_3 \) acting on \( \mathbb{H}^3 \). \( \square \)

The other subgroups except for \( \mathbb{Z}/3, \mathbb{Z}/4 \) and \( \mathbb{Z}/6 \) can not occur either, since \( D_2 \) occurs as a subgroup of each of them and \( C_{\Gamma_3}(D_2) \) is finite.
Next, we study the centralizers of the remaining finite groups: $\mathbb{Z}/3$, $\mathbb{Z}/4$, and $\mathbb{Z}/6$.

As was mentioned at the beginning of this section, hyperbolic $n$-simplex groups arise as groups of symmetries of regular tessellations of hyperbolic $n$-space. The symmetric group $\Gamma_3 = [3, 4, 4]$ of a honeycomb $\{3, 4, 4\}$ is generated by reflections $S_i$ in four planes $P_i$ (say) which form a 3-simplex with dihedral angles

$$\theta(P_1, P_2) = \pi/3, \quad \theta(P_2, P_3) = \pi/4, \quad \theta(P_3, P_4) = \pi/4,$$

and the remaining three angles $\pi/2$. Since the group of isometries of $\mathbb{H}^3$ is isomorphic to the group of Möbius transformations of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and any isometry of hyperbolic 3-space is generated by reflections, then $[3, 4, 4]$ can be represented by a group of Möbius transformations generated by the inversions in four circles cutting one another at the same angles as the corresponding reflection planes. Therefore $\Gamma_3$ can be represented by the antilinear fractional transformations:

$$\Gamma_3 = [3, 4, 4] = \langle S_1(z) = 1/z, \quad S_2(z) = 1 - z, \quad S_3(z) = iz, \quad S_4(z) = z \rangle.$$

Recall that an antilinear fractional transformation is a continuous map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$\phi(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d$ are in $\mathbb{C}$ and $ad - bc \neq 0$.

The generators of $\Gamma_3$ can be represented by antilinear fractional transformations determined by the following matrices

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Using this representation an elementary calculation shows that the centralizer of $F$ in $\Gamma_3$ contains an element of infinite order only if $F = \mathbb{Z}/3$.

Therefore the possible infinite virtually cyclic subgroups of $\Gamma_3$ of type $F \rtimes_{\alpha} \mathbb{Z}$ are:

$$\mathbb{Z}, \quad \mathbb{Z}/2 \times \mathbb{Z}, \quad \mathbb{Z}/3 \times \mathbb{Z}, \quad \mathbb{Z}/3 \rtimes \mathbb{Z}.$$ 

Next, we classify the groups that map onto $D_{\infty}$ with finite kernel.

Let $Q$ map onto $D_{\infty}$, with nontrivial kernel, i.e., we are given the following exact sequence

$$1 \to F \to Q \to D_{\infty} \to 1.$$ 

Now, this exact sequence gives rise to another exact sequence

$$1 \to F \to Q' \to \mathbb{Z} \to 1,$$

where $\mathbb{Z}$ is the infinite cyclic subgroup of index two in $D_{\infty}$ and $Q'$ is just the inverse of $Z$ under the map $Q \to D_{\infty}$. The sequence $1 \to F \to Q' \to Z \to 1$ splits since $Z$ is free. Hence $F$ is isomorphic to either $\mathbb{Z}/2$ or $\mathbb{Z}/3$ and $Q'$ is isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}$, $\mathbb{Z}/3 \times \mathbb{Z}$, or $\mathbb{Z}/3 \rtimes \mathbb{Z}$ by the classification of the groups of type $F \rtimes_{\alpha} \mathbb{Z}$. Since $Q'$ is a subgroup of index two in $Q$, we have the exact sequence

$$1 \to Q' \to Q \to \mathbb{Z}/2 \to 1.$$ 

Thus we have reduced the classification of the groups that map onto $D_{\infty}$ to finding solutions to the extension problems given above.
Using group cohomology (see [Bro82, Theorem IV.3.12, Theorem IV.6.6]) a fairly straightforward calculation shows that $Q$ must be one of the following groups:

$$D_{\infty}, \ Z/2 \times D_{\infty}, \ Z/4 \ast Z/2 \ Z/4, \ Z/4 \ast Z/2 (Z/2)^2,$$

$$Z/3 \times D_{\infty} \equiv Z/6 \ast Z/3 \ Z/6, \ S_3 \ast Z/3 S_3.$$

Hence we have a proof of the following lemma:

**Lemma 3.4.** Let $Q$ be infinite virtually cyclic subgroup of $\Gamma_3$, then $Q$ is one of the following groups:

$$\mathbb{Z}, \ \mathbb{Z}/2 \times \mathbb{Z}, \ \mathbb{Z}/3 \times \mathbb{Z}, \ D_{\infty}, \ \mathbb{Z}/2 \times D_{\infty}, \ \mathbb{Z}/4 \ast \mathbb{Z}/2 \mathbb{Z}/4, \ \mathbb{Z}/4 \ast \mathbb{Z}/2 (\mathbb{Z}/2)^2, \ \mathbb{Z}/3 \times D_{\infty} \equiv \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6, \ S_3 \ast \mathbb{Z}/3 S_3.$$


4. The Reduction to Finite Subgroups

In Theorem 4.7, we show that for $\Gamma = \Gamma_3$, $\mathcal{F} = \mathcal{FLN}$ and $\mathcal{F}' = \mathcal{VC}$ the relative assembly map given in Theorem 2.2 is an isomorphism for $n < 2$. Thus it suffices to compute $H_n^{\text{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{F}); \mathbb{KK}^{-\infty})$ when $\mathcal{F} = \mathcal{FLN}$ to get the main result.

**Theorem 4.1.** The assembly map

$$\mathcal{A}_\mathcal{F}_Q : H_n^{\text{Or}(\mathcal{F}_Q)}(E(Q, \mathcal{F}_Q); \mathbb{KK}^{-\infty}) \to K_n(\mathbb{Z}Q)$$

is an isomorphism for any $n \in \mathbb{Z}$ if $Q = \mathbb{Z}$ or $D_{\infty}$, and an isomorphism for $n < 2$ if $Q = \mathbb{Z}/2 \times \mathbb{Z}, \ \mathbb{Z}/3 \times \mathbb{Z}, \ \mathbb{Z}/3 \times \mathbb{Z}, \ \mathbb{Z}/2 \times D_{\infty}, \ \mathbb{Z}/4 \ast \mathbb{Z}/2 \mathbb{Z}/4, \ \mathbb{Z}/4 \ast \mathbb{Z}/2 (\mathbb{Z}/2)^2, \ \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6, \ or \ S_3 \ast \mathbb{Z}/3 S_3$.

Note that the family $\mathcal{F}_Q$ consist of the finite subgroups of $Q$.

**Remark 4.2.** The claim for $\mathbb{Z}$ and $D_{\infty}$ has been proved in [LS00, Lemma 2.4], for any $n$.

To prove Theorem 4.1 for the remaining subgroups we need a sequence of Lemmas. In Lemma 4.3, we compute the $K$-theory of each possible $Q$, the range of $\mathcal{A}_\mathcal{F}_Q$, and we observe in Proposition 4.5 that the assembly map

$$\mathcal{A} : H_n^{\text{Or}(\mathcal{F}_Q)}(E(Q, \mathcal{F}_Q); \mathbb{KK}^{-\infty}) \to K_n(\mathbb{Z}Q)$$

is an isomorphism for $n < 2$ in each case except for $Q = \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6$ for which $\mathcal{A}$ fails to be an isomorphism at $n = -1$, (see Remark 4.6). Then to complete the proof of the theorem, we compute $H_n^{\text{Or}(Q)}(E(Q, \mathcal{F}_Q); \mathbb{KK}^{-\infty})$, the domain of $\mathcal{A}_\mathcal{F}_Q$, and note that the domain and the range of $\mathcal{A}_\mathcal{F}_Q$ are finitely generated abelian groups, and are isomorphic as abstracts groups for $n < 2$. We use the fact that the assembly map $\mathcal{A}$ factors through $\mathcal{A}_\mathcal{F}_Q$; this guarantees that $\mathcal{A}_\mathcal{F}_Q$ is surjective. For $Q = \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6$, [FJ95, Theorem 2.6] guarantees the surjectivity of $\mathcal{A}_\mathcal{F}_Q$ at $n = -1$. For finitely generated abelian groups, any surjective endomorphism is automatically an isomorphism completing the proof.

**Lemma 4.3.** Let $Q$ be an infinite virtually cyclic subgroup of $\Gamma_3$. If $Q \neq \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6$, then $\text{Wh}(Q) = 0$, $K_0(\mathbb{Z}Q) = \mathbb{Z}$, and $K_n(\mathbb{Z}Q) = 0$ for all $n < 0$. If $Q = \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6$, then $\text{Wh}(Q) = 0$, $K_0(\mathbb{Z}Q) = \mathbb{Z}$, $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$, and $K_n(\mathbb{Z}Q) = 0$ for all $n \leq -2$. 
In order to carry out our computations, we need information on the $K$-theory of the finite subgroups of each infinite virtually cyclic group. These finite subgroups are: $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/2 \times \mathbb{Z}/2 \cong D_2, \mathbb{Z}/6, S_3 \cong D_3$.

It is a result of Carter [C80a] that $K_n(\mathbb{Z}G) = 0$ if $n < -1$ for any finite group $G$. In [Bas68, Theorem 10.6], Bass determines $K_{-1}(\mathbb{Z}G)$ for finite abelian groups. For $G = \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$ and $(\mathbb{Z}/2)^2$, $K_{-1}(\mathbb{Z}G) = 0$. For $G = \mathbb{Z}/6, K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) = \mathbb{Z}$. To compute $K_{-1}(\mathbb{Z}[D_3])$ we need the following formula due to Carter [C80b, Theorem 3], the reader is referred to Section 5.

$$0 \to K_0(\mathbb{Z}) \to (\bigoplus_{p|n} K_0(\mathbb{Z}_p D_3)) \oplus K_0(\mathbb{Q}D_3) \to (\bigoplus_{p|n} K_0(\mathbb{Q}_p D_3)) \to K_{-1}(\mathbb{Z}D_3) \to 0.$$  

The group algebra $\mathbb{Q}D_3$ is isomorphic to $\mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$, and the same statement is true if $\mathbb{Q}$ is replaced by $\mathbb{Q}_2$ or $\mathbb{Q}_3$. Hence $K_0(\mathbb{Q}_2[\mathbb{D}_3]) \cong K_0(\mathbb{Q}_3[\mathbb{D}_3]) \cong K_0(\mathbb{Q}[\mathbb{D}_3]) \cong \mathbb{Z}^3$. Using techniques described in [CuR81, Section 5], we have that $K_0(\mathbb{Z}_3[D_3]) \cong K_0(\mathbb{Z}_2[D_3]) \cong K_0(F_3[\mathbb{Z}/2]) = K_0(F_3 \times F_3) = \mathbb{Z}^2$. Also $K_0(\mathbb{Z}_2[D_3]) \cong K_0(F_2[D_3]) = K_0(F_2[\mathbb{Z}/2] \times M_2(F_2)) = K_0(F_2 \times M_2(F_2)) = \mathbb{Z}^2$. Another result of Carter [C80a, Theorem 1] states that $K_{-1}(\mathbb{Z}G)$ is torsion-free for any symmetric group $S_n$. In particular $K_{-1}(\mathbb{Z}D_3)$ is torsion-free, so by counting ranks in the exact sequence above we have that $K_{-1}(\mathbb{Z}D_3) = 0$.

It is well known that $K_0(\mathbb{Z}G) = 0$ when $G$ is any of the above finite groups (see [Re76], [Ro94] and that $Wh(G) = 0$ for $G$ cyclic of order 2, 3, 4 and 6. For $Wh(D_n)$ we use the following formula given in [Bas65]: $Wh(G) = (\mathbb{Z}^y \oplus SK_1(\mathbb{Z}G))$, where $y$ is the number of real representations of $\mathbb{R}G$ minus the number of rational representations of $\mathbb{Q}G$. When $G = D_2$ or $D_3$, $y$ can be shown to be zero, and Oliver [O89] proves $SK_1(\mathbb{Z}G)$ vanishes for all finite dihedral groups. For each of the finite groups in question, $K_1(\mathbb{Z}G) = G^{ab} \oplus \mathbb{Z}/2$ (see, [O89]).

We are now ready to prove Lemma 4.3.

**Proof of Lemma 4.3.** If $Q = \mathbb{Z}/p \times \mathbb{Z}$ with $p = 2$ or 3, we write $\mathbb{Z}[\mathbb{Z}/p][\mathbb{Z}]$, and apply the Fundamental Theorem of algebraic $K$-theory (see [Bas68, Theorem 10.6]):

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_n(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_{n-1}(\mathbb{Z}[\mathbb{Z}/p]) \oplus 2NK_n(\mathbb{Z}[\mathbb{Z}/p]), \quad n < 2.$$  

In [LS00, Theorem 3.1 (e)] Lück and Stamm show that the nil terms $NK_n(\mathbb{Z}[\mathbb{Z}/p])$ are zero for $n < 2$. Thus for $p = 2$ or 3

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_n(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_{n-1}(\mathbb{Z}[\mathbb{Z}/p]) = 0, \quad n \leq -1,$$

$$K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_0(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/p]) \cong \mathbb{Z},$$

$$K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_1(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/p]) \cong \mathbb{Z}/p \oplus \mathbb{Z} \oplus \mathbb{Z}.$$  

Hence it follows that both $Wh(Q)$ and $\tilde{K}_0(Q)$ vanish for $Q = \mathbb{Z}/2 \times \mathbb{Z}$ or $\mathbb{Z}/3 \times \mathbb{Z}$.

If $Q = \mathbb{Z}/3 \times \mathbb{Z}$, we use a twisted version of a surjective Minoh square [M71]. Let $\xi_3$ be a primitive third root of unity, let $R = \mathbb{Z}[\xi_3]$ and let $\alpha$ be the automorphism of $R$ defined by $\alpha(\xi_3) = \xi_3^{-1}$. We have

$$\begin{array}{ccc}
\mathbb{Z}[\mathbb{Z}/3 \times \mathbb{Z}] & \longrightarrow & R_0(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathbb{Z}[\mathbb{Z}] & \longrightarrow & F_3[\mathbb{Z}]
\end{array}$$
which yields a Mayer-Vietoris sequence of $K$-groups

$$\cdots \to K_2(F_3[Z]) \to K_1(\mathbb{Z}/3 \times \mathbb{Z}) \to K_1(R\alpha[Z]) \oplus K_1(\mathbb{Z}[Z]) \to$$

$$K_1(F_3[Z]) \to K_0(\mathbb{Z}/3 \times \mathbb{Z}) \to \cdots$$

Now the projection map $\text{pr} : K_n(\mathbb{Z}) \to K_n(F_3)$ is an isomorphism for $n = 0, 1$ and an epimorphism for $n = 2$. The same is true for $\text{pr}_{\alpha} : K_n(\mathbb{Z}[Z]) \to K_n(F_3[Z])$, due to the Fundamental Theorem and the fact that $\mathbb{Z}$ and $F_3$ are regular rings. Therefore the Mayer-Vietories sequence shows that

$$K_n(\mathbb{Z}/3 \times \mathbb{Z}) \to K_n(R\alpha[Z])$$

is an isomorphism for $n = 0, 1$. Now Farrell and Hsiang prove in [24] that

$$K_1(R\alpha[Z]) \cong X \oplus \tilde{C}(R, \alpha) \oplus \tilde{C}(R, \alpha^{-1}),$$

where $X$ is an abelian group which fits into an exact sequence

$$0 \longrightarrow K_1(R)/I \longrightarrow X \longrightarrow (K_0(R))^{\alpha*} \longrightarrow 0,$$

with $I = \{\gamma - \alpha_{\gamma}(\gamma) \mid \gamma \in K_1(R)\}$ and $(K_0(R))^{\alpha*}$ denotes the fixed points of $K_0(R)$ under $\alpha_*$. In [FH70] Farrell and Hsiang also show that $\tilde{C}(R, \alpha)$ and $\tilde{C}(R, \alpha^{-1})$ vanish when $R$ is a regular ring. Thus

$$0 \longrightarrow K_1(R)/I \longrightarrow K_1(R\alpha[Z]) \longrightarrow (K_0(R))^{\alpha*} \longrightarrow 0$$

is exact. The map $\alpha_*$ acts trivially on $K_0(R) \cong \mathbb{Z}$, $K_1(R) \cong \mathbb{Z}/6$ and the order of $I$ is 3. Hence $K_1(\mathbb{Z}/3 \times \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. It follows that $Wh(\mathbb{Z}/3 \times \mathbb{Z})$ is trivial.

To show that $K_0(\mathbb{Z}/3 \times \mathbb{Z})$ vanishes, we use the fact that for $F$ a regular ring the inclusion map $K_0(F) \to K_0(\mathbb{Z}/3 \times \mathbb{Z})$ is an epimorphism (see [FH70]). Since $K_0(R)$ vanishes for $R = \mathbb{Z}[\xi_3]$, we have the desired result.

In [FJ95] Farrell and Jones show that if $G$ is infinite virtually cyclic, then $K_n(\mathbb{Z}G)$ is zero for $n < -1$ and that $K_{-1}(\mathbb{Z}G)$ is generated by the images of $K_{-1}(\mathbb{Z}F)$ where $F$ ranges over all finite subgroups $F \subset G$. Then it follows that $K_n(\mathbb{Z}/3 \times \mathbb{Z})$ is zero for $n < 0$.

The other cases are groups of the shape $Q = Q_0 \ast_{\mathbb{Z}/p} Q_1$, with $Q \neq \mathbb{Z}/6 \ast_{\mathbb{Z}/3} \mathbb{Z}/6$, and finite groups $Q_i$ such that $Wh(Q_i) = 0$ for $q \leq 1$ (to clarify this notation we refer the reader to Section 5). Using the results of Farrell and Jones mentioned in the previous paragraph, we have that $K_n(\mathbb{Z}Q) = 0$ for all $n < 0$.

Prassids and Munkholm in [MuPr01, Corollary 3.6] show that there are exact sequences

$$K_1(\mathbb{Z}/p) \to K_1(\mathbb{Z}Q_0) \oplus K_1(\mathbb{Z}Q_1) \to K_1(\mathbb{Z}Q) \to K_0(\mathbb{Z}/p) \to \cdots,$$

and

$$Wh(\mathbb{Z}/p) \to Wh(\mathbb{Z}Q_0) \oplus Wh(\mathbb{Z}Q_1) \to Wh(\mathbb{Z}Q) \to Wh(\mathbb{Z}/p) \to \cdots$$

After working through the exact sequences, we see that for all four groups $Q$, $K_0(\mathbb{Z}Q) = \mathbb{Z}$, $K_1(\mathbb{Z}Q) = Q^{ab} \oplus \mathbb{Z}/2$, and $Wh(\mathbb{Z}Q) = 0$.

The claim for $Q = \mathbb{Z}/6 \ast_{\mathbb{Z}/3} \mathbb{Z}/6$ follows from the arguments used in the last two paragraphs: $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_n(\mathbb{Z}Q) = 0$ for $n \leq -2$ by [FJ95].

$K_0(\mathbb{Z}Q) = \mathbb{Z}$ and $Wh(\mathbb{Z}Q) = 0$ by [MuPr01].

We will make repeated use of the following well known lemma for an arbitrary group $G$, (see [Pe98, Lemma 1.1]):
Lemma 4.4. For \( n < 0 \), the assembly map \( H_n(BG; \mathbb{K}^{-\infty}(Z)) \rightarrow K_n(ZG) \) is an isomorphism if and only if \( K_n(ZG) = 0 \). It is an isomorphism for \( n = 0 \) if and only if \( K_0(ZG) = 0 \), and it is an isomorphism for \( n = 1 \) if and only if \( Wh(G) = 0 \).

Proposition 4.5. Let \( Q \) be an infinite virtually cyclic subgroup of \( \Gamma_3 \). If \( Q \neq \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6 \), then the assembly map

\[
\mathcal{A} : H_n^{or}(Q, T_R)(E(Q, T_R); \mathbb{K}^{-\infty}(Z)) \rightarrow H_n(BQ; \mathbb{K}^{-\infty}(Z)) \rightarrow K_n(ZQ)
\]

is an isomorphism for \( n < 2 \). If \( Q = \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6 \), then \( \mathcal{A} \) is an isomorphism for \( n = 0, 1 \), and \( n \leq -2 \).

Proof. This follows immediately from Lemma 4.3 and Lemma 4.4.

Remark 4.6. If \( Q = \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6 \), then the assembly map \( \mathcal{A} \) can not be an isomorphism for \( n = -1 \), since \( K_{-1}(ZQ) = \mathbb{Z} \oplus \mathbb{Z} \).

Now we are ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. In Lemma 4.3, we compute the target of the map for each possible \( Q \). We now compute the domain using the spectral sequence given in Theorem 2.2.

\[
E^2_{p,q} = H_p(E(Q, F)/Q; \{K_q(ZQ_o)\}) \Longrightarrow H^{or}_{p+q}(E(Q, F); \mathbb{K}^{-\infty})
\]

with \( F = F_Q \). Note that the family \( F_Q \) consists of the finite subgroups of \( Q \).

The infinite cyclic group acts on the real line (with an appropriate simplicial decomposition) in an obvious way with finite isotropy, and this action can be extended to the groups with \( \mathbb{Z}/2 \) and \( \mathbb{Z}/3 \) summands by letting \( \mathbb{Z}/2 \) and \( \mathbb{Z}/3 \) act trivially. Hence for the groups of type \( F \times \mathbb{Z} \), \( E(Q, F_Q) = \mathbb{R} \), and \( E(Q, F_Q)/Q = S^1 \) with \( \mathbb{Z}/2 \), and \( \mathbb{Z}/3 \) stabilizers at each simplex respectively.

For the groups of the shape \( Q = Q_0 \ast_{Z/p} Q_1 \) with \( p = 2 \) or \( 3 \), \( E(Q, F_Q)/Q \) is an interval, with \( \mathbb{Z}/p \) stabilizer at the 1-simplex, and \( Q_i \) stabilizers at each vertex (see [S80, Theorem 7]).

The rings \( ZQ_o \) for each of the stabilizers all have trivial negative \( K \)-groups, (except for \( Q_o = \mathbb{Z}/6 \) for which \( K_{-1}(Z[Z/6]) = \mathbb{Z} \), \( K_0(ZQ_o) = \mathbb{Z} \) and \( K_1(ZQ_o) = Q_o^{ab} \oplus \mathbb{Z}/2 \).

With the above information, the \( E^2 \) term of the homotopy colimit spectral sequence can be easily computed for \( n \) less than two. Since \( E(Q, F_Q)/Q \) is one dimensional, the spectral sequence collapses at \( E^2 \), and in each case

\[
H^{or}_{n}(E(Q, F_Q); \mathbb{K}^{-\infty}) \cong K_n(ZQ) \text{ for } n < 2.
\]

It remains to show that \( \mathcal{A}_{F_Q} \) gives the isomorphism.

Recall that the assembly map \( \mathcal{A} \) factors through the assembly map:

\[
H_n(BQ; \mathbb{K}^{-\infty}(Z)) \rightarrow H^{or}_{n}(E(Q, F_Q); \mathbb{K}^{-\infty}) \rightarrow K_n(ZQ).
\]

To finish the argument, we consider the following two cases:

(1) If \( Q \neq \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6 \), then Proposition 4.5 implies that the assembly map is an isomorphism for \( n < 2 \), thus the composite is an isomorphism. This guarantees the assembly map \( \mathcal{A}_{F_Q} \) is surjective in this case.

(2) If \( Q = \mathbb{Z}/6 \ast \mathbb{Z}/3 \mathbb{Z}/6 \), then Proposition 4.5 implies that the assembly map \( \mathcal{A}_{F_Q} \) is surjective for all \( n < 2 \), with \( n \neq -1 \) (see Remark 4.6). [FJ95, Theorem 2.6] guarantees that \( \mathcal{A}_{F_Q} \) is surjective at \( n = -1 \).
Since $K_0(\mathbb{Z}Q), K_1(\mathbb{Z}Q), \text{and } K_{-1}(\mathbb{Z}[\mathbb{Z}/6\oplus\mathbb{Z}/6])$ (for the other infinite virtually cyclic groups $K_{-1}(\mathbb{Z}Q) = 0$) are finitely generated abelian groups, any surjective endomorphism is an automorphism completing the proof of Theorem 4.1.

**Theorem 4.7.** The relative assembly map

$$H^0_{\text{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{F}LN); \mathbb{KZ}^{-\infty}) \to H^0_{\text{Or}(\Gamma_3)}(E(\Gamma_3, N); \mathbb{KZ}^{-\infty})$$

is an isomorphism for $n < 2$.

**Proof.** It follows from Theorem 4.1 and Theorem 2.2.

**Corollary 4.8.** The assembly map

$$H^0_{\text{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{F}LN); \mathbb{KZ}^{-\infty}) \to K_n(\mathbb{Z}\Gamma_3)$$

is an isomorphism for all $n < 2$.

**Proof.** It follows from Theorem 4.7 and Theorem 2.1.

5. **Proof of Main Theorem**

**Main Theorem.** Let $\Gamma_3 = O^+(3,1) \cap GL(4,\mathbb{Z})$. Then the lower algebraic $K$-theory of the integral group ring of $\Gamma_3$ is given as follows:

$$Wh(\Gamma_3) = 0,$$

$$K_0(\mathbb{Z}\Gamma_3) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4,$$

$$K_{-1}(\mathbb{Z}\Gamma_3) \cong \mathbb{Z} \oplus \mathbb{Z}, \text{ and}$$

$$K_n(\mathbb{Z}\Gamma_3) = 0, \text{ for } n < -1.$$

**Proof.** We use the homotopy colimit spectral sequence discussed in Section 2,

$$E^2_{p,q} = H_p(E(G, F)/G; \{K_q(\mathbb{Z}G_\sigma)\}) \Rightarrow K_{p+q}(\mathbb{Z}G).$$

All the information needed to compute the $E^2$-term is encoded in $E(\Gamma_3, \mathcal{F}LN)/\Gamma_3$ and in the algebraic $K$-groups of the finite subgroups of $\Gamma_3$.

Since $\Gamma_3$ is a hyperbolic 3-simplex reflection group, the fundamental 3-simplex $X_3$ of $\Gamma_3$ satisfies the requirements to be a model for $E(\Gamma_3, \mathcal{F}LN)/\Gamma_3$. $X_3$ has 4 faces with stabilizer $\mathbb{Z}/2$, 6 edges with stabilizers $D_2, D_3, D_4$, 4, three vertices with stabilizers $D_6, \mathbb{Z}/2 \times D_4, \mathbb{Z}/2 \times S_4$, and one ideal vertex with stabilizer the crystallographic group $P4m$ (in our calculations the ideal vertex is being ignored; in [Pe98] Pearson shows that the lower algebraic $K$-theory of $P4m$ vanishes.

The complex that gives the homology of $X_3$ with local coefficients $\{Wh_q(F_\sigma)\}$ has the form

$$\bigoplus_{\sigma} Wh_q(F_{\sigma_2}) \to \bigoplus_{\sigma} Wh_q(F_{\sigma_1}) \to \bigoplus_{\sigma} Wh_q(F_{\sigma_0}),$$

where $\sigma_i$ denotes the cells in dimension $i$, and $Wh_q(F_{\sigma_i})$ occurs in the summand as many times as the numbers of conjugacy classes of the subgroup $F_{\sigma_i}$ in $\Gamma_3$. The homology of this complex gives the data for the $E^2$-term. Let us recall that

$$Wh_q(F) = \begin{cases} Wh(\mathbb{Z}F) = Wh_1(F), & q = 1 \\ K_0(\mathbb{Z}F) = Wh_0(F), & q = 0 \\ K_n(\mathbb{Z}F) = Wh_q(F), & q \leq -1. \end{cases}$$

So we analyze this complex for each of the following cases: $q < -1, q = -1, 0, 1$. 


$q < -1$. Carter shows in [C80a] that $K_q(\mathbb{Z} F) = 0$ when $F$ is a finite group. Hence the whole complex consists of zero terms and we obtain $E^2_{p,q} = 0$ for $q < -1$.

$q = -1$. Again using Carter’s result in [C80a], $K_{-1}(\mathbb{Z} F) = 0$, for all the groups which occur as stabilizers of the 2-cells and the 1-cells, therefore $E^2_{p,q} = 0$ for $p = 1, 2$. For $p = 0$ the complex may have non-zero terms in dimension zero, and the resulting homology group is

$$
H_0(X_3 ; \{ K_{-1}(\mathbb{Z} F_o) \} ) = \bigoplus_{o^0} K_{-1}(\mathbb{Z} F_o).
$$

Since there is only one conjugacy class for each of the subgroups $D_6$, $\mathbb{Z}/2 \times D_4$, $\mathbb{Z}/2 \times S_4$ of $\Gamma_3$ occurring as stabilizers of the 0-cells, then

$$
H_0(X_3 ; \{ K_{-1}(\mathbb{Z} F_o) \} ) = K_{-1}(\mathbb{Z}[D_6]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4]).
$$

To calculate the $K$-groups: $K_{-1}(\mathbb{Z}[D_6])$, $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4])$, and $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4])$, we use the following formula due to Carter [C80b, Theorem 3].

Let $G$ be a group of order $n$, let $p$ denote a prime number, let $\mathbb{Z}_p$ denote the $p$-adic integers and let $\mathbb{Q}_p$ denote the $p$-adic numbers. Then the following sequence is exact:

$$
0 \to K_0(\mathbb{Z}) \to \left( \bigoplus_{p|n} K_0(\mathbb{Z}_p G) \right) \oplus K_0(\mathbb{Q} G) \to \left( \bigoplus_{p|n} K_0(\mathbb{Q}_p G) \right) \to K_0(\mathbb{Z} G) \to 0.
$$

The group algebra $\mathbb{Q} D_6$ is isomorphic to $\mathbb{Q}^4 \times M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$, and the same statement is true if $\mathbb{Q}$ is replaced by $\mathbb{Q}_2$ and $\mathbb{Q}_3$. Hence $K_0(\mathbb{Q} D_6) \cong K_0(\mathbb{Q}_2 [D_6]) \cong K_0(\mathbb{Q}_3 [D_6]) \cong \mathbb{Z}^6$. Using techniques describe in [Cu81, Section 3], $K_0(\mathbb{Z}_2 [D_6]) \cong K_0(\mathbb{Z}_2 [D_6]) \cong K_0(\mathbb{F}_2 [D_6]) \cong K_0(\mathbb{F}_2 [D_6]) \cong K_0(\mathbb{F}_2 [Z/2] \times M_2(\mathbb{F}_2)) \cong K_0(\mathbb{F}_2 \times F_2) \cong \mathbb{Z}^2$. Also $K_0(\mathbb{Z}_3 [D_6]) \cong K_0(\mathbb{F}_3 [D_6]) \cong K_0(\mathbb{F}_3 [D_6]) \cong K_0(\mathbb{F}_3 \times F_3 \times F_3 \times F_3) \cong \mathbb{Z}^4$. Carter also shows in [C80a] that $K_{-1}(\mathbb{Z} G)$ is torsion free for any of the groups above, so counting ranks in the exact sequence, we have that $K_{-1}(\mathbb{Z}[D_6]) \cong \mathbb{Z}$.

The computations for $\mathbb{Z}/2 \times D_4$, and $\mathbb{Z}/2 \times S_4$ are nearly identical. The algebra $\mathbb{Q}[\mathbb{Z}/2 \times D_4]$ is isomorphic to $\mathbb{Q}^6 \times M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$ and the same decomposition, so $K_0(\mathbb{Q}[\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{Q}^6 \times M_2(\mathbb{Q}) \times M_2(\mathbb{Q})) \cong \mathbb{Z}^6$. Using techniques describe in [Cu81, Section 3], $K_0(\mathbb{Z}_2 [\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{Z}_2 [\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{F}_2 [\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{F}_2 [\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{F}_2 [\mathbb{Z}/2 \times D_4]) \cong \mathbb{Z}^8$. Counting ranks, $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) \cong 0$. The algebra $\mathbb{Q}[\mathbb{Z}/2 \times S_4]$ is isomorphic to $\mathbb{Q}^4 \times (M_2(\mathbb{Q}))^2 \times (M_3(\mathbb{Q}))^4$ and the same statement is true if $\mathbb{Q}$ is replaced by $\mathbb{Q}_2$ and $\mathbb{Q}_3$. We have $K_0(\mathbb{Q}[\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{Q}^6 \times M_2(\mathbb{Q}) \times M_2(\mathbb{Q})) \cong \mathbb{Z}^8$. The integral $p$-adic terms are $K_0(\mathbb{Z}_2 [\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_2 [\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_2 [\mathbb{Z}/2 \times S_4]) \cong \mathbb{Z}^2$, and $K_0(\mathbb{Z}_3 [\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_3 [\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_3 [\mathbb{Z}/2 \times S_4]) \cong \mathbb{Z}^8$. Counting ranks, $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) \cong 0$.

It follows that

$$
E^3_{0,-1} = H_0(X_3 ; \{ K_{-1}(\mathbb{Z} F_o) \} ) = \mathbb{Z} \oplus \mathbb{Z}.
$$

$q = 0$. It is well know that $K_0(\mathbb{Z} F) = 0$ when $F$ is one of the groups that occur as stabilizers of the 2-cells and the 1-cells (see for example [Re76], [Ro94]), so $E^3_{p,q} = 0$ for $p = 1, 2$. For $p = 0$ the complex may have non-zero terms in dimension zero, and the resulting homology is

$$
H_0(X_3 ; \{ K_0(\mathbb{Z} F_o) \} ) = K_0(\mathbb{Z}[D_6]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/2 \times S_4]).
$$
In [Re76] Reiner shows that $K_0(Z[D_6]) = 0$, then
$$E_{0,0}^2 = \tilde{K}_0(Z[Z/2 \times D_4]) \oplus \tilde{K}_0(Z[Z/2 \times S_4]).$$

To compute the $K$-groups: $\tilde{K}_0(Z[Z/2 \times G])$ for $G = D_4$ or $S_4$, consider the following Cartesian square

$$
\begin{array}{ccc}
Z[Z/2][G] & \longrightarrow & Z[G] \\
\downarrow & & \downarrow \\
Z[G] & \longrightarrow & F_2[G]
\end{array}
$$

which yields the Mayer-Vietories sequence (see [40, Theorem 49.27])

$$K_1(ZG) \oplus K_1(ZG) \to K_1(F_2[G]) \to K_0(Z[|G/2|]) \to \tilde{K}_0(ZDG) \oplus \tilde{K}_0(ZG) \to 0$$

In [Re76] Reiner shows that $K_0(ZG)$ is trivial for $G = D_4$ and $S_4$ and the $K$-groups $K_1(ZG)$ can be computed as follows: For any group $G$, it is well known that the rank of $K_1(ZG)$ is equal to the rank of $Wh(G)$ (see, for example [O89]) where the rank of $Wh(G)$ is the number of real representations of $ZG$ minus the rational representations of $ZG$ (see case $q = 1$ below), and that the torsion part of $K_1(ZG)$ is \{0\} $\oplus G^{ab} \oplus S K_1(ZG)$ (see [O89, Theorem 7.4]). For $G = D_4$ or $S_4$, Oliver in [O89] shows that both $Wh(G)$ and $SK_1(ZG)$ are zero. Since $(D_4)^{ab} = Z/2 \oplus Z/2$, and $(S_4)^{ab} = Z/2$, then it follows that $K_1(Z[D_4]) = (Z/2)^3$, and $K_1(Z[S_4]) = (Z/2)^2$. We thank B. Magurn who kindly inform us that $K_1(F_2[D_4]) = Z/4 \oplus (Z/2)^2$, and $K_1(F_2[S_4]) = Z/4 \oplus Z/2$. Therefore the exact sequence in (1) yields the following exact sequences

$$K_1(ZG) \oplus K_1(ZG) \to K_1(F_2[G]) \to K_0(Z[|G/2|]) \to 0,$$

$$K_1(ZG) \oplus K_1(ZG) \to K_1(F_2[G]) \to K_0(Z[|G/2|]) \to 0.$$

Next, we study the image of $\varphi : K_1(ZG) \oplus K_1(ZG) \to K_1(F_2[G])$. Let us consider first the case $G = D_4$ for which $\text{im}(\varphi) = Z/2 \oplus Z/2$. This can be seen as follows: First $\text{im}(\varphi) = \text{im}(\psi)$ where $\psi : K_1(Z[D_4]) \to K_1(F_2[D_4])$ is induced by the canonical ring homomorphism $Z \to F_2$. Note the $K_1(Z)$ is a direct summand of $K_1(Z[D_4])$ and isomorphic to $Z/2$; but this summand goes to zero in $K_1(F_2[D_4])$ since it factors through the following commutative square

$$
\begin{array}{ccc}
Z/2 & \longrightarrow & K_1(Z) \\
\downarrow & & \downarrow \\
K_1(Z[D_4]) & \longrightarrow & K_1(F_2[D_4])
\end{array}
$$

Since $K_1(Z[D_4]) = (Z/2)^3$, then $\text{dim}_{F_2}(\text{im}(\psi)) \leq 2$. But from the exact sequence given in (2), $\text{dim}_{F_2}(\text{im}(\psi)) \geq 2$, thus $\text{im}(\varphi) = Z/2 \oplus Z/2$ and it follows that $\text{coker}(\varphi) \cong Z/4$.

Next, for $G = S_4$ a nearly identical argument shows that the image of the map $\varphi : K_1(Z[S_4]) \oplus K_1(Z[S_4]) \to K_1(F_2[S_4])$ is isomorphic to $Z/2$ and $\text{coker}(\varphi) \cong Z/4$. Hence after working through the exact sequences given in (2) and (3), we have that $\tilde{K}_0(Z[Z/2][D_4]) = \tilde{K}_0(Z[Z/2][S_4]) = Z/4$. 

\[\text{17}\]
It follows that

\[ E^2_{0,0} = H_0(X_3 ; \{ K_0(\mathbb{Z}F_\sigma) \}) = \mathbb{Z}/4 \oplus \mathbb{Z}/4. \]

\( q = 1 \). Oliver in [O89] has shown that \( \text{Wh}(F) = 0 \) when \( F \) is one of the groups that occur as stabilizers of the 2-cells and the 1-cells, so it follows that \( E^2_{p,q} = 0 \) for \( p = 1, 2 \). As before for \( p = 0 \), the complex may have non-zero terms. The resulting homology groups are

\[ E^2_{0,1} = H_0(X_3 ; \{ \text{Wh}(F_\sigma) \}) = \text{Wh}(D_0) \oplus \text{Wh}(\mathbb{Z}/2 \times D_4) \oplus \text{Wh}(\mathbb{Z}/2 \times S_4). \]

To calculate: \( \text{Wh}(D_0) \), \( \text{Wh}(\mathbb{Z}/2 \times D_4) \), and \( \text{Wh}(\mathbb{Z}/2 \times S_4) \), we use the following formula given in [Bas68]: \( \text{Wh}(F) = \mathbb{Z}^y \oplus \text{SK}_1(\mathbb{Z}G) \), where \( y \) is the number of real representations of \( \mathbb{R}G \) minus the number of rational representations of \( \mathbb{Q}G \). When \( G = D_0 \), \( \mathbb{Z}/2 \times D_4 \) or \( \mathbb{Z}/2 \times S_4 \), \( y \) can be shown to be zero and Oliver in [O89] proves that \( \text{SK}_1 \) vanishes for all finite dihedral groups. Oliver also shows in [O89, Example 9.9] that if \( |G| = 16 \), then

\[ \text{SK}_1(\mathbb{Z}G) = \begin{cases} 1 & \text{if } G^{ab} \cong (C_2)^2 \text{ or } (C_2^3) \\
\mathbb{Z}/2 & \text{if } G^{ab} \cong C_4 \times C_2. \end{cases} \]

In particular for \( G = \mathbb{Z}/2 \times D_4 \), \( G^{ab} \cong (C_2)^3 \), and we have \( \text{SK}_1(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) = 1 \).

For the remaining case of \( \text{SK}_1(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) \), we use the following formula due to Oliver [O89, Theorem 3.9].

Let \( G \) be a finite group of order \( n \). For each prime \( p \), let \( \hat{\mathbb{Z}}_p G \) and \( \hat{\mathbb{Q}}_p G \) denote the \( p \)-adic completions of \( \mathbb{Z}G \) and \( \mathbb{Q}G \), and set \( \text{SK}_1(\hat{\mathbb{Z}}_p G) = \ker(K_1(\hat{\mathbb{Z}}_p G) \rightarrow K_1(\hat{\mathbb{Q}}_p G)) \). Then set

\[ CL_1(\mathbb{Z}G) = \ker\{ \text{SK}_1(\hat{\mathbb{Z}}_p G) \rightarrow \bigoplus_p \text{SK}_1(\hat{\mathbb{Z}}_p G) \}. \]

The sum \( \bigoplus_p \text{SK}_1(\hat{\mathbb{Z}}_p G) \) is, in fact, a finite sum, \( \text{SK}_1(\hat{\mathbb{Z}}_p G) = 1 \) whenever \( p \nmid |G| \), and the localization homomorphism \( l \) is onto. In particular \( \text{SK}_1(\mathbb{Z}G) \) sits in an extension

\[ 1 \rightarrow CL_1(\mathbb{Z}G) \rightarrow \text{SK}_1(\mathbb{Z}G) \rightarrow \bigoplus_p \text{SK}_1(\hat{\mathbb{Z}}_p G) \rightarrow 1. \]

Wall in [W74, Theorem 2.5] shows that \( \text{SK}_1(\hat{\mathbb{Z}}_p G) \) is a \( p \)-group for any prime \( p \) and any finite group \( G \), and \( \text{SK}_1(\hat{\mathbb{Z}}_p G) = 1 \) if the \( p \)-Sylow subgroup of \( G \) is abelian. Also Oliver in [O89, Proposition 12.7] shows that \( \text{SK}_1(\hat{\mathbb{Z}}_p G) = 1 \) if the \( p \)-Sylow subgroup of \( G \) has a normal abelian subgroup with cyclic quotient. In particular for \( G = \mathbb{Z}/2 \times S_4 \), the 3-Sylow subgroup of \( G \) is of type \( C_3 \), and the 2-Sylow subgroup of \( G \) is of type \( C_2 \times D_4 \). Then it follows that both \( \text{SK}_1(\hat{\mathbb{Z}}_2 G) \) and \( \text{SK}_1(\hat{\mathbb{Z}}_2 G) \) vanish, and we conclude \( \text{SK}_1(\mathbb{Z}G) \cong CL_1(\mathbb{Z}G) \). Since the group algebra \( \mathbb{R}[\mathbb{Z}/2 \times S_4] \) splits as \( \mathbb{R}^2 \times (M_2(\mathbb{R}))^2 \times (M_3(\mathbb{R}))^4 \), then by [O89, Theorem 5.4] \( CL_1(\mathbb{Z}G) = 1 \), then it follows that \( \text{SK}_1(\mathbb{Z}G) \) vanishes. Hence the whole complex consists of zero terms and \( E^2_{p,q} = 0 \) for \( q = 1 \).

Thus the spectral sequence collapses at \( E^2 \), completing our computations of the algebraic \( K \)-groups \( K_n(\mathbb{Z}G) \) for \( n < 2 \). \( \square \)
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REFERENCES

[BM] P. Bahls, and M. Mihalik Centralizers of parabolic subgroups of even Coxeter groups, to appear.


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