MOTIVES AND ÉTALE MOTIVES WITH FINITE COEFFICIENTS

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Abstract. We prove that for suitable base fields, inverting the Bott element in Voevodsky’s category of motives with finite coefficients yields the category of étale motives with finite coefficients.

1. Introduction

Many important problems in algebraic geometry and number theory can be phrased as comparison results between the Zariski (or Nisnevich) and the étale topology. Important examples include the Quillen-Lichtenbaum conjectures, and Voevodsky’s [10] recent proof of the Milnor conjecture signifies big progress in this direction. Already in the 80’s, Thomason [7] observed that with finite coefficients (prime to the characteristic of the base field), algebraic and étale K-theory become isomorphic after inverting the Bott element. More recently, Levine [2] proved a similar result about motivic and étale cohomology.

The purpose of this article is to establish the corresponding result for Voevodsky’s category of motives. These categories of motives are defined both for the Nisnevich and the étale topology and denoted by $DM^{eff}_{gm}$ and $DM^{eff}_{gm,et}$, respectively. With rational coefficients, they are known to be equivalent [9, Proposition 3.3.2]. With finite coefficients, one obtains the category $DM^{eff}_{gm}([\mathbb{Z}/n][W^{-1}_{mod}])$ when inverting the Bott element in $DM^{eff}_{gm}([\mathbb{Z}/n])$ (see 2.18 for the precise definition). We prove the following result (see Theorem 2.19):

Theorem 1.1. Assume that the base field $k$ is of characteristic 0 and that $n$ is an odd integer such that $k$ has finite étale dimension with respect to $n$ and contains the $n$th roots of unity. Then the change of topologies induces an equivalence

$$DM^{eff}_{gm}([\mathbb{Z}/n][W^{-1}_{mod}]) \stackrel{\cong}{\longrightarrow} DM^{eff}_{gm,et}([\mathbb{Z}/n])$$

of triangulated categories.

Observe (see Proposition 2.25) that $DM^{eff}_{gm,et}([\mathbb{Z}/n])$ is equivalent to the derived category of complexes of $\mathbb{Z}/n$-modules with finitely generated homology if $k$ is algebraically closed. The proof of our Theorem relies on Levine’s result, of course. It also involves duality and a detailed comparison of the categories of geometric motives and the “large” category of motives (that is the one built from complexes of sheaves with transfers). Similar methods also allow us to rephrase rigidity (see Theorem 2.27):

Date: February 11, 2004.
The first author was partially supported by the Clay Mathematics Institute.
Theorem 1.2. If $L/F$ is an extension of algebraically closed fields, then the base change functor induces a full embedding

$$DM^\text{eff}_m(F, \mathbb{Z}/n) \to DM^\text{eff}_m(L, \mathbb{Z}/n).$$

2. Inverting the Bott element in $DM_m(\mathbb{Z}/n)$

In what follows, $F$ is a perfect field admitting resolution of singularities, and $n > 1$ is an integer prime to the characteristic of $F$ such that the $n$th roots of unity are contained in $F$. We further assume that $n$ is either odd or the fourth roots of unity are in $F$, and that $F$ has finite étale $n$-dimension (i.e., for any étale $G$ of $\mathbb{Z}/n$-modules and any smooth variety $X$ over $F$, $H^n(X, G) = 0$ for $n > 1$.

It suffices (see [1, Exp. X, Corollaire 4.3]) to check this for $Spec(F)$. See e.g., [5, sections II.2-4] for a discussion when this holds. For instance, one has finite étale $n$-dimension if $F$ are the complex numbers, or if $F$ is a number field and $n$ is odd or if $F$ has no real embeddings.). By $DM_m$ we denote the triangulated category of geometric motives of Voevodsky (cf. [9]).

Recall that $DM_m$ is a rigid triangulated tensor category, it has a full [9, Theorem 4.3.1] triangulated tensor subcategory $DM^\text{eff}_m$ of effective motives, and there is another full embedding $DM^\text{eff}_m \to DM^\text{eff}_m$ into the derived category of complexes of Nisnevich sheaves with transfers on $Sm/F$ with homotopy invariant homology sheaves [9, Theorem 3.2.6]. Recall that this embedding maps $M_m(X)$ to $C_\ast tr(X) = C_\ast L(X)$ which we sometimes also denote by $M(X)$. If not stated explicitly otherwise, $DM^\text{eff}_m$ and all variants of it always denote the subcategory and not the localization of $D^-(Sh_{\text{Nis}}(SmCor))$.

In the same way as $DM^\text{eff}_m$, we can define a category $DM^\text{eff}_m(\mathbb{Z}/n)$ by starting from the category $SmCor^d(F)/n$ of smooth correspondences with $\mathbb{Z}/n$-coefficients (that is the category obtained from $SmCor$ by tensoring the morphism groups with $\mathbb{Z}/n$), taking the homotopy category of complexes, localizing with respect to the thick subcategory generated by the homotopy $(T_{\text{hom}})$ and Mayer-Vietoris $(T_{\text{MV}})$ complexes and then taking the idempotent completion. This category has a triangulated tensor structure induced by products of varieties. Denote the obvious functor from $Sm/F$ to $DM^\text{eff}_m(\mathbb{Z}/n)$ by $M_m/n$, and the "reduced motive" [9, p. 192] by $\tilde{M}_m/n$.

Inverting the Tate object $\mathbb{Z}/n(1)$ (defined as usual by $\mathbb{Z}/n(1) = M_m/n(\mathbb{P}^1)[-2]$) formally gives a new triangulated category $DM_m(\mathbb{Z}/n)$. As in [9, Corollary 2.1.5], the permutation on $\mathbb{Z}/n(1) \otimes \mathbb{Z}/n(1)$ is the identity and hence $DM_m(\mathbb{Z}/n)$ is again tensor triangulated. Do not confuse the object $\mathbb{Z}/n(1)$ of $DM^\text{eff}_m(\mathbb{Z}/n)$ with the object of $DM^\text{eff}_m$ that is also denoted by $\mathbb{Z}/n(1)$.

Along the same lines, we define $DM^\text{eff}_m(\mathbb{Z}/n)$ and $DM^\text{eff}_m(\mathbb{Z}/n)$ from the category of $\mathbb{Z}/n$-sheaves with transfers. That is, we proceed exactly as in [9, section 3.1], but starting with sheaves of $\mathbb{Z}/n$-modules instead of $\mathbb{Z}$-modules and $\mathbb{Z}/n$-linear homomorphisms between them.

Clearly, the results of [9, sections 3.1, 3.2, 3.5] hold in this context. In particular, we have the following:

Lemma 2.1. The Yoneda embedding for $SmCor^d(F)/n$ induces a full tensor triangulated embedding.
\[ DM_{gm}^{\eff}(\mathbb{Z}/n) \rightarrow DM_{gm}^{\eff}(\mathbb{Z}/n) \]

of tensor triangulated categories with dense image.

(Recall that a triangulated subcategory \( S \subset T \) is called “dense” if \( T \) is the smallest triangulated category containing \( S \) and being closed under arbitrary direct sums.)

**Proof.** Similar to [9, Theorem 3.2.6].

Likewise, the results of [9, sections 4.1, 4.2, 4.3] are independent of the coefficients used, so that the natural functor

\[ DM_{gm}^{\eff}(\mathbb{Z}/n) \rightarrow DM_{gm}(\mathbb{Z}/n) \]

is a full embedding.

There are natural functors \( DM_i \rightarrow DM_{gm}^{\eff}(\mathbb{Z}/n) \) that will all be written as \( M \rightarrow M/n \). Explicitly, for the categories \( DM_{gm}^{\eff} \) and \( DM_{gm} \), these functors are induced by the obvious functor \( \text{SmCor} \rightarrow \text{SmCor}/n \), and for \( DM_{gm}^{\eff} \), it is induced by the functor on \( D^- (\text{Shv}_{N,\omega}(\text{SmCor})) \) that maps a complex \( \mathcal{F} \) to \( \mathcal{F} \otimes \mathbb{Z}/n \), where \( \mathcal{F} \) is a complex of free sheaves with transfers quasiisomorphic to \( \mathcal{F} \), \( \otimes \) is the ordinary tensor product of sheaves and \( \mathbb{Z}/n \) is the constant sheaf with value \( \mathbb{Z}/n \); for \( DM_{gm}^{\eff} \), we proceed in the same way. From the explicit description of the functors, it is evident that they commute with the various full embeddings of categories of motives.

**Lemma 2.2.** Denote by \( \mathbb{Z}/n \) the object of \( DM_{gm}^{\eff} \) given by the complex \([n]: \text{Spec}(F) \rightarrow \text{Spec}(F)\) over \( \text{SmCor} \) and also its image in \( DM_{gm}^{\eff} \). Let \( M \) and \( N \) be objects of \( DM_{gm}^{\eff} \) resp. \( DM_{gm}^{\eff} \). Then there are natural isomorphisms

\[ \text{Hom}_{DM_{gm}^{\eff}}(M, N \otimes^L \mathbb{Z}/n) \cong \text{Hom}_{DM_{gm}}^{\eff}(M/n, N/n) \]

and

\[ \text{Hom}_{DM_{gm}^{\eff}}(M, N \otimes^r \mathbb{Z}/n) \cong \text{Hom}_{DM_{gm}^{\eff}}(M/n, N/n). \]

Here, \( \otimes^r \) denotes the tensor product structure on \( DM_{gm}^{\eff} \) resp. \( DM_{gm}^{\eff} \).

**Proof.** By Lemma 2.1 and [9, Theorem 3.2.6], the second isomorphism follows from the first one. We may assume that \( M \) and \( N \) are both represented by complexes of free sheaves with transfers (i.e., sheaves of the form \( \mathbb{Z}_{tr}(X) \)) having homotopy invariant cohomology sheaves (because bounded above complexes of presheaves on any additive category are quasiisomorphic to complexes of representables). If \( \otimes \) just denotes the tensor product of complexes and \( \otimes^L \) the left derived tensor product, then there are isomorphisms \( N \otimes \mathbb{Z}/n \cong N \otimes^L \mathbb{Z}/n \cong N \otimes^r \mathbb{Z}/n \) in \( D^- (\text{Shv}_{N,\omega}(\text{SmCor})) \) and all those quasiisomorphic complexes have homotopy invariant cohomology sheaves. Indeed, the first isomorphism holds because the sheaves \( \mathbb{Z}_{tr}(X) \) are flat, the homotopy invariance of the cohomology sheaves of \( N \otimes^L \mathbb{Z}/n \) is Lemma 2.3 below and the second isomorphism follows from that and the definition of \( \otimes^r \). This implies that

\[ \text{Hom}_{DM_{gm}^{\eff}}(M, N \otimes^r \mathbb{Z}/n) \cong \text{Hom}_{D^- (\text{Shv}(\text{SmCor}))}(M, N \otimes^L \mathbb{Z}/n) \]

\[ \cong \text{Hom}_{D^- (\text{Shv}(\text{SmCor}/\mathbb{Z}/n))}(M/n, N/n) \]

\[ \cong \text{Hom}_{DM_{gm}^{\eff}}(\mathbb{Z}/n)(M/n, N/n). \]
as asserted. Here the first and third isomorphisms follow from [9, Proposition 3.2,3] resp. the analogous statement with $\mathbb{Z}/n$-coefficients, and the second isomorphism is a consequence of the fact that $\text{Hom}_{\text{Ab}}(A,B/n) \cong \text{Hom}_{\mathbb{Z}/n-\text{Mod}}(A/n,B/n)$ for any abelian groups $A$ and $B$.

\begin{proof}
Let $\mathcal{F}$ be a homotopy invariant sheaf with transfers. Then the sheaves $\text{Tor}_i(\mathcal{F}, R)$ are homotopy invariant sheaves with transfers. Indeed, using that the category of homotopy invariant sheaves is abelian [9, Proposition 3.1,13] and looking at the double complex, we may assume that $R$ is flat and $i = 0$. The presheaf $\mathcal{F} \otimes R$ is clearly homotopy invariant and has transfers, hence [9, Theorem 3.1,12] implies that the sheaf $\mathcal{F} \otimes R$ is homotopy invariant with transfers as asserted.

Now let $A$ be as in the statement of the lemma. There is a strongly convergent spectral sequence

$$\text{Tor}_p(H^{-q}(A), R) \Rightarrow H^{p-q}(A \otimes^L R)$$

in the category of sheaves with transfers. We proved that the $E^2$-terms of this spectral sequence are homotopy invariant. Now the assertion of the lemma follows from [6, Lemma 1.3].

\end{proof}

We recall that, if we construct a category $\mathcal{DM}^{\text{eff}}_{\text{et}}(\mathbb{Z}/n)$ using the étale topology instead of the Nisnevich one, the following result holds (cf. [9, Proposition 3.3.3]):

\begin{proposition}[Voevodsky]
The natural triangulated functor (induced by restriction)

$$\Phi : \mathcal{DM}^{\text{eff}}_{\text{et}}(\mathbb{Z}/n) \to D^-(\text{Shv}(\text{F}_{\text{et}}), \mathbb{Z}/n)$$

to the derived category of $\mathbb{Z}/n$-sheaves on the small étale site of $F$ is an equivalence of categories.

\end{proposition}

\begin{remark}
The corresponding result for unbounded complexes also holds.

\end{remark}

This result follows from Suslin rigidity, and the inverse functor associates to a sheaf on $\text{Spec}(F)_{\text{et}}$ the corresponding locally constant sheaf on $\text{Sm}/F$; see [8, Lemma 6.11] and [1, XV 2.2]. The equivalence respects the tensor structure (see the proof of Theorem 2.19). We need a more explicit description of this equivalence; first, some notation.

\begin{notation}
For an $F$-scheme $X$, let $p_X : X \to \text{Spec}(F)$ denote the structural morphism.

Let $X$ be a smooth connected $F$-scheme of dimension $d$ and $\mathcal{F}$ a complex of sheaves of $\mathbb{Z}/n$-modules (necessarily locally constant) on $\text{Spec}(F)_{\text{et}}$. We denote the corresponding object of $\mathcal{DM}^{\text{eff}}_{\text{et}}(\mathbb{Z}/n)$ again by $\mathcal{F}$.

\end{notation}

\begin{lemma}
There is a natural (in $X$ and $\mathcal{F}$) isomorphism

$$\text{Hom}_{\mathcal{DM}^{\text{eff}}_{\text{et}}(\mathbb{Z}/n)}(M(X)/n_{\text{et}}, \mathcal{F}) \cong \text{Hom}_{D^-(\text{Shv}(\text{F}_{\text{et}}), \mathbb{Z}/n)}(\mathbb{R}p_X!(\mathbb{Z}/n)(d)[2d], \mathcal{F})$$

\end{lemma}
Proof. First, we observe that $M(X)/n_{et}$ and $R_{P_{X!}}(\mathbb{Z}/n)(d)[2d]$ are compact objects in their respective categories (recall that an object $A$ is called compact if $\text{Hom}(A, -)$ commutes with coproducts). Indeed, by [8, Lemma 6.23] and the étale version of Lemma 2.2, 

$$\text{Hom}_{DM^{-}_{d}(\mathbb{Z}/n)}(M(X)/n_{et}, F) \cong H^{0}_{et}(X, F)$$

(note that $F$ is automatically $\mathbb{A}^{1}$-local and that $F \otimes^L \mathbb{Z}/n \cong F$), and the latter functor commutes with arbitrary direct sums. On the other hand, $R_{P_{X!}}(\mathbb{Z}/n)(d)[2d]$ is a bounded complex with constructible cohomology (see [1, Exp. IX, Définition 2.3] for what that means). This follows as $\mathbb{Z}/n(d)[2d]$ has constructible cohomology, from [1, Exp. XVII, Théorème 5.3,6]. Now [1, Exp. IX, Corollaire 2.7,3] implies that $R_{P_{X!}}(\mathbb{Z}/n)(d)[2d]$ is a compact object.

So we may assume that $F$ has constructible cohomology. Using Verdier-Grothendieck duality [1, Théorème XVIII.3.2,5], the right hand side is naturally isomorphic to

$$\text{Hom}_{D^{-}(\text{Shv}_{et}(\mathbb{Z}/n))}(\mathbb{Z}/n, p_{X!}(F)).$$

We can assume (as we have finite cohomological dimension) that $F = G[s]$ for a sheaf $G$ (of $\mathbb{Z}/n$-modules). Thus, the right hand side is further isomorphic to

$$\text{Ext}^{s}_{\text{Shv}(\mathbb{Z}^{-1})}(\mathbb{Z}, G) \cong \text{Ext}^{s}_{\text{Shv}(\text{SmCor})}(\mathbb{Z}_{tr}(X), G)$$

$$\cong \text{Hom}_{DM^{-}_{d}(\mathbb{Z}/n)}(M(X)/n_{et}, G[s])$$

where the first isomorphism is [8, Lemma 6.11] and the second isomorphism combines the usual expression of Ext-groups as homomorphisms in the derived category, the fact that $G$ is an $\mathbb{A}^{1}$-local object of $D^{-}(\text{Shv}_{et}(\text{SmCor}))$, and Lemma 2.2. This proves the assertion. \hfill \Box

Corollary 2.8. Let $X/F$ be smooth. There is a natural isomorphism (in $D^{-}(\text{Shv}(F_{et}), \mathbb{Z}/n)$) of the form

$$\Phi(M(X)/n_{et}) \cong R_{P_{X!}}(\mathbb{Z}/n)(d)[2d].$$

Proof. That follows immediately from Lemma 2.7, the Yoneda lemma and the fact (Proposition 2.4) that $\Phi$ is an equivalence of categories. \hfill \Box

Remark 2.9. This identification of the functor $\Phi$ enables us to compare duality in the category of motives with duality in the category of étale sheaves.

Lemma 2.10. For $q \geq 0$, there is a natural isomorphism

$$\text{Hom}_{DM^{-}_{d}(\mathbb{Z}/n)}(M(X)/n_{et}, \mathbb{Z}/n(q)_{et}[p]) \cong H^{p}_{et}(X, \mu_{n}^{\otimes q}).$$

Proof. This follows using Lemma 2.7 and the isomorphism (cf. [6, Proposition 6.7]) $\mathbb{Z}/n(q)_{et}[p] \cong \mu_{n}^{\otimes q}[p]$ for $q \geq 0$. \hfill \Box

Recall that an object $X$ in a triangulated category $\mathcal{T}$ having all small coproducts is called compact ($\aleph_{0}$-small in the terminology of [3]) provided that $\text{Hom}_{\mathcal{T}}(X, -)$ commutes with all coproducts. Equivalently, any map $X \rightarrow \oplus A_{i}$ factors through a finite direct sum.

Recall [3, Definition 8.1.6, Remark 4.2.6] also that we say that $\mathcal{T}$ is generated by a set $S$ of objects if an object $A$ of $\mathcal{T}$ is 0 if and only if $\text{Hom}_{\mathcal{T}}(X[n], A) = 0$ for all $n \in \mathbb{Z}$ and $X \in S$. Equivalently (see Lemma 2.12 below), if the smallest localizing (that is, triangulated, thick, and closed under coproducts) subcategory of $\mathcal{T}$ containing $S$ is $\mathcal{T}$ itself. If all objects of $S$ are compact, we say that $\mathcal{T}$ is compactly generated.
For a triangulated category with all small coproducts $\mathcal{T}$, we write $\mathcal{T}^c$ for the (automatically triangulated and thick) subcategory of compact objects.

We have the following general result, see [3, Theorem 4.4.9].

**Theorem 2.11.** Let $\mathcal{T}$ be a triangulated category with all small coproducts, $S$ a localizing subcategory and assume that both $\mathcal{T}$ and $S$ are compactly generated. Further assume that $S^c \subset \mathcal{T}^c$. Then $\mathcal{T}/S$ is compactly generated and the idempotent completion of $\mathcal{T}^c/S^c$ is precisely $(\mathcal{T}/S)^c$.

The following criterion identifying the subcategory of compact objects is taken from [3, Lemma 4.4.5].

**Lemma 2.12.** Assume $\mathcal{T}$ is a triangulated category with all small coproducts and $S$ is a thick triangulated subcategory of $\mathcal{T}^c$ containing a set of generators of $\mathcal{T}$. Then $S = \mathcal{T}^c$.

**Definition 2.13.** We write $DM_{gm,et}^{eff}(\mathbb{Z}/n)$ for the subcategory of compact objects in $DM_{et}^{eff}(\mathbb{Z}/n)$.

The above definition is justified by the following result.

**Lemma 2.14.** The compact objects of $DM_{gm,et}^{eff}(\mathbb{Z}/n)$ are precisely the objects of $DM_{gm}^{eff}(\mathbb{Z}/n)$ (and similar for étale motives by definition).

**Proof.** We can obtain $DM_{gm,et}^{eff}(\mathbb{Z}/n)$ from $D(PShv(SmCor, \mathbb{Z}/n))$ by localizing with respect to the localizing subcategory generated by complexes of the form $Z_{tr}(X \times \mathbb{A}^1)/n \rightarrow Z_{tr}(X)/n$ and complexes of the form $Z_{tr}(U \cap V)/n \rightarrow Z_{tr}(U)/n \oplus Z_{tr}(V)/n \rightarrow Z_{tr}(X)/n$ for open covers $X = U \cup V$ of smooth schemes. We observe that the compact objects in $D(PShv(SmCor, \mathbb{Z}/n))$ are (up to isomorphism) precisely those in the homotopy category of bounded complexes over the category of smooth correspondences, $\mathcal{H}^b(SmCor/n)$ (this follows from Lemma 2.12). Now Theorem 2.11 applies to prove the assertion.

**Remark 2.15.** It seems natural to ask if it is possible to give a more explicit description of $DM_{gm,et}^{eff}(\mathbb{Z}/n)$ similar to the one of $DM_{gm}^{eff}(\mathbb{Z}/n)$. One might consider the category $\mathcal{H}^b(SmCor/n)$ and try to localize with respect to complexes $X \times \mathbb{A}^1 \rightarrow X$ and some other class of complexes playing the role of Mayer-Vietoris complexes in the Nisnevich case. The obvious candidate is the class of Čech complexes for all possible étale covers. Unfortunately, those are not bounded and we see no apparent way of replacing them with bounded complexes.

There is an obvious localization functor $DM_{gm,et}^{eff}(\mathbb{Z}/n) \rightarrow DM_{et}^{eff}(\mathbb{Z}/n)$.

Choose a primitive $n$th root of unity in $F$, that is, an isomorphism $\beta : \mathbb{Z}/n \rightarrow \mu_n$. Such a choice determines an isomorphism $H^{0,1}_{\mathcal{M}}(F, \mathbb{Z}/n) \cong H^{0}_{\mathcal{M}}(F, \mu_n) \cong \mathbb{Z}/n.$ Denote the element of $H^{0,1}_{\mathcal{M}}(F, \mathbb{Z}/n)$ corresponding to 1 via this isomorphism also by $\beta$ (called “Bott element”). In the category $DM_{gm}^{eff}(\mathbb{Z}/n)$, $\beta$ corresponds to an element of $\text{Hom}(\mathbb{Z}/n(0), \mathbb{Z}/n(1))$; its image in $H^{0}_{\mathcal{M}}(F, \mu_n) \cong \mathbb{Z}/n$ gives the morphism $\beta : \mathbb{Z}/n \rightarrow \mu_n$ in the category $D^{-}(Shv(F_{et}), \mathbb{Z}/n)$. Tensoring with the identity, we obtain, for each object $A$ of $DM_{gm}^{eff}(\mathbb{Z}/n)$ (resp. of $D^{-}(Shv(F_{et}), \mathbb{Z}/n)$) a morphism...
\( \beta_A : A \to A(1) \). We write \( W_{mol} \) for the class of morphisms \( \beta_A \) in \( DM^{eff}_n(Z/n) \) (and also for its image in \( DM_{gm}(Z/n) \)) and \( W_{el} \) for the class of \( \beta_A \) in \( D^- (\text{Shv}(F_{et}), Z/n) \).

Recall from [3, Definition 1.6.4] the following definition:

**Definition 2.16.** Let \((X_i, f_i : X_i \to X_{i+1})_{i \geq 0} \) be a sequence of objects in a triangulated category \( T \) having countable coproducts. A **sequential colimit** (often called telescope, or homotopy colimit) of the sequence is a cone of the morphism

\[
\sum_i (1 - f_i) : \bigoplus_i X_i \to \bigoplus_i X_i.
\]

**Remark 2.17.** Note that for any compact object \( A \) and any sequential colimit \( X \) of a sequence \( X_i \), the natural map \( \text{colim}_i \text{Hom}(A, X_i) \to \text{Hom}(A, X) \) is an isomorphism. Indeed,

\[
\text{colim}_i \text{Hom}(A, X_i) = \text{coker} \left( \sum_i (1 - f_i) : \bigoplus_i \text{Hom}(A, X_i) \to \bigoplus_i \text{Hom}(A, X_i) \right)
\]

and the cokernel is equal to \( \text{Hom}(A, X) \) since the map

\[
\sum_i (1 - f_i) : \bigoplus_i \text{Hom}(A[-1], X_i) \to \bigoplus_i \text{Hom}(A[-1], X_i)
\]

is injective.

Now we invert the Bott element in our category:

**Definition 2.18.** We define \( DM^{eff}_n(Z/n)[W_{mol}^{-1}] \) to be the localization with respect to the localising subcategory \( S \) generated by the cones of the morphisms \( \beta_A \), where \( A \) ranges over a set of compact generators of \( DM^{eff}_n(Z/n) \). Note that, if \( A \to B \to C \) is a distinguished triangle such that the cones of \( \beta_A \) and \( \beta_B \) are in \( S \), then so is the cone of \( \beta_C \). The category \( DM^{eff}_n(Z/n)[W_{mol}^{-1}] \) is by definition the full subcategory of \( DM^{eff}_n(Z/n)[W_{mol}^{-1}] \) whose objects are in \( DM^{eff}_n(Z/n) \). By Lemma 2.14, Theorem 2.11 and Lemma 2.12, this is the idempotent completion of the localization of \( DM^{eff}_n(Z/n) \) with respect to the thick subcategory generated by the cones of the \( \beta_A \).

Our goal is to prove the following result:

**Theorem 2.19.** There is a triangulated equivalence

\[
DM^{eff}_n(Z/n)[W_{mol}^{-1}] \cong DM^{eff}_n(Z/n)[W_{el}^{-1}] = DM^{eff}_n(Z/n).
\]

**Proof.** Observe that in the étale setting \( DM^{eff}_n(Z/n) = DM^{eff}_n(Z/n)[W_{et}^{-1}] \). Throughout the proof, we use the fact that \( DM^{eff}_n(Z/n) \) is the full subcategory of compact objects of \( DM^{eff}_n(Z/n) \), and likewise for étale motives (see Lemma 2.14). We will show that the functor (induced by étale sheafification followed by \( \Phi \))

\[
DM^{eff}_n(Z/n)/S \to D(\text{Shv}(F_{et}), Z/n),
\]

where \( S \) is the localizing subcategory generated by the cones of \( \beta_A \) with \( A \) ranging over a set of generators (in particular, \( S \) is compactly generated) is an equivalence of categories. By Proposition 2.4 and Theorem 2.11, the assertion of the theorem follows.

The proof will be broken into several steps:
Step 1: Using the main result of [2], we show that for any \( A = M(X)/n \) a motive of a smooth scheme, the map
\[
\text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(A, \mathbb{Z}/n(p)[q]) \to \text{Hom}_{\mathcal{D}(\text{Shv}(F_{et}, \mathbb{Z}/n))}(\Phi(A_{et}), \Phi(\mathbb{Z}/n(p)_{et}[q]))
\]
is an isomorphism.
Indeed, we will see that
\[
\text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(A, \mathbb{Z}/n(p)[q]) = \text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(A, \mathbb{Z}/n(p)[q])[\beta^{-1}].
\]
Here we define
\[
\text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(A, \mathbb{Z}/n(p)[q])[\beta^{-1}] := \text{colim}_n \text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(A, \mathbb{Z}/n(p+i)[q]).
\]
This in turn is isomorphic to \( H^q_{mot}(X, \mathbb{Z}/n(p))[\beta^{-1}] \) if \( A = M(X)/n \) because of Lemma 2.1 and Lemma 2.2, and so [2, Theorem 1.1] shows that the map 2.21 is an isomorphism, because
\[
\text{Hom}_{\mathcal{D}(\text{Shv}(F_{et}, \mathbb{Z}/n))}(\Phi((M(X)/n)_{et}), \Phi(\mathbb{Z}/n(p)_{et}[q])) \cong H^q_{et}(X, \mu_n^{\otimes p})
\]
by Corollary 2.8 and Lemma 2.10.
In light of the fact that \( M(X)/n \) is a compact object in \( DM^{eff}(\mathbb{Z}/n) \), the isomorphism 2.22 is implied by the following assertion: For any object \( C \) and any sequential colimit \( \hat{C} \) of \( C \to C(1) \to C(2) \to \cdots \) (with maps multiplication by Bott), the map \( C \to \hat{C} \) is a \( W_{mot} \)-equivalence, and \( \hat{C} \) is \( W_{mot} \)-local, so that
\[
\text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(\mathbb{Z}/n)[\beta^{-1}] = \text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(-, \hat{C}).
\]
The first part of this assertion is obvious; the second part follows from the fact that for any \( B \), the natural map \( \beta^* : \text{colim}_n \text{Hom}(B(1), C(n)) \to \text{colim}_n \text{Hom}(B, C(n)) \) is an isomorphism - which in turn is a consequence of the relation \( \beta f = f(1) \beta \) for any morphism \( f \) - and compact generation of the localizing subcategory.

Step 2: The map 2.21 is an isomorphism for any object \( A \) in \( DM^{eff}(\mathbb{Z}/n) \) as this category is generated as a triangulated category by direct sums of direct summands of objects \( M(X)/n \) for \( X \) smooth over \( F \).

Step 3: We claim that the equivalence of Proposition 2.4 is a tensor equivalence. Indeed, by [8, Lemma 6.9, Theorem 7.20 and Theorem 9.32] we have a chain of tensor triangulated equivalences from \( DM^{eff}_{gm}(\mathbb{Z}/n) \to D^-(\text{Shv}(F_{et}, \mathbb{Z}/n)) \) which coincides with the forgetful functor \( \Phi \). (Recall that in the Nisnevich setting the embedding of \( DM^{eff}_{gm} \) into the derived category of Nisnevich sheaves with transfers is not preserving the tensor structure, cf. [9, Remark on page 210].) Hence, the functor \( \Phi \) also respects the respective internal homomorphism functors (which are adjoints to the tensor functors and as such uniquely determined by them), that is, there is a natural isomorphism
\[
\Phi(\text{Hom}_{DM^{eff}_{gm}(\mathbb{Z}/n)}(A, B)) \cong R\text{Hom}_{\mathcal{D}(\text{Shv}(F_{et}, \mathbb{Z}/n))}(\Phi(A), \Phi(B))
\]
if \( \text{Hom}_{DM^{eff}_{gm}(\mathbb{Z}/n)}(A, B) \) exists, which it does when \( A \) is compact by Lemma 2.14 and [9, Proposition 3.2.8].
Assume that \( A \) and \( B \) are in \( DM^{eff}_{gm}(\mathbb{Z}/n) \). Then,
where the first isomorphism follows (because $A$ is compact) as in step 1, the second is just the definition of internal homomorphisms, the third is in step 2, the fourth is 2.23, and the last is the property of $\mathbf{RHom}$ of being an internal hom with respect to the derived tensor structure.

The equivalence 2.20 now follows by taking limits. First, we can replace $B$ by a general (non-compact) object $B'$. Indeed, for $A$ compact, the full subcategory of objects $B$ such that the isomorphism

$$\text{Hom}_{DM^eff}(\mathbb{Z}/n)[W_{m1}^{-1}](A, B) \cong \text{Hom}_{DM^eff}(\mathbb{Z}/n)(A, B)[j^{-1}]$$

holds is localizing, hence all of $DM^eff(\mathbb{Z}/n)$. Next, we can do the same with $A$, using the Milnor sequence. This shows that the functor 2.20 is fully faithful. Since it is evidently essentially surjective, the proof is complete.

\[ \square \]

**Corollary 2.24.** The triangulated functor

$$DM^eff_{gm}(\mathbb{Z}/n)[W_{m1}^{-1}] \rightarrow D^-(\mathbf{Sh}(F_{et}), \mathbb{Z}/n)$$

is a full embedding with dense image.

**Proof.** Follows from Theorem 2.19, Definition 2.13 and Proposition 2.4. $\square$

**Proposition 2.25.** If $F$ is algebraically closed, then there is an equivalence of tensor triangulated categories

$$DM^eff_{et}(\mathbb{Z}/n) \rightarrow D^-(\mathbb{Z}/n - \mathbf{Mod}).$$

**Proof.** This follows from Proposition 2.4 and the fact that the étale site of $\text{Spec}(F)$ is trivial. $\square$

**Corollary 2.26.** Let $F$ be algebraically closed. Then there is a full embedding

$$DM^eff_{gm}(F, \mathbb{Z}/n)[W_{m1}^{-1}] \rightarrow D^-(\mathbb{Z}/n - \mathbf{Mod}).$$

with dense image.

**Proof.** This follows from Corollary 2.24 and Proposition 2.25. $\square$

We can also prove the following categorical version of rigidity:

**Theorem 2.27.** If $L/F$ is an extension of algebraically closed fields, then the base change functor induces a full embedding

$$DM^eff_{gm}(F, \mathbb{Z}/n) \rightarrow DM^eff_{gm}(L, \mathbb{Z}/n).$$
Proof. By rigidity (see e.g. [4, Theorem 2.17]), we have an isomorphism $H^{p,q}(X,\mathbb{Z}/n) \cong H^{p,q}(X \times_F L, \mathbb{Z}/n)$ for any variety $X$ smooth over $F$. Using arguments of Step 1 and Step 2 of the proof of Theorem 2.19, we conclude that
\[
\text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(A/\mathbb{Z}/n[\mathbb{Q}]) \to \text{Hom}_{DM^{eff}(L,\mathbb{Z}/n)}(A/\mathbb{Z}/n[\mathbb{Q}])
\]
is an isomorphism. By Friedlander-Voevodsky duality [9, section 4.2], we have (for any $r \geq 0$) natural isomorphisms
\[
\text{Hom}_{DM^{eff}}(M(Y)(r)[2i], M^c(X)) \cong A_{r+i}(Y, X) \cong A_{r+n}(\text{Spec}(K), Y \times X) \cong \text{Hom}_{DM^{eff}}(M(Y \times X)(r + n)[2i], \mathbb{Z}/n)
\]
where $M^c(X) = C_\bullet(Z_{et}, X, 0)$ is the motive of $X$ with compact support (compare [9, p. 228]). By Lemma 2.2, we therefore have a natural isomorphism $\text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(M(Y)/n(r)[2i], M^c(X)/n) \cong \text{Hom}_{DM^{eff}(\mathbb{Z}/n)}(M(Y \times X)/n(r + n)[2i], \mathbb{Z}/n)$. By Lemma 2.1 and [9, Corollary 3.5], we may assume that $X$ is projective and thus $M^c(X) = M(X)$. Hence the Theorem follows. \hfill \Box

References


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