HOMOLOGY OF $GL_3(F)$ FOR INFINITE FIELD $F$

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ABSTRACT. The homology of $GL_3(F)$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}\left[\frac{1}{2}\right]$ is studied, where $F$ is an infinite field. The main theorem states that the natural map $H_3(GL_3(F), \mathbb{Z}[\frac{1}{2}]) \to H_3(GL_3(F), \mathbb{Z}[\frac{1}{2}])$ is injective. Using this we will study the indecomposable part of $K_3(F)$.

1. INTRODUCTION

The Hurewicz theorem relates homotopy groups to homology groups, which are much easier to calculate. This in turn provides a homomorphism from the Quillen $K_n$-group of a field $F$ to the $n$-th integral homology of stable group $GL(F)$, $h_n : K_n(F) \to H_n(GL(F), \mathbb{Z})$. One of the approaches to investigate these groups is by means of the homology stability.

Suslin’s stability theorem states that for an infinite field $F$ the natural map $H_i(GL_n(F), \mathbb{Z}) \to H_i(GL(F), \mathbb{Z})$ is bijective if $n \geq i$ [8]. In [8], Suslin constructed a map from $H_n(GL_n(F), \mathbb{Z})$ to Milnor’s $K_n$-group $K^M_n(F)$ such that the sequence

$$H_n(GL_{n-1}(F), \mathbb{Z}) \to H_n(GL_n(F), \mathbb{Z}) \to K^M_n(F) \to 0, \quad (1)$$

is exact. Combining these two results he constructed a map from $K_n(F)$ to $K^M_n(F)$ such that the composite homomorphism

$$K^M_n(F) \to K_n(F) \to K^M_n(F)$$

coincides with the multiplication by $(-1)^{n-1}(n-1)!$ [8, Sec. 4].

In the case of $n < i$, Suslin leaves it as a conjecture that for any infinite field $F$ the natural homomorphism

$$H_i(\text{inc}) : H_i(GL_n(F), \mathbb{Q}) \to H_i(GL(F), \mathbb{Q})$$

is injective. This conjecture is easy if $i = 1, 2$. For $i = 3$ the conjecture was proved positively by Elbaz-Vincent [3].

One even suspects that this injectivity would be true not only “rationally” but in a stronger form, namely,

**Conjecture 1.1.** For any infinite field $F$, the homomorphism of homology groups

$$H_n(\text{inc}) : H_n(GL_{n-1}(F), \mathbb{Z}\left[\frac{1}{(n-1)!}\right]) \to H_n(GL_n(F), \mathbb{Z}\left[\frac{1}{(n-1)!}\right])$$

is injective.
This conjecture is trivial if $n = 1$. It is true for $n = 2$ due to the decomposition $H_2(GL_2(F), \mathbb{Z}) = H_2(GL_1(F), \mathbb{Z}) \oplus K^M_2(F)$. The proof of this conjecture for $n = 3$ is the main goal of this paper (Theorem 5.2).

Combining the main result with (1), one obtains the following split exact sequence

$$0 \to H_3(GL_2(F), \mathbb{Z}[\frac{1}{2}]) \to H_3(GL_3(F), \mathbb{Z}[\frac{1}{2}]) \to K^M_3(F) \otimes \mathbb{Z}[\frac{1}{2}] \to 0.$$ 

We will give an explicit splitting map $K^M_3(F) \otimes \mathbb{Z}[\frac{1}{2}] \to H_3(GL_3(F), \mathbb{Z}[\frac{1}{2}])$. Applying these results we will prove that

$$K_3(F)_{\text{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \cong H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]),$$

where $K_3(F)_{\text{ind}} = \text{coker}(K^M_3(F) \to K_3(F))$ is the indecomposable part of $K_3(F)$.

Our general strategy will be the same as in [3]. We will introduce some spectral sequences similar to ones in [3], but smaller and still big enough to do some computation. Our result will come out of analyzing these spectral sequences.

As a result of our computation, we shall prove that the complex

$$H_3(F^{*2} \times GL_1(F), \mathbb{Z}) \xrightarrow{\beta_1^{(3)}} H_3(F^* \times GL_2(F), \mathbb{Z}) \xrightarrow{\beta_2^{(3)}} H_3(GL_3(F), \mathbb{Z}) \to 0$$

is exact, where $\beta_1^{(3)} = H_3(\text{inc})$ and $\beta_2^{(3)} = H_3(\alpha_{1,2}) - H_3(\alpha_{2,2})$, $\alpha_{1,2} : F^{*2} \times GL_1(F) \to F^* \times GL_2(F)$, $\text{diag}(a, b, c) \mapsto \text{diag}(b, a, c)$ and $\alpha_{2,2} = \text{inc} : F^{*2} \times GL_1(F) \to F^* \times GL_2(F)$. Using this we shall prove that if $F$ is an algebraically closed field, the so called classical Bloch group $\mathfrak{g}^3(F)_{cl}$ is divisible (Proposition 3.7). This gives a positive answer to conjecture 0.2 in [11] for $n = 3$ (Remark 2).

In the last section we will study the map $H_n(\text{inc}) : H_n(GL_{n-1}(F), k) \to H_n(GL_n(F), k)$ when $k$ is a field and $(n - 1)! \in k^*$.

Here we establish some notation. In this note by $H_i(G)$ we mean the $i$-th integral homology of the group $G$. We use the bar resolution to define the homology of a group [1, Chap. 1, Section 5]. Define $c(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \sigma(g_{\sigma(1)}) \sigma(g_{\sigma(2)}) \cdots \sigma(g_{\sigma(n)}) \in H_n(G)$, where the elements $g_i \in G$ commute with each other and $\Sigma_n$ is the symmetric group of degree $n$. By $G_n$ we mean the general linear group $GL_n(F)$, where $F$ is an infinite field. Note that $G_0$ is the trivial group and $G_1 = F^*$. The $i$-th factor of $F^{*m}$ is denoted by $F^*_i$.

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2. THE SPECTRAL SEQUENCES

Let $C_i(F^m)$ and $D_i(F^m)$ be the free abelian groups with a basis consisting of $(\langle v_0 \rangle, \ldots, \langle v_i \rangle)$ and $(\langle w_0 \rangle, \ldots, \langle w_i \rangle)$ respectively, where every $\min\{l+1, n\}$ of $v_i \in F^m$ and every $\min\{l+1, n-1\}$ of $w_i \in F^m$ are linearly independent. By
\langle v_i \rangle$ we mean the line passing through vectors $v_i$ and $0$. Let $\partial_0 : C_0(F^n) \to C_{-1}(F^n) := \mathbb{Z}, \sum_i n_i(\langle v_i \rangle) \mapsto \sum_i n_i$ and $\partial_l = \sum_{i=0}^l (-1)^i d_i : C_i(F^n) \to C_{l-1}(F^n)$, $l \geq 1$, where
\[ d_i((\langle v_0 \rangle, \ldots, \langle v_i \rangle)) = (\langle v_0 \rangle, \ldots, \langle v_i \rangle, \ldots, \langle v_i \rangle). \]

Define the differential $\tilde{\partial}_l = \sum_{i=0}^l (-1)^i \tilde{d}_i : D_l(F^n) \to D_{l-1}(F^n)$ similar to $\partial_l$. Set $L_0 = \mathbb{Z}$, $M_0 = \mathbb{Z}$, $L_l = C_{l-1}(F^n)$ and $M_l = D_{l-1}(F^n)$, $l \geq 1$. It is easy to see that the complexes
\[ L_* : \quad 0 \leftarrow L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_l \leftarrow \cdots \]
\[ M_* : \quad 0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_l \leftarrow \cdots \]
are exact. Take a $G_n$-resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ with trivial $G_n$-action. From the double complexes $L_* \otimes_{G_n} P_*$ and $M_* \otimes_{G_n} P_*$ we obtain two first quadrant spectral sequences converging to zero with
\[ E^1_{p,q}(n) = \begin{cases} H_q(F^{*p} \times G_{n-p}) & \text{if } 0 \leq p \leq n, \\ H_q(G_n, C_{p-1}(F^n)) & \text{if } p \geq n + 1, \end{cases} \]
\[ \tilde{E}^1_{p,q}(n) = \begin{cases} H_q(F^{*p} \times G_{n-p}) & \text{if } 0 \leq p \leq n - 1, \\ H_q(G_n, D_{p-1}(F^n)) & \text{if } p \geq n. \end{cases} \]

For $1 \leq p \leq n$, and $q \geq 0$ the differential $d^{p,n}_{p,0}(n)$ equals $\sum_{i=1}^p (-1)^{i+1} H_q(\alpha_{i,p})$, where $\alpha_{i,p} : F^{*p} \times G_{n-p} \to F^{*p-1} \times G_{n-p+1}$, diag$(a_1, \ldots, a_p, A) \mapsto \text{diag}(a_1, \ldots, \alpha_i, \ldots, a_p \left( \begin{array}{cc} a_i & 0 \\ 0 & A \end{array} \right))$. In particular for
\[ 0 \leq p \leq n, \quad d^{p,n}_{p,0}(n) = \begin{cases} \text{id}_\mathbb{Z} & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even}, \end{cases} \]
so $E^2_{n,0}(n) = 0$ for $p \leq n - 1$. It is also easy to see that $E^2_{n,0}(n) = E^2_{n+1,0}(n) = 0$. See the proof of [5, Thm. 3.5] for more details.

In this note we will use $\tilde{E}^i_{p,q}(n)$ only for $n = 3$, so from now on by $\tilde{E}^i_{p,q}$ we mean $\tilde{E}^i_{p,q}(3)$. We describe $\tilde{E}^1_{p,q}$ for $p = 3, 4$. Let $w_1 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle), w_2 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle) \in D_2(F^3)$ and $u_1, \ldots, u_5, u_{6, a} \in D_3(F^3), a \in F^* - \{1\}$, where
\[ u_1 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_1 + e_2 + e_3 \rangle), \quad u_2 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_1 + e_2 \rangle), \]
\[ u_3 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_2 + e_3 \rangle), \quad u_4 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_1 + e_3 \rangle), \]
\[ u_5 = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_3 \rangle), \quad u_{6,a} = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle). \]

By the Shapiro lemma $\tilde{E}^1_{3, q} = H_q(\text{Stab}_{G_3}(w_1)) \oplus H_q(\text{Stab}_{G_3}(w_2))$ and $\tilde{E}^1_{4, q} = \bigoplus_{j=1}^5 H_q(\text{Stab}_{G_3}(u_j)) \oplus \bigoplus_{a \in F^* - \{1\}} H_q(\text{Stab}_{G_3}(u_{6,a}))$. Applying the
so called center killer lemma [8, Thm. 1.9], one gets

\[ \hat{E}^{3}_{q} = H_{q}(F^{*}I_{2} \times F^{*}) \oplus H_{q}(F^{*}I_{2} \times F^{*}) \]

\[ \hat{E}^{q}_{4} = H_{q}(F^{*}I_{3}) \oplus H_{q}(F^{*}I_{2} \times F^{*}) \oplus H_{q}(F^{*} \times F^{*}I_{2}) \oplus H_{q}(T) \]

\[ \oplus H_{q}(F^{*}I_{2} \times F^{*}) \oplus \bigoplus_{a \in F^{*} \setminus \{1\}} H_{q}(F^{*}I_{2} \times F^{*}) \]

where \( T = \{(a, b, a) \in F^{3} : a, b \in F^{*}\} \). Note that \( \hat{d}_{p,q}^{1} = d_{p,q}^{1}(3) \) for \( p = 1, 2 \),
\( \hat{d}_{3,q}^{1}(F^{*}I_{3}) = d_{3,q}^{1}(3) \) and \( \hat{d}_{3,q}^{1}(F^{*}I_{2} \times F^{*}) = H_{q}(\text{inc}) \), where \( \text{inc} : F^{*}I_{2} \times F^{*} \to F^{*} \to F^{*} \).

3. Some computation

**Lemma 3.1.** The group \( \hat{E}^{2}_{p,0} \) is trivial for \( 0 \leq p \leq 5 \).

**Proof.** Triviality of \( \hat{E}^{2}_{p,0} \) is easy for \( 0 \leq p \leq 2 \). To prove the triviality of \( \hat{E}^{2}_{3,0} \), note that \( \hat{E}^{1}_{3,0} = Z, \hat{E}^{1}_{3,0} = Z \oplus Z \) and \( \hat{d}_{3,0}^{1}(n_{1}, n_{2}) = n_{1} + n_{2} \), so
if \( (n_{1}, n_{2}) \in \ker(\hat{d}_{3,0}^{1}) \), then \( n_{2} = -n_{1} \). It is easy to see that this sits in
\( \text{im}(\hat{d}_{3,0}^{1}) \). We prove the triviality of \( \hat{E}^{2}_{5,0} \). Triviality of \( \hat{E}^{2}_{4,0} \) is similar but
much easier. This proof is just taken from [3, Sec. 1.3.3].

**Triviality of \( \hat{E}^{2}_{5,0} \).** The proof will be in four steps;

**Step 1.** The sequence \( 0 \to C_{*}(F^{3} \otimes G_{3}) \to D_{*}(F^{3} \otimes G_{3}) \to Q_{3}(F^{3}) \to 0 \) is exact, where \( Q_{*}(F^{3}) := D_{*}(F^{3})/C_{*}(F^{3}) \).

**Step 2.** The group \( H_{*}(Q_{*}(F^{3}) \otimes G_{3}) \) is trivial.

**Step 3.** The map induced in homology by \( C_{*}(F^{3}) \otimes G_{3} \to D_{*}(F^{3}) \otimes G_{3} \) is zero in degree 4.

**Step 4.** The group \( \hat{E}^{2}_{5,0} \) is trivial.

**Proof of step 1.** For \( i \geq -1 \), \( D_{i}(F^{3}) \simeq C_{i}(F^{3}) \oplus Q_{i}(F^{3}) \). This decomposition is compatible with the action of \( G_{3} \), so we get an exact sequence of
\( \mathbb{Z}[G_{3}] \)-modules

\[ 0 \to C_{i}(F^{3}) \to D_{i}(F^{3}) \to Q_{i}(F^{3}) \to 0 \]

which splits as a sequence of \( \mathbb{Z}[G_{3}] \)-modules. One can easily deduce the
desired exact sequence from this. Note that this exact sequence does not
split as complexes.

**Proof of step 2.** The complex \( Q_{*}(F^{3}) \) induces a spectral sequence

\[ \hat{E}^{1}_{p,q} = \begin{cases} 0 & \text{if } 0 \leq p \leq 2 \\ H_{q}(G_{3}, Q_{p-1}(F^{3})) & \text{if } p \geq 3 \end{cases} \]

which converges to zero. To prove the claim it is sufficient to prove that
\( \hat{E}^{2}_{5,0} = 0 \) and to prove this it is sufficient to prove that \( \hat{E}^{2}_{3,1} = 0 \). One can see that \( \hat{E}^{1}_{3,1} = H_{1}(F^{*}I_{2} \times F^{*}) \). If \( w = (e_{1}, e_{2}, e_{3}, e_{1} + e_{2}) \in Q_{3}(F^{3}) \), then \( H_{1}(\text{Stab}_{G_{3}}(w)) \simeq H_{1}(F^{*}I_{2} \times F^{*}) \) is a summand of \( \hat{E}^{1}_{4,1} \) and \( \hat{d}_{4,1}^{1} : H_{1}(\text{Stab}_{G_{3}}(w)) \to \hat{E}^{1}_{3,1} \) is an isomorphism. So \( \hat{d}_{4,1}^{1} \) is surjective and
therefore $\tilde{E}_{3,1}^2 = 0$.

**Proof of step 3.** Consider the following commutative diagram

$$
\begin{array}{c}
C_5(F^3) \otimes_{G_3} \mathbb{Z} \rightarrow C_4(F^3) \otimes_{G_3} \mathbb{Z} \rightarrow C_3(F^3) \otimes_{G_3} \mathbb{Z} \\
D_5(F^3) \otimes_{G_3} \mathbb{Z} \rightarrow D_4(F^3) \otimes_{G_3} \mathbb{Z} \rightarrow D_3(F^3) \otimes_{G_3} \mathbb{Z}.
\end{array}
$$

The generators of $C_4(F^3) \otimes_{G_3} \mathbb{Z}$ are of the form $x_{a,b} \otimes 1$, where $x_{a,b} = \langle \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_3 \rangle, \langle e_1 + e_2 \rangle \rangle$, $a, b \in F^* - \{1\}$ and $a \neq b$.

Since $C_3(F^3) \otimes_{G_3} \mathbb{Z} = \mathbb{Z}$, $(x_{a,b} - x_{c,d}) \otimes 1 \in \ker(\partial_4 \otimes 1)$ and the elements of this form generate $\ker(\partial_4 \otimes 1)$. Hence to prove this step it is sufficient to prove that $(x_{a,b} - x_{c,d}) \otimes 1 \in \im(\partial_5 \otimes 1)$.

Set $w_{a,b} = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_3 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle) \in D_5(F^3)$, where $a, b \in F^* - \{1\}$ and $a \neq b$. Let $g$, $g'$, and $g''$ be the matrices

$$
\begin{pmatrix}
0 & a^{-1} & 0 \\
-1 & 1 + a^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & -b^{-1} \\
0 & 1 & -ab^{-1} \\
0 & 0 & b^{-1}
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & a^{-1} & 0 \\
0 & 0 & b^{-1}
\end{pmatrix},
$$

respectively, then

$$g(\tilde{a}_1(w_{a,b})) = \tilde{a}_0(w_{a,b}), \quad g'(\tilde{a}_3(w_{a,b})) = \tilde{a}_2(w_{a,b}), \quad g''(\tilde{a}_4(w_{a,b})) = v_{1,1},$$

and so $(\tilde{\partial}_5 \otimes 1)(w_{a,b} \otimes 1) = (v_{1,1} - v_{a,b}) \otimes 1$, where

$$v_{a,b} = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_3 \rangle, \langle e_1 + e_2 \rangle \rangle).$$

Note that the elements of the form $(gw - w) \otimes 1$ are zero in $D_4 \otimes_{G_3} \mathbb{Z}$. If

$$u_{a,b} = (\langle e_3 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle \rangle),$$

$$u'_{a,b} = (\langle e_1 + ae_2 + be_3 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle \rangle),$$

where $a, b \in F^* - \{1\}$, then

$$gu_{a,b} = (\langle e_3 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle \rangle),$$

$$g'u'_{a,b} = (\langle e_3 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle \rangle).$$

So if $a, b, c, d \in F^* - \{1\}$, $a \neq b, c \neq d$, then $(\tilde{\partial}_5 \otimes 1)((z_{a,b} - z_{c,d}) \otimes 1) = (t_{c,d} - t_{a,b}) \otimes 1$, where

$$z_{a,b} = (\langle e_3 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + ae_2 \rangle \rangle),$$

$$t_{a,b} = (\langle e_3 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle \rangle).$$

If $g_1$, $g_2$ and $g_3$ are the matrices

$$
\begin{pmatrix}
-1 & 0 & 1 \\
-1 & 0 & 0 \\
-1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -1 & 1 \\
0 & -1 & 0 \\
1 & -1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & -1
\end{pmatrix},
$$
respectively, then \( g_1(\tilde{d}_0(y_{a,b})) = t \frac{1}{1-s}, \frac{1-s}{s} \), \( g_2(\tilde{d}_1(y_{a,b})) = t \frac{s-1}{s}, \frac{1-s}{s} \) and \( g_3(\tilde{d}_3(y_{a,b})) = v_{a-b}, \frac{1-s}{s} \), where
\[
y_{a,b} = (\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + ae_2 + be_3 \rangle, \langle e_3 \rangle, \langle e_1 + e_2 + e_3 \rangle, \langle e_1 + e_2 \rangle).
\]
By an easy computation
\[
(\tilde{d}_5 \otimes 1)(y_{a,b} \otimes 1) = t \frac{1}{1-s}, \frac{1-s}{s} \otimes 1 - t \frac{s-1}{s}, \frac{1-s}{s} \otimes 1 + v_{a-1} \otimes 1
- v_{a-b}, \frac{1-s}{s} \otimes 1 + v_{a,b} \otimes 1 - x_{a,b} \otimes 1.
\]
Now it is easy to see that \((x_{a,b} - x_{a,d}) \otimes 1 \in (\tilde{d}_5 \otimes 1)(D_5(F^3) \otimes_G \mathbb{Z}) \). This completes the proof of step 3.

**Proof of Step 4.** Applying the homology long exact sequence to the short exact sequence obtained in the first step, we get the exact sequence
\[
H_4(C_*(F^3) \otimes_G \mathbb{Z}) \to H_4(D_*(F^3) \otimes_G \mathbb{Z}) \to H_4(Q_*(F^3) \otimes_G \mathbb{Z}).
\]
By steps 2 and 3, \( H_4(D_*(F^3) \otimes_G \mathbb{Z}) = 0 \), but \( \tilde{E}^2_{5,0} = H_4(D_*(F^3) \otimes_G \mathbb{Z}) \). This completes the proof of the triviality of \( \tilde{E}^2_{5,0} \). \( \square \)

**Lemma 3.2.** The group \( \tilde{E}^2_{p,1} \) is trivial for \( 0 \leq p \leq 4 \).

**Proof.** Triviality of \( \tilde{E}^2_{p,1}, p = 0, 1 \), is a result of lemma 3.1 and the fact that the spectral sequence converges to zero (one can also prove this directly).

If \((a, b, c) \in \ker(\tilde{d}_{1,1}^2), a, b, c \in H_1(F^*)\), then \( a = b \). It is easy to see that this element sits in \( \text{im}(\tilde{d}_{1,1}^3) \). Let \( x = (x_1, \ldots, x_5, (x_{6a})) \in \tilde{E}_{4,1}^1 \), where \( x_2 = (a_2, a_2, b_2), x_3 = (a_3, b_3, b_3), x_4 = (a_4, b_4, a_4), x_5 = (a_5, a_5, b_5), a_i, b_i \in H_1(F^*) \). By a direct calculation \( \tilde{d}_{4,1}^1(x) = (z_1, z_2) \), where
\[
z_1 = -(a_2, a_2, b_2) - (a_3, b_3, b_3) + (b_4, a_4, a_4) + (a_5, a_5, b_5),
z_2 = (a_2, a_2, b_2) + (b_3, b_3, a_3) - (a_4, a_4, b_4) - (a_5, a_5, b_5).
\]
If \( y = ((a, b, c), (d, d, e)) \in \ker(\tilde{d}_{1,1}^3), a, b, c, d, e \in H_1(F^*) \), then \( b + d = a - b + c + e = 0 \). Let \( x_2 = (-b, b, -c), x_3 = (-a + b, 0, 0) \) and set \( x' = (0, x_2, x_3, 0, 0, 0) \in \tilde{E}_{4,1}^1 \), then \( y = \tilde{d}_{4,1}^1(x') \).

To prove the triviality of \( \tilde{E}_{4,1}^2 \); let \( x \in \ker(\tilde{d}_{4,1}^1) \) and set \( w_1 = ((\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_3 + e_4 \rangle), w_2 = ((\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_3 + e_4 \rangle), w_3 = ((\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 + e_3 \rangle, \langle e_2 + e_3 \rangle), w_4 = ((\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 + e_3 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + e_2 + e_3 \rangle), w_5 = ((\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 + e_3 \rangle, \langle e_1 + e_2 + e_3 \rangle, \langle e_1 + e_2 + e_3 \rangle), a, b \in F^* - \{1\} \) and \( a \neq b \). The groups \( T_i = H_1(\text{Stab}_{G_1}(w_i)), i = 0, 1, 2, 3, 4 \) and \( T_4 = \bigoplus_{a \in F^* - \{1\}} H_1(\text{Stab}_{G_1}(w_{4a})) \) are summands of \( \tilde{E}_{5,1}^1 \). Note that \( T_1 = H_1(F^* I_2 \times F^*), T_2 = H_1(T), T_3 = T_5 = H_1(F^* I_3) \) and \( T_4 = \bigoplus_{a \in F^* - \{1\}} H_1(F^* I_2 \times F^*). \) The restriction of \( \tilde{d}_{5,1}^1 \) on these summands is
as follow;
\[
\begin{align*}
\bar{d}_{5,1}^2 v_1 & = (0, (c_1, c_1, d_1), 0, (c_1, c_1, d_1), - (c_1, c_1, d_1)), \\
\bar{d}_{5,1}^2 v_2 & = (0, 0, (d_2, c_2, c_2), (c_2, d_2, c_2), 0, (c_2, d_2, c_2)), \\
\bar{d}_{5,1}^2 T_5 & = ((c_3, c_3, c_3), (c_3, c_3, c_3), -(c_3, c_3, c_3), 0, 0, 0), \\
\bar{d}_{5,1}^2 T_4 & = (0, 0, 0, 0, 0, (c_4, c_4, d_4)), \\
\bar{d}_{5,1} T_5 & = \text{id}_{H_1(F^* \cdot I_5)}.
\end{align*}
\]

Let \( z_1 = (a_5, a_5, b_5) \in T_1 \) and \( z_2 = (a_4, b_4, a_4) \in T_2 \), then \( x - \bar{d}_{5,1}^2(z_1 + z_2) = (x', x', x', 0, 0, (x'_6, a)) \), so we can assume that \( x_4 = x_5 = 0 \). An easy calculation shows that \( x_2 = b_2 = -a_3 = -b_3 \). If \( z_3 = (a_2, a_2, a_2) \in T_3 \), then \( x - \bar{d}_{5,1}^2(z_3) = (x'_1, 0, 0, 0, 0, (x'_6, a)) \). Again we can assume that \( x_2 = x_3 = 0 \). If \( z_4 = (x_6, a) \in T_4 \), then \( x - \bar{d}_{5,1}^2(z_4) = (x'_1, 0, 0, 0, 0, 0) \). Once more we can assume that \( x_6 = 0 \). These reduce \( x \) to an element of the form \( (x_1, 0, 0, 0, 0, 0) \). If \( x_1 \in T_5 \), then \( \bar{d}_{5,1}^2(x_1) = (x_1, 0, 0, 0, 0, 0) \). This completes the triviality of \( \tilde{E}^2_{4,1} \). \hfill \Box

**Lemma 3.3.** The group \( \tilde{E}^2_{p,2} \) is trivial for \( 0 \leq p \leq 3 \).

**Proof.** Triviality of \( \tilde{E}^2_{0,2} \) and \( \tilde{E}^2_{1,2} \) is a result of lemmas 3.1, 3.2 and the fact that the spectral sequence converges to zero. Let \( \tilde{E}_{1,2}^1 = H_2(F^* \times G_2) = H_2(F^* \otimes H_2(G_2) \otimes H_1(F^*) \otimes H_1(G_2), \tilde{E}_{1,2}^2 = H_2(F^* \otimes H_1(F^*) \otimes H_1(G_2), \tilde{E}_{2,2}^1 = \bigoplus_{i=1}^6 T_i \) and \( \tilde{E}_{3,2}^2 = H_3(F^* \otimes H_2(F^* \otimes G_2) \otimes H_1(F^*) \otimes H_1(G_2), \tilde{E}_{3,2}^3 = \bigoplus_{i=1}^9 T_i \), where \( T_i = H_2(F^*) \) for \( i = 1, 2, 3 \), \( T_4 = H_1(F_2^*) \otimes H_1(F_3^*), T_5 = H_1(F_1^*) \otimes H_1(F_3^*), T_6 = H_1(F_2^*) \otimes H_1(F_3^*), T_7 = H_2(F_1^* \otimes F_2^*), T_8 = H_2(F_1^* \otimes F_2^*), T_9 = H_1(F_2^* \otimes F_1^* \otimes F^*), \) and \( T_9 = H_1(F_2^* \otimes F_1^*) \otimes H_1(F_2^* \otimes F^*) \). If \( y = (y_1, y_2, y_3, \sum r \otimes s, \sum t \otimes u, \sum v \otimes w) \in \tilde{E}_{1,2}^1 \) and \( x = (x_1, x_2, x_3, \sum a \otimes b, \sum c \otimes d, \sum e \otimes f, x_7, x_8, \sum g \otimes h) \in \tilde{E}_{3,2}^2 \), then \( \bar{d}_{3,2}^1(y) = (h_1, h_2, h_3) \), where \( h_1 = -y_1 + y_2, h_3 = -\sum s \otimes \text{diag}(1, r) - \sum r \otimes \text{diag}(1, s) - \sum t \otimes \text{diag}(1, u) + \sum v \otimes \text{diag}(1, w) \) and \( \bar{d}_{3,2}^1(z_1 \otimes i_{6,0}, \ldots, z_9 \otimes f, z_5 = -\sum b \otimes a - \sum a \otimes b + \sum c \otimes d + \sum g \otimes h, z_6 = -\sum d \otimes c + \sum f \otimes e + \sum e \otimes f + \sum g \otimes h). \)

If \( y \in \text{ker}(\bar{d}_{3,2}^1) \), then \( y_1 = y_2 \) and \( h_3 = 0 \). By \( H_1(F^*) \otimes H_1(G_2) \cong H_1(F^*) \otimes H_1(G_2) \) and \( h_3 = 0 \) we have \( -\sum s \otimes r - \sum r \otimes s - \sum t \otimes u + \sum v \otimes w = 0 \). If \( z = (y_1, y_1, y_3, 0, \sum t \otimes u, \sum r \otimes s + \sum t \otimes u, 0, 0, 0) \in \tilde{E}_{3,2}^1 \), then \( y = \bar{d}_{3,2}^1(z) \) and therefore \( \tilde{E}_{2,2}^2 = 0 \).

Let \( \bar{d}_{3,2}^1(x) = 0 \). Consider the summands \( S_2 = H_2(\text{Stab}_{G_2}(u_2)) = H_2(F^* \otimes I_2^* \times F^*) \) and \( S_3 = H_2(\text{Stab}_{G_3}(u_3)) = H_2(F^* \times F^* \otimes I_2^* \times F^* \otimes I_2^*) \) of \( \tilde{E}_{4,2}^1 \). Then \( S_i \simeq H_2(F^* \otimes H_2(F^*) \otimes H_2(F^*) \otimes H_1(F^*) \otimes H_1(F^*) \) and by a direct calculation
\[
\begin{align*}
\bar{d}_{4,2}^1 s_2((y_1, y_2, s \otimes t)) &= (-y_1, -y_1, -y_2, 0, -s \otimes t, -s \otimes t, y_1, y_2, s \otimes t), \\
\bar{d}_{4,2}^1 s_3((u_1, u_2, p \otimes q)) &= (-u_1, -u_2, -u_2, -p \otimes q, -p \otimes q, 0, u_2, u_1, -q \otimes p).
\end{align*}
\]
Choose $z_2' = (-x_2, -x_3, -\sum e \otimes f) \in S_2$ and $z_3' = (x_3 + x_8, 0, -\sum a \otimes b) \in S_3$. Then $x = d_{1,2}^3(z_2' + z_3')$ and therefore $\tilde{E}_{3,2}^2 = 0$. 

**Lemma 3.4.** The groups $\tilde{E}_{0,3}^2$, $\tilde{E}_{1,3}^2$ and $\tilde{E}_{0,4}^2$ are trivial.

**Proof.** These follow from 3.1, 3.2 and 3.3 and the fact that the spectral sequence converges to zero. 

**Proposition 3.5.** (i) The complex 

$$H_2(F^* \times G_0) \xrightarrow{d_{1,2}^3} H_2(F^* \times G_1) \xrightarrow{d_{1,3}^3} H_2(F^* \times G_2) \xrightarrow{d_{1,3}^3} H_2(F^* \times G_3) \to 0$$

is exact, where $d_{1,2}^3(3) = H_2(\alpha_{1,2}) - H_2(\alpha_{2,3}) + H_2(\alpha_{3,3})$, $d_{1,2}^3(3) = H_2(\alpha_{1,2}) - H_2(\alpha_{2,3})$ and $d_{1,2}^3(3) = H_2(\text{inc})$.

(ii) The complex 

$$H_3(F^* \times G_1) \xrightarrow{d_{1,3}^3} H_3(F^* \times G_2) \xrightarrow{d_{1,3}^3} H_3(F^* \times G_3) \to 0$$

is exact, where $d_{1,3}^3(3) = H_3(\alpha_{1,2}) - H_3(\alpha_{2,3})$ and $d_{1,3}^3(3) = H_3(\text{inc})$.

(iii) (stability) The map $H_3(\text{inc}) : H_2(G_2) \to H_2(G_3)$ is an isomorphism and the map $H_3(\text{inc}) : H_3(F^* \times G_2) \to H_3(F^* \times G_3)$ is surjective.

**Proof.** The only case that remains to prove is that $H_2(G_2) \to H_2(G_3)$ is an isomorphism. The proof is similar to the proof of [5, lem. 4.2] using (i). 

**Remark 1.** (i) By a similar approach as the above proposition one can prove that $H_2(G_n) \to H_2(G_{n+1})$ is an isomorphism for $n \geq 3$. For this one should work with $E^1_{p,q}(n)$, $n \geq 3$. This combined with 3.5 will prove the homology stability for the functor $H_2$.

(ii) A similar result as 3.5(ii) is not true for $n = 2$, that is the complex 

$$H_2(F^* \times G_0) \xrightarrow{d_{1,2}^3(2)} H_2(F^* \times G_1) \xrightarrow{d_{1,2}^3(2)} H_2(G_2) \to 0$$

is not exact. In fact 

$$\ker(d_{1,2}^3(2))/\text{im}(d_{1,2}^3(2)) \simeq \langle x \wedge (x - 1) - x \otimes (x - 1) : x \in F^* \rangle$$

is a subset of $H_2(F^*) \oplus (F^* \otimes F^*)$, where $(F^* \otimes F^*) = (F^* \otimes F^*)/\langle a \otimes b + b \otimes a : a, b \in F^* \rangle$. To prove this let $Q(F)$ be the free abelian group with the basis $\{[x] : x \in F^* - \{1\}\}$. Denote by $p(F)$ the factor group of $Q(F)$ by the subgroup generated by the elements of the form $[x] - [y] + [y/x] - [(1 - x^{-1})/(1 - y^{-1})] + [(1 - x)/(1 - y)]$. The homomorphism $\psi : Q(F) \to F^* \otimes F^*$, $[x] \mapsto x \otimes (x - 1)$ induces a homomorphism $p(F) \to (F^* \otimes F^*)$, [9, 1.1]. By [9, 2.2], $E^2_{p,q}(2) = p(F)$ and $E^2_{p,q}(2)$ has the following form

$$
\begin{array}{cccc}
* & E^2_{2,0}(2) & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & p(F) & *
\end{array}
$$
An easy calculation shows that $E^2_{1,2}(2) \subseteq H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$. By [9, 2,4] $d^3_{4,0}(2) : E^3_{4,0}(2) \rightarrow E^3_{1,2}(2)$ is defined by $d^3_{4,0}(2)([x]) = x \wedge (x - 1) - x \otimes (x - 1)$. Because the spectral sequence converges to zero we see that $d^3_{4,0}(2)$ is surjective and so $E^2_{1,2}(2)$ is generated by the elements of the form $x \wedge (x - 1) - x \otimes (x - 1) \in H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$.

Following [11, Section 3] we define;

**Definition 3.6.** We call $\varrho^3(F)_{cl} := H(C_{n+2}(F^n)_{G_n} \rightarrow C_{n+1}(F^n)_{G_n} \rightarrow C_n(F^n)_{G_n})$ the $n$-th classical Bloch group.

It is well known that $\varrho^2(F)_{cl} \simeq p(F)$ [9, 1.1], where $p(F)$ is defined in remark 1.

**Proposition 3.7.** We have an isomorphism $\varrho^3(F)_{cl} \simeq F^*$. In particular if $F$ is algebraically closed, then $\varrho^3(F)_{cl}$ is divisible.

**Proof.** Using 3.5 one sees that $E^2_{p,q} (3)$ is of the form

$$
\begin{array}{ccccccc}
* & * &   &   &   &   &   \\
0 & 0 & * & * & 0 & * &   \\
0 & 0 & 0 & * & 0 & * &   \\
0 & 0 & 0 & F^* & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & \varrho^3(F)_{cl} & *  \\
\end{array}
$$

From this we obtain the exact sequence

$$0 \rightarrow E^3_{0,0} \rightarrow \varrho^3(F)_{cl} \overset{d^3_{0,0}}{\rightarrow} F^* \rightarrow 0.
$$

Comparing $E^3_{0,0}(3)$ with $E^3_{0,0}(3)$ and applying lemma 3.4 one sees that $E^3_{0,0}(3) = 0$. Now it is easy to see that $E^3_{0,0}(3) = 0$. This proves the first part of the proposition. The second part follows from the fact that for an algebraically closed field $F$, $F^*$ is divisible. \hfill \Box

**Remark 2.** From 3.7 and the existence of a surjective map $\varrho^3(F)_{cl} \rightarrow \varrho^3(F)$ [11, Prop. 3.11] we deduce that $\varrho^3(F)$ is divisible. See [11, 2.7] for the definition of $\varrho^3(F)$. This gives a positive answer to conjecture 0.2 in [11] for $n = 3$.

4. KÜNNETH THEOREM FOR $H_3(F^* \times F^*)$

The Künneth theorem claims that the group $H_n(F^* \times F^*)$, $n \geq 1$, sits in the following exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H_i(F^*) \otimes H_j(F^*) \rightarrow H_n(F^* \times F^*) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}^2_{i+j}(H_i(F^*), H_j(F^*)) \rightarrow 0
$$

which splits. But most of the time this splitting is not canonical. Here we will see that if $n \leq 3$, then it splits canonically. This is clear for $n = 1$. 
For $n = 2$ it follows from the fact that $\text{Tor}_1^F(H_i(F^*), H_j(F^*)) = 0$ if $(i, j) = (1, 0), (0, 1)$.

So let $n = 3$. If $\mu_F$ is the group of roots of unity of $F$, then $\mu_F = \lim_{n \to F} \mu_n, F$, where $\mu_n, F$ is the group of $n$-th roots of unity. By what we know about the homology of finite cyclic groups we obtain $H_2(\mu_F) = 0$. Hence the Künneth theorem for $H_3(\mu_F \times \mu_F)$ finds the following form

$$0 \rightarrow H_3(\mu_F) \oplus H_3(\mu_F) \rightarrow H_3(\mu_F \times \mu_F) \rightarrow \text{Tor}_1^F(\mu_F, \mu_F) \rightarrow 0.$$

Clearly $H_3(\mu_F) \oplus H_3(\mu_F) \rightarrow H_3(\mu_F \times \mu_F)$ is defined by $H_3(i_1) + H_3(i_2)$, where $i_k : \mu_F \rightarrow \mu_F \times \mu_F$ is the usual injection, $k = 1, 2$. Define the map $\beta : H_3(p_1) \oplus H_3(p_2) : H_3(\mu_F \times \mu_F) \rightarrow H_3(\mu_F) \oplus H_3(\mu_F)$, where $p_k : \mu_F \times \mu_F \rightarrow \mu_F$ is the usual projection, $k = 1, 2$. From $\alpha \circ \beta = \text{id}$ one deduce that the above exact sequence splits canonically. Thus we have the canonical decomposition

$$H_3(\mu_F \times \mu_F) = H_3(\mu_F) \oplus H_3(\mu_F) \oplus \text{Tor}_1^F(\mu_F, \mu_F).$$

We construct the splitting map $\text{Tor}_1^F(\mu_F, \mu_F) \rightarrow H_3(\mu_F \times \mu_F)$. The elements of $\text{Tor}_1^F(\mu_F, \mu_F) = \text{Tor}_1^F(H_1(\mu_F), H_1(\mu_F))$ are of the form $\langle \xi, n, [\xi] \rangle = \langle \langle \xi, n, [\xi] \rangle \rangle$ for some $n$, where $\xi$ is an $n$-th root of unity in $F$ [4, Chap. V, Section 6]. It is easy to see that $\partial_j \sum_{i=1}^n [\xi^i] = n[\xi]$ in $B_1(\mu_F)$. See [1, Chap. I, section 5] for the definition of $\partial_2$ and $B_*$. By [4, Chap. V, Prop. 10.6] the map $\phi : \text{Tor}_1^F(H_1(\mu_F), H_1(\mu_F)) \rightarrow H_3(B_* \mu_F \otimes B_* \mu_F)$ can be defined by

$$a := \langle [\xi], n, \langle [\xi] \rangle \rangle \mapsto \langle [\xi] \rangle \otimes \sum_{i=1}^n \langle [\xi^i] \rangle = \sum_{i=1}^n [\xi^i] \otimes [\xi].$$

Considering the isomorphism $B_* \mu_F \otimes B_* \mu_F \simeq B_*(\mu_F \times \mu_F)$ we have $\phi(a) = \chi(\xi) \in H_3(\mu_F \times \mu_F)$, where

$$\chi(\xi) := \sum_{i=1}^n (\langle (\xi, 1) \rangle \langle (1, \xi) \rangle [\xi^i]) - \langle (1, \xi), (\xi, 1) \rangle \langle (1, \xi^i) \rangle [\xi, 1] + \langle (1, \xi) \rangle \langle (\xi, 1) \rangle [\xi^i, 1] + \langle (1, \xi^i) \rangle \langle (\xi, 1) \rangle \langle (\xi^i, 1) \rangle].$$

Consider the following commutative diagram

$$0 \rightarrow H_3(\mu_F) \oplus H_3(\mu_F) \rightarrow H_3(\mu_F \times \mu_F) \rightarrow \text{Tor}_1^F(\mu_F, \mu_F) \rightarrow 0$$

$$0 \rightarrow \bigoplus_{i+j=3} H_i(F^*) \otimes H_j(F^*) \rightarrow H_3(F^* \times F^*) \rightarrow \text{Tor}_1^F(F^*, F^*) \rightarrow 0.$$
5. The injectivity theorem

Let $A := \mathbb{Z}[\frac{1}{2}]$ and let $P_{\ast} \to A$ be a $A[G_{3}]$-resolution of $A$ with trivial $G_{3}$-action. Consider the complex

$$D'_{0} : 0 \to D'_{1}(F^{3}) \to D'_{2}(F^{3}) \to \cdots \to D'_{i}(F^{3}) \to \cdots,$$

where $D'_{i}(F^{3}) := D_{i}(F^{3}) \otimes A$. The double complex $D'_{\ast} \otimes_{G_{3}} P_{\ast}$ induces a first quadrant spectral sequence $E^{1}_{p,q} \Rightarrow H_{p+q}(G_{3}, A)$, where $E^{1}_{p,q} = \tilde{E}^{1}_{p+1,q}(3) \otimes A$ and $\partial^{1}_{p,q} = \tilde{d}^{1}_{p+1,q} \otimes id_{A}$.

**Lemma 5.1.** The groups $E^{2}_{3,0}$, $E^{2}_{4,0}$, $E^{2}_{2,1}$, $E^{2}_{3,1}$, $E^{2}_{1,2}$ and $E^{2}_{2,2}$ are trivial.

**Proof.** This follows from the above spectral sequence and lemmas 3.1, 3.2, 3.3.

**Theorem 5.2.** The map $H_{3}(\text{inc}) : H_{3}(G_{2}, \mathbb{Z}[\frac{1}{2}]) \to H_{3}(G_{3}, \mathbb{Z}[\frac{1}{2}])$ is injective.

**Proof.** By lemma 5.1, $E_{0,3}^{2} \simeq E_{0,3}^{\infty} \simeq H_{3}(G_{3}, A)$, so to prove the theorem it is sufficient to prove that $H_{3}(G_{2}, A)$ is a summand of $E_{0,3}^{2}$. To prove this it is sufficient to define a map $\varphi : H_{3}(F^{*} \times G_{2}, A) \to H_{3}(G_{2}, A)$ such that $\varphi|H_{3}(G_{2}, A)$ is the identity map and $\partial_{3,3}^{1}(H_{3}(F^{*} \times G_{1}, A)) \subseteq \ker(\varphi)$.

By a similar argument as in the previous section we have the canonical decompositions $H_{3}(F^{*} \times G_{2}, A) = \bigoplus_{i=0}^{4} S_{i}$, where $S_{i} = H_{1}(F^{*}, A) \otimes H_{3-i}(G_{2}, A)$ for $0 \leq i \leq 3$ and $S_{4} = \text{Tor}_{1}^{H_{1}}(H_{1}(F^{*}, A), H_{1}(G_{2}, A))$. Note that the splitting map is

$$S_{4} \simeq \text{Tor}_{1}^{H_{1}}(\mu_{F}, \mu_{F}) \otimes A \xrightarrow{\phi} H_{3}(F^{*} \times F^{*}, A) \xrightarrow{q} H_{3}(F^{*} \times G_{2}, A),$$

where $\phi$ is defined in the previous section and $q : F^{*} \times F^{*} \to F^{*} \times G_{2}$, $(a, b) \mapsto \text{diag}(a, b, 1)$.

Define $\varphi : S_{0} \to H_{3}(G_{2}, A)$ the identity map, $\varphi : S_{2} \simeq H_{2}(F^{*}, A) \otimes H_{1}(G_{1}, A) \to H_{3}(F^{*} \times G_{1}, A)$ the shuffle product, $\varphi : S_{3} \to H_{3}(G_{2}, A)$ the map induced by $F^{*} \to G_{2}$, $a \mapsto \text{diag}(a, 1)$, and $\varphi : S_{4} \to H_{3}(G_{2}, A)$ the composite map

$$S_{4} \xrightarrow{\phi} H_{3}(F^{*} \times F^{*}, A) \xrightarrow{\text{inc}} H_{2}(G_{2}, A).$$

Consider the decomposition $H_{3}(G_{2}, A) = H_{2}(G_{1}, A) \oplus K_{2}^{M}(F) \otimes A$ [3, Prop. A. 11, p. 67]. Then $S_{1} = S_{1}' \oplus S_{1}''$, where $S_{1}' = H_{1}(F^{*}, A) \otimes H_{2}(G_{1}, A)$ and $S_{1}'' = H_{1}(F^{*}, A) \otimes K_{2}^{M}(F) \otimes A$. Define $\varphi : S_{1}' \to H_{3}(G_{2}, A)$ the shuffle product and let $\varphi : S_{1}'' \to H_{3}(G_{2}, A)$ be the composite map

$$H_{1}(F^{*}, A) \otimes K_{2}^{M}(F) \otimes A \xrightarrow{f} H_{1}(F^{*}, A) \otimes H_{2}(G_{2}, A) \xrightarrow{g} H_{3}(F^{*} \times G_{2}, A) \xrightarrow{h} H_{3}(G_{2}, A),$$

where $f$ is induced by

$$K_{2}^{M}(F) \otimes A \to H_{2}(G_{2}, A),$$

$$\{a, b\} \mapsto \frac{1}{2}\text{c}(\text{diag}(a, 1), \text{diag}(b, b^{-1})).$$
are the shuffle product and \( h \) is induced by the map \( F^* \times G_2 \to G_2 \), \( \text{diag}(a, A) \mapsto aA \). By proposition 4.1 we have the canonical decomposition 
\[
H_3(F^* \times G_1, A) = \bigoplus_{i=0}^{8} T_i,
\]
where
\[
\begin{align*}
T_0 &= H_3(G_1, A), \\
T_1 &= \bigoplus_{i=1}^{3} H_i(F^*_1, A) \otimes H_{3-i}(G_1, A), \\
T_2 &= \bigoplus_{i=1}^{3} H_i(F^*_2, A) \otimes H_{3-i}(G_1, A), \\
T_3 &= H_1(F^*_1, A) \otimes H_1(F^*_2, A) \otimes H_1(G_1, A), \\
T_4 &= \text{Tor}_1^A(H_1(F^*_1, A), H_1(F^*_2, A)), \\
T_5 &= \text{Tor}_1^A(H_1(F^*_1, A), H_1(G_1, A)), \\
T_6 &= \text{Tor}_1^A(H_1(F^*_2, A), H_1(G_1, A)), \\
T_7 &= H_1(F^*_1, A) \otimes H_2(F^*_2, A), \\
T_8 &= H_2(F^*_1, A) \otimes H_1(F^*_2, A).
\end{align*}
\]

We know that \( \bar{d}_{1,3} = \sigma_1 - \sigma_2 \), where \( \sigma_i = H_3(\alpha_{i,2}) \). It is not difficult to see that \( \bar{d}_{1,3}(T_0 \oplus T_1 \oplus T_2 \oplus T_7 \oplus T_8) \subseteq \ker(\varphi) \). Here one should use the isomorphism \( H_1(G_1, A) \cong H_1(G_2, A) \). Now \( (\sigma_1 - \sigma_2)(T_4) \subseteq S_4 \), \( \sigma_1(T_5) \subseteq S_0 \) and \( \sigma_2(T_5) \subseteq S_1 \), \( \sigma_1(T_6) \subseteq S_4 \) and \( \sigma_2(T_6) \subseteq S_0 \). With this description one can see that \( \bar{d}_{1,3}(T_4 \oplus T_5 \oplus T_6) \subseteq \ker(\varphi) \). To finish the proof of the claim we have to prove that \( \bar{d}_{1,3}(T_3) \subseteq \ker(\varphi) \). Let \( x = a \otimes b \otimes c \in T_3 \), then
\[
\begin{align*}
\bar{d}_{1,3}(x) &= -b \otimes c(\text{diag}(a, 1), \text{diag}(1, c)) - a \otimes c(\text{diag}(b, 1), \text{diag}(1, c)) \in S_1 \\
&= (-b \otimes c(a, c) - a \otimes c(b, c), b \otimes \{a, c\} + a \otimes \{b, c\}) \in S'_1 \oplus S''_1
\end{align*}
\]
So
\[
\begin{align*}
\varphi(\bar{d}_{1,3}(x)) &= -c(\text{diag}(b, 1), \text{diag}(1, a), \text{diag}(1, c)) \\
&- c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(1, c)) \\
&+ \frac{1}{2} c(\text{diag}(b, b), \text{diag}(a, 1), \text{diag}(c, c^{-1})) \\
&+ \frac{1}{2} c(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})).
\end{align*}
\]

Set \( p := \text{diag}(p, 1), \bar{p} := \text{diag}(1, q), p\bar{q}r := c(\text{diag}(p, 1), \text{diag}(1, q), \text{diag}(1, r)) \), etc. Conjugation by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) induces the equality \( p\bar{q}r = \bar{p}qr \) and it is easy to see that \( pqr = -qpr \) and \( p^{-1}qr = -\bar{p}qr \). With these notations and the
above relations we have
\[
\varphi(v_{1,3}^1) = -abc - \overline{abc} + \frac{1}{2}(bac + \overline{bac} + \overline{bac} + \overline{bac}) \\
+ \frac{1}{2}(abc + \overline{abc} + \overline{abc} + \overline{abc}) = 0.
\]
This proves that $H_3(G_2, A)$ is a summand of $\mathcal{E}^2_{0,3}$. 

\[\square\]

**Theorem 5.3.** We have a short exact sequence
\[
0 \to H_3(G_2, \mathbb{Z}[rac{1}{2}]) \to H_3(G_3, \mathbb{Z}[rac{1}{2}]) \to K_3^M(F) \otimes \mathbb{Z}[rac{1}{2}] \to 0
\]
which splits. The splitting map $K_3^M(F) \otimes \mathbb{Z}[rac{1}{2}] \to H_3(G_3, \mathbb{Z}[rac{1}{2}])$ is defined by
\[
\{a, b, c\} \mapsto [a, b, c] := \frac{1}{2}c(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(c, 1, c^{-1})).
\]

**Proof.** The exactness follows from 5.2 and [6, 3.25]. For the splitting map see proposition 6.3 and remark 3 in the next section. 

\[\square\]

6. **Indecomposable part of $K_3(F)$**

**Lemma 6.1.** Let $G$ be a group and let $g_1, g_2, h_1, \ldots, h_n \in G$ such that each pair commute. Let $C_G(\langle h_1, \ldots, h_n \rangle)$ be the subgroup of $G$ consists of all elements of $G$ that commute with all $h_i, i = 1, \ldots, n$. If $c(g_1, g_2) = 0$ in $H_2(C_G(\langle h_1, \ldots, h_n \rangle))$, then $c(g_1, g_2, h_1, \ldots, h_n) = 0$ in $H_{n+2}(G)$. 

**Proof.** The homomorphism $C_G(\langle h_1, \ldots, h_n \rangle) \times \langle h_1, \ldots, h_n \rangle \to G$ defined by $(g, h) \mapsto gh$ induces the map $H_2(C_G(\langle h_1, \ldots, h_n \rangle)) \otimes H_n(\langle h_1, \ldots, h_n \rangle) \to H_{n+2}(G)$. The claim follows from the fact that $c(g_1, g_2, h_1, \ldots, h_n)$ is the image of $c(g_1, g_2) \otimes c(h_1, \ldots, h_n)$ under this map.

\[\square\]

**Definition 6.2.** Let $A_{i,n} := \text{diag}(a_i, \ldots, a_i, a_i^{-1}, I_{n-i}) \in G_n$. We define $[a_1, \ldots, a_n] := c(A_{1,n}, \ldots, A_{n,n}) \in H_n(G_n)$. 

**Proposition 6.3.** (i) The map $\nu_n : K_3(F) \to H_n(G_n)$ defined by $\{a_1, \ldots, a_n\} \mapsto [a_1, \ldots, a_n]$ is a homomorphism of groups.

(ii) Let $\kappa_n : H_n(G_n) \to K_3(F)$ be the map defined by Suslin. Then the composite map $\kappa_n \circ \nu_n$ coincides with the multiplication by $(-1)(n-1)(n-1)!$. 

**Proof.** (i) The map $K_2^M(F) \to H_2(G_2)$ is defined by $\{a, b\} \mapsto [a, b]$ [3, A. 11]. So by lemma 6.1
\[
[a_1, 1 - a_1, a_3, \ldots, a_n] = 0.
\]
To complete the proof of (i) it is sufficient to prove that
\[
[a_1, \ldots, a_{n-2}, a_{n-1}, a_n] = -[a_1, \ldots, a_{n-2}, a_n, a_{n-1}].
\]
This can be done in the following way;

\[ [a_1, \ldots, a_{n-2}, a_{n-1}, a_n] = \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, A_{n-1,n}, A_{n,n}) \]
\[ = \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(a_{n-1}I_{n-2}, a_{n-1}^{-1}, a_{n-1}^{-1}), A_{n,n}) \]
\[ + \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(I_{n-2}, a_{n-1}^{-1}, a_{n-1}^{-1}), A_{n,n}) \]
\[ = \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(a_{n-1}I_{n-2}, a_{n-1}^{-1}, a_{n-1}^{-1}), \text{diag}(a_nI_{n-2}, a_n^{-1}, a_n^{-1}), 1) \]
\[ + \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(a_{n-1}I_{n-2}, a_{n-1}^{-1}, a_{n-1}^{-1}), \text{diag}(I_{n-2}, a_n^{-1}, a_n^{-1})) \]
\[ + \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(I_{n-2}, a_n^{-1}, a_n^{-1}), A_{n,n}) \]
\[ = -[a_1, \ldots, a_{n-2}, a_{n-1}, a_n] \]
\[ + \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(I_{n-2}, a_{n-1}^{-1}, a_{n-1}^{-1}), \text{diag}(I_{n-2}, a_n^{-1}, a_n^{-1})) \]
\[ + \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(I_{n-2}, a_n^{-1}, a_n^{-1}), \text{diag}(I_{n-2}, a_n^{-1}, a_n^{-1})) \]
\[ + \mathbf{c}(A_{1,n}, \ldots, A_{n-2,n}, \text{diag}(I_{n-2}, a_n^{-1}, a_n^{-1}), \text{diag}(a_nI_{n-2}, 1, 1)) \]
\[ = -[a_1, \ldots, a_n, a_{n-1}] \]

(ii) Let \( \tau_n \) be the composite map \( K_n^M(F) \to K_n(F) \xrightarrow{h} H_n(G_n) \). Then \( \kappa_n \circ \tau_n \) coincides with the multiplication by \( (-1)^{(n-1)(n-1)!} \). [8, section 4]. It is well known that the composite map \( K_n^M(F) \xrightarrow{\tau} H_n(G_n) \to H_n(G_n)/H_n(G_{n-1}) \) is an isomorphism and it is defined by \( \{a_1, \ldots, a_n\} \mapsto (a_1 \cup \cdots \cup a_n) \mod H_n(G_{n-1}), \) where

\[ a_1 \cup a_2 \cup \cdots \cup a_n = \mathbf{c}(\text{diag}(a_1, I_{n-1}), \text{diag}(1, a_2, I_{n-2}), \ldots, \text{diag}(I_{n-1}, a_n)) \]

(See [6, Remark 3.27].) Also we know that \( \kappa_n \) factor as

\[ H_n(G_n) \to H_n(G_n)/H_n(G_{n-1}) \to K_n^M(F). \]

Our claim follows from the fact that \([a_1, \ldots, a_n] \mod H_n(G_{n-1}) = (-1)^{n-1}(n-1)! (a_1 \cup \cdots \cup a_n) \mod H_n(G_{n-1}) \].

\[ \square \]

**Remark 3.** It is easy to see that in \( H_3(G_3) \)

\[ \mathbf{c}(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(1, c^{-1}, c)) = 0. \]

Using this one can prove that

\[ [a, b, c] = \mathbf{c}(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(c, 1, c^{-1})). \]

From this one can deduce that \([a, b, c] = -[c, b, a] \).
Lemma 6.4. (i) We have the following isomorphisms
\[ H_i(SL(F)) \cong H_0(F^*, H_i(SL_n(F))) \text{ for } \ n \geq i, \]
\[ H_3(G_3) \cong H_0(F^*, H_3(SL_3(F))) \oplus K_2(F) \otimes F^* \oplus H_3(F^*), \]
\[ H_3(G_2, Z[\frac{1}{2}]) \cong H_0(F^*, H_3(SL_2(F), Z[\frac{1}{2}])) \oplus K_2(F) \otimes F^* \otimes Z[\frac{1}{2}] \oplus H_3(F^*, Z[\frac{1}{2}]). \]

(ii) Let \( H_3(\text{inc}) : H_3(G_2, Z[\frac{1}{2}]) \to H_3(G_3, Z[\frac{1}{2}]) \). Then on the summands
\[ H_3(\text{inc}) = \begin{pmatrix}
\text{inc}_* & \beta & 0 \\
0 & 2, \text{id} & 0 \\
0 & 0 & \text{id}
\end{pmatrix}, \]
where \( \beta : K_2(F) \otimes F^* \otimes Z[\frac{1}{2}] \to H_0(F^*, H_3(SL_3(F), Z[\frac{1}{2}])) \) is induced by
\[ \{a, b\} \otimes c \mapsto c(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(c, 1, c^{-1})). \]

Proof. The part (i) of this lemma is rather well known (see [7, 2.7], [9, p. 233], [2, Rem. 1.2.8]). We will include the proofs to clarify the proof of (ii).

For the first isomorphism see [7, 2.7, p. 284]. Each element \( M \in G \) can be written as \( M = \text{diag}(\det(M)^{-1}, M), \text{diag}(\det(M), 1) \). This induces the homotopy equivalence \( BG^+ \cong BSL(F)^+ \times BF^+ \) [9, Lemma 5.3]. The second isomorphism in (i) follows from applying the Künneth theorem to \( BG^+ \), the first isomorphism and the stability theorem \( H_3(G_3) \cong H_3(G) \) [8, Thm. 3.4]. The inclusions \( H_0(F^*, H_3(SL_3(F))) \to H_3(G_3) \) and \( H_3(F^*) \to H_3(G_3) \) are induced by the maps \( SL_3(F) \to G_3, M \mapsto M, \) and \( F^* \to G_3, a \mapsto \text{diag}(a, 1, 1) \), respectively. The inclusion \( K_2(F) \otimes F^* \to H_3(G_3) \) is defined by
\[ \{a, b\} \otimes c \mapsto c(\text{diag}(a, 1, 1), \text{diag}(b, b^{-1}, 1), \text{diag}(1, 1, c)). \]

Now we prove the last isomorphism in (i). Set \( A := Z[\frac{1}{2}] \). From the map \( \gamma : SL_2(F) \times F^* \to G_2, (M, a) \mapsto a M \), we obtain two short exact sequences
\[ 1 \to \mu_2, F \to SL_2(F) \times F^* \to \text{im(}\gamma\text{)} \to 1, \]
\[ 1 \to \text{im}(\gamma) \to G_2 \to F^*/F^2 \to 1. \]

Writing the Lyndon-Hochschild-Serre spectral sequence of the above exact sequences and carrying out not difficult analysis, one gets
\[ H_3(\text{im}(\gamma), A) \cong H_3(SL_2(F) \times F^*, A), \]  
\[ H_0(F^*/F^2, H_3(\text{im}(\gamma), A)) \cong H_3(G_2, A). \]

(2) The action of \( F^2 \) on \( H_3(\text{im}(\gamma), A) \) is trivial because for every \( M \in \text{im}(\gamma) \),
\[ \text{diag}(a^2, 1).M.\text{diag}(a^{-2}, 1) = \text{diag}(a, a).\text{diag}(a, a^{-1}).M, \]
\[ \text{diag}(a^{-1}, a).\text{diag}(a^{-1}, a^{-1}), \]
so from (3) we obtain
\[ H_0(F^*, H_3(\text{im}(\gamma), A)) \cong H_3(G_2, A). \]  
(4)
Relations (2) and (4) imply
\[ H_3(G_2, A) \simeq H_0(F^*, H_3(SL_2(F) \times F^*, A)). \]

Now applying the Künneth theorem we get the isomorphism that we are looking for. The inclusions \( H_0(F^*, H_3(SL_2(F), A)) \to H_3(G_2, A) \) and \( H_3(F^*, A) \to H_3(G_2, A) \) are defined in natural way. (See the proof of the second isomorphism.) The inclusion \( K_2(F) \otimes F^* \otimes A \to H_3(G_2, A) \) is defined by
\[ \{a, b\} \otimes c \mapsto \mathbf{c}(\text{diag}(a, 1), \text{diag}(b, b^{-1}), \text{diag}(c, c)). \]

Using remark 3
\[ H_3(\text{inc})(\{a, b\} \otimes c) = \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(b, b^{-1}, 1), \text{diag}(c, c, 1)) \]
\[ = \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(b, b^{-1}, 1), \text{diag}(c, c, c^{-2})) \]
\[ + \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(b, b^{-1}, 1), \text{diag}(1, 1, c^2)) \]
\[ = \mathbf{c}(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(c, 1, c^{-1})) \]
\[ + \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(b, b^{-1}, 1), \text{diag}(1, 1, c^2)). \]

Therefore on the summands
\[ H_3(\text{inc})(0, \{a, b\} \otimes c, 0) = ([a, b, c], 2\{a, b\} \otimes c, 0). \]

\[ \square \]

**Corollary 6.5.** There is an exact sequence
\[ 0 \to H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])) \to \]
\[ H_0(F^*, H_3(SL_3(F), \mathbb{Z}[\frac{1}{2}])) \to K_3^M(F) \otimes \mathbb{Z}[\frac{1}{2}] \to 0 \]
which splits. The splitting map is induced by \( \{a, b, c\} \mapsto \frac{1}{2}[a, b, c]. \)

**Proof.** The proof follows from 6.4, 6.3 and 5.3. \[ \square \]

Let \( K^M_3(F) \to K_3(F) \) be the natural map from the Milnor \( K \)-group to the Quillen \( K \)-group. Define \( K_3(F)_{\text{ind}} := \text{coker}(K^M_3(F) \to K_3(F)). \) This group is called the indecomposable part of the Quillen \( K_3 \)-group.

**Theorem 6.6.** We have an isomorphism
\[ K_3(F)_{\text{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \simeq H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])). \]

**Proof.** Suslin constructed a map \( \varphi : K_3(F) \to K^M_3(F) \) such that \( K^M_3(F) \xrightarrow{\psi} K_3(F) \xrightarrow{\varphi} K^M_3(F) \) coincides with the multiplication by 2. This implies that
\[ 0 \to K^M_3(F) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\psi} K_3(F) \otimes \mathbb{Z}[\frac{1}{2}] \to K_3(F)_{\text{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \to 0. \]
is exact and splits. Set \( L_i = H_0(F^*, H_3(SL_i(F), \mathbb{Z}[^1_2])) \) for \( i = 2, 3 \). We have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_3(F)_{\text{ind}} \otimes \mathbb{Z}[^1_2] \\
\downarrow f & & \downarrow g \\
0 & \longrightarrow & K_3(F) \otimes \mathbb{Z}[^1_2] \\
\end{array}
\longrightarrow K_3^M(F) \otimes \mathbb{Z}[^1_2] \longrightarrow 0
\]

where \( g \) is the map

\[
K_3(F) \otimes \mathbb{Z}[^1_2] \cong H_3(SL_i(F), \mathbb{Z}[^1_2]) \cong H_0(F^*, H_3(SL_3(F), \mathbb{Z}[^1_2]))
\]

(see [7, Prop. 2.5]) and \( f \) is induced by the commutativity of the right part of the diagram. The results follows from the Five lemma.

\[\square\]

Remark 4. Theorem 6.6 generalizes theorem [7, Thm. 4.1], where three torsion was not treated.

7. Homology of \( GL_n(F) \)

Let \( k \) be a field and \( C_i(F^n) := C_i(F^n) \otimes k \). Consider the following commutative diagram of two complexes

\[
\begin{array}{ccc}
0 & \leftarrow & C_0'(F^n) \\
\downarrow & & \downarrow \\
0 & \leftarrow & C_1'(F^n) \\
\downarrow & & \downarrow \\
& \leftarrow & C_2'(F^n) \\
\downarrow & & \downarrow \\
& & \cdots
\end{array}
\]

where the first vertical map is zero and the other vertical maps are just identity maps. This gives a map of the first quadrant spectral sequences

\[
E_{pq}^1(n) \otimes k \rightarrow \mathcal{E}_{pq}^1(n),
\]

where \( \mathcal{E}_{pq}^1(n) \Rightarrow H_{p+q-1}(G_n, k) \) with \( \mathcal{E}^1 \)-terms

\[
\mathcal{E}_{pq}^1(n) = \begin{cases} 
E_{pq}^1(n) \otimes k & \text{if } p \geq 1 \\
0 & \text{if } p = 0
\end{cases}
\]

and differentials \( \partial_{pq}^1(n) \) = \( \begin{cases} 
d_{pq}^1(n) \otimes \text{id}_k & \text{if } p \geq 2 \\
0 & \text{if } p = 1
\end{cases} \). It is not difficult to see that \( E_{pq}^\infty \otimes k = \mathcal{E}_{pq}^\infty \) if \( p \neq 1, q \leq n \) and \( p + q \leq n + 1 \). Hence \( \mathcal{E}_{pq}^\infty = 0 \) if \( p \neq 1, q \leq n \) and \( p + q \leq n + 1 \).

We look at the second spectral sequence in a different way. The complex

\[
0 \leftarrow C_0'(F^n) \leftarrow C_1'(F^n) \leftarrow \cdots \leftarrow C_i'(F^n) \leftarrow \cdots
\]

induces a first quadrant spectral sequence \( \mathcal{E}_{pq}^1(n) \Rightarrow H_{p+q}(G_n, k) \), where \( \mathcal{E}_{pq}^1(n) = \mathcal{E}_{p+1,q}^1(n) \) and \( \partial_{pq}^1(n) = \partial_{p+1,q}^1(n) \). Thus \( \mathcal{E}_{pq}^\infty(n) = 0 \) if \( p \geq 1, q \leq n - 1 \) and \( p + q \leq n \).
Proposition 7.1. Let \( n \geq 3 \) and let \( k \) be a field such that \((n-1)! \in k^*\). Let the complex

\[
H_n(F^{*2} \times G_{n-2}, k) \xrightarrow{\beta^{(n)}_2} H_n(F^* \times G_{n-1}, k) \xrightarrow{\beta^{(n)}_1} H_n(G_{n}, k) \to 0
\]

be exact, where \( \beta^{(n)}_2 = H_n(\alpha_1; 2) - H_n(\alpha_2; 2) \) and \( \beta^{(n)}_1 = H_n(\text{inc}) \). If the map \( H_n(\text{inc}) : H_n(G_{n-1}, k) \to H_n(G_{m}, k) \) is injective for \( m = n-1, n-2 \), then \( H_n(\text{inc}) : H_n(G_{n-1}, k) \to H_n(G_{n}, k) \) is injective.

Proof. The exactness of (5) shows that the differentials \( \partial^{r}_{r,n-r+1}(n) : E^{r}_{r,n-r+1}(n) \to E^{r}_{0,n}(n) \) are zero for \( r \geq 2 \). This proves that \( E^{\infty}_{0,n}(n) \cong E^{2}_{0,n}(n) \). To complete the proof it is sufficient to prove that the group \( H_n(G_{n-1}, k) \) is a summand of \( E^{\infty}_{0,n}(n) \). To prove this it is sufficient to define a map \( \varphi : H_n(F^* \times G_{n-1}, k) \to H_n(G_{n-1}, k) \) such that \( \partial^{1}_{1,n}(H_n(F^{*2} \times G_{n-2}, k)) \subseteq \text{ker}(\varphi) \). Consider the decompositions \( H_n(F^* \times G_{n-1}, k) = \bigoplus_{i=0}^{n} S_i \), where \( S_i = H_i(F^*, k) \otimes H_{n-i}(G_{n-1}, k) \). For \( 2 \leq i \leq n \), the stability theorem gives the isomorphisms \( H_i(F^*, k) \otimes H_{n-i}(G_{n-2}, k) \cong S_i \). Define \( \varphi : S_0 \to H_n(G_{n-1}, k) \) the identity map and for \( 2 \leq i \leq n \), \( \varphi : S_i \simeq H_i(F^*, k) \otimes H_{n-i}(G_{n-2}, k) \to H_n(G_{n-1}, k) \) the shuffle product. To complete the definition of \( \varphi \) we must define it on \( S_1 \). By a theorem of Suslin [8, 34] and the assumption, we have the decomposition \( H_{n-1}(G_{n-1}, k) \cong H_{n-1}(G_{n-2}, k) \oplus K_{n-1}^M(F) \otimes k \). So \( S_1 \simeq H_1(F^*, k) \otimes H_{n-1}(G_{n-2}, k) \oplus H_1(F^*, k) \otimes K_{n-1}^M(F) \otimes k \). Now define \( \varphi : H_1(F^*, k) \otimes H_{n-1}(G_{n-2}, k) \to H_n(G_{n-1}, k) \) the shuffle product and \( \varphi : H_1(F^*, k) \otimes K_{n-1}^M(F) \to H_n(G_{n-1}, k) \) the composite map

\[
H_1(F^*, k) \otimes K_{n-1}^M(F) \otimes k \xrightarrow{f} H_1(F^*, k) \otimes H_{n-1}(G_{n-1}, k) \xrightarrow{g} H_n(F^* \times G_{n-1}, k) \xrightarrow{h} H_n(G_{n-1}, k),
\]

where \( f = \frac{1}{n-1}(\text{id} \otimes \kappa_{n-1}) \), \( g \) is the shuffle product and \( h \) is induced by the map \( F^* \times G_{n-1} \to G_{n-1} \), \( \text{diag}(a, A) \mapsto aA \). By the K"unneth theorem we have the decomposition

\[
T_0 = H_n(G_{n-2}, k),
\]

\[
T_1 = \bigoplus_{i=1}^{n} H_i(F^*, k) \otimes H_{n-i}(G_{n-2}, k),
\]

\[
T_2 = \bigoplus_{i=1}^{n} H_i(F^*, k) \otimes H_{n-i}(G_{n-2}, k),
\]

\[
T_3 = H_1(F^*, k) \otimes H_1(F^*, k) \otimes H_{n-2}(G_{n-2}, k),
\]

\[
T_4 = \bigoplus_{i+j \geq 3} H_i(F^*, k) \otimes H_j(F^*, k) \otimes H_{n-i-j}(G_{n-2}, k).
\]

By lemma 6.3, \( T_3 = T_3^* \otimes T_3^* \), where \( T_3^* = H_1(F^*, k) \otimes H_1(F^*, k) \otimes H_{n-2}(G_{n-3}, k) \) and \( T_3^* = H_1(F^*, k) \otimes H_1(F^*, k) \otimes K_{n-2}^M(F) \otimes k \). It is not
difficult to see that \( \partial^1_{1,n}(T_0 \oplus T_1 \oplus T_2 \oplus T_3 \oplus T_4) \subseteq \ker(\varphi) \). Here one should use the stability theorem. To prove \( \partial^1_{1,n}(T_3^n) \subseteq \ker(\varphi) \) we apply 6.3;

\[
\partial^1_{1,n} \left( a \otimes b \otimes \{c_1, \ldots, c_{n-2}\} \right) \\
= \frac{(-1)^{n-3}}{(n-3)!} \left( b \otimes c(\text{diag}(a, I_{n-2}), \text{diag}(1, C_{1,n-2}), \ldots, \text{diag}(1, C_{n-2,n-2})) \\
+ a \otimes c(\text{diag}(b, I_{n-2}), \text{diag}(1, C_{1,n-2}), \ldots, \text{diag}(1, C_{n-2,n-2})) \right) \\
= \frac{1}{(n-2)!} \left( b \otimes [c_1, \ldots, c_{n-2}, a] + a \otimes [c_1, \ldots, c_{n-2}, b]; \\
- b \otimes c(\text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(a I_{n-2}, 1)) \\
- a \otimes c(\text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(b I_{n-2}, 1)) \right).
\]

Therefore \( \partial^1_{1,n} \left( a \otimes b \otimes \{c_1, \ldots, c_{n-2}\} \right) = (x_1, x_2) \in T_3^n \oplus T_3^n \), where

\[
x_1 = \frac{1}{(n-2)!} \left( b \otimes c(\text{diag}(C_{1,n-2}, \ldots, \text{diag}(C_{n-2,n-2}), \text{diag}(a I_{n-2}))) \\
+ a \otimes c(\text{diag}(C_{1,n-2}, \ldots, \text{diag}(C_{n-2,n-2}), \text{diag}(b I_{n-2}))) \right),
\]

\[
x_2 = (-1)^{n-2} \left( b \otimes \{c_1, \ldots, c_{n-2}, a\} + a \otimes \{c_1, \ldots, c_{n-2}, b\} \right).
\]

We have \( \phi(x_1) = -\frac{1}{(n-2)!}y \), where

\[
y = c(\text{diag}(b, I_{n-2}), \text{diag}(1, C_{1,n-2}), \ldots, \text{diag}(1, C_{n-2,n-2}), \text{diag}(1, a I_{n-2})) \\
+ c(\text{diag}(a, I_{n-2}), \text{diag}(1, C_{1,n-2}), \ldots, \text{diag}(1, C_{n-2,n-2}), \text{diag}(1, b I_{n-2}))
\]

and \( \phi(x_2) = \frac{(-1)^{n-2}}{n-1} \frac{(-1)^{n-2}}{(n-2)!} z = \frac{1}{(n-2)!} z \), where

\[
z = \\
c(\text{diag}(b I_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(a I_{n-2}, a^{-(n-2)})) \\
+ c(\text{diag}(a I_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(b I_{n-2}, b^{-(n-2)})) \\
= c(\text{diag}(b I_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(a I_{n-2}, a)) \\
+ c(\text{diag}(b I_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, a^{-(n-1)})) \\
+ c(\text{diag}(a I_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(b I_{n-2}, b)) \\
+ c(\text{diag}(a I_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, b^{-(n-1)})).
\]
Hence $\phi(x_2) = -\frac{1}{(n-2)!}z'$, where
\[
z' = c\left(\text{diag}(bI_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, a)\right) \\
+ c\left(\text{diag}(aI_{n-1}), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, b)\right) \\
= c\left(\text{diag}(bI_{n-2}, 1), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, a)\right) \\
+ c\left(\text{diag}(aI_{n-2}, b), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, b)\right) \\
+ c\left(\text{diag}(I_{n-2}, a), \text{diag}(C_{1,n-2}, 1), \ldots, \text{diag}(C_{n-2,n-2}, 1), \text{diag}(I_{n-2}, b)\right) \\
= -y.
\]

Therefore $\varphi(x_2) = -\frac{1}{(n-2)!}z' = -\frac{1}{(n-2)!}y = -\varphi(x_1)$. This completes the proof of the fact that $\varphi_1^n(\mathcal{H}(F^{*2} \times G_{n-2}, k)) \subseteq \ker(\varphi)$. 

So it is reasonable to conjecture

**Conjecture 7.2.** Let $n \geq 3$. Then the following complex is exact
\[
H_n(F^{*2} \times G_{n-2}, k) \overset{\varphi_1^n}{\longrightarrow} H_n(F^* \times G_{n-1}, k) \overset{\varphi_1^n}{\longrightarrow} H_n(G, k) \rightarrow 0.
\]

**Remark 5.** The surjectivity of $\varphi_1^n$ is already proven by Suslin [8].

**Remark 6.** All the results of this note is true if one replace the infinite field with a semi-local ring with infinite residue fields.

**References**


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