KUNNETH DECOMPOSITIONS FOR QUOTIENT VARIETIES

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Abstract. In this paper we discuss Küneth decompositions for finite quotients of several classes of smooth projective varieties. We establish the strong Küneth decomposition for finite quotients of projective smooth linear varieties and also Chow Küneth decomposition for certain finite quotients of abelian varieties.

1. Introduction

Strong Küneth decompositions are known to exist for linear varieties (cf. [8]), a class that includes projective spaces, flag schemes, toric schemes and spherical schemes (all over a field $k$). Previous work of the second author ([9], [10]) shows in particular that for varieties possessing a strong Küneth decomposition, the cycle map to étale cohomology is an isomorphism. Chow-Küneth decompositions, on the other hand, are conjectured (at least by optimists) to exist for all smooth projective varieties $X$; so far, they are known to exist (over $\mathbb{Q}$) for curves [14], surfaces [18], projective spaces [14], abelian varieties ([23], [3], [13]) certain types of threefolds [4], and other isolated examples. In each of these cases, though, one must construct the projectors explicitly and then check the various orthogonality conditions; this process depends heavily on particular properties of the variety under consideration. In this paper we extend the strong Küneth decomposition and the Chow-Küneth decomposition to varieties that are finite quotients of smooth projective varieties that admit the corresponding decomposition, working of course with rational coefficients throughout.

In the next section, we extend the strong Küneth decomposition to finite quotients of projective smooth linear varieties. The main result is that if $X$ is a variety possessing a strong Küneth decomposition and $f : X \to Y$ is a finite surjective map, then $Y$ possesses a strong Küneth decomposition which we may describe explicitly in terms of that for $X$. The methods used are all elementary and require little more than the definitions and basic properties of Chow groups. As an application, we describe a strong Küneth decomposition for symmetric products of projective spaces. We also discuss some formal consequences of such a strong Küneth decomposition: we show that a strong Küneth decomposition implies a Chow Küneth decomposition and that the rational higher Chow groups are determined by the rational higher Chow groups of the base field and the rational (ordinary) Chow groups of the given variety.

In the following section we establish Chow Küneth decomposition for finite quotients of abelian varieties for the action of a finite group so that the hypothesis in Theorem (3.2) are satisfied. The techniques are a modification of those of Deninger-Murre [3] and also [Be].

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This gives Chow-Kühneth decompositions for a number of interesting examples of nonabelian quotients of abelian varieties; examples of these may be found in [7], [16] some of which we briefly discuss. The existence of such a Chow-Kühneth decomposition is guaranteed by the work of Kimura [11] on finite-dimensionality of motives; however, his results do not yield explicit formulas for the projectors, which is part of our goal.

We discuss, in an appendix, certain formal consequences of the existence of strong Kühneth decompositions. This discussion is done in a somewhat more general setting so that it applies to other situations not considered in the body of the paper.

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Through the paper we will fix a field $k$ of arbitrary characteristic and restrict to the category of quasi-projective schemes over $k$.

1.1. Group Scheme Actions and Quotients. Following the treatment in [17], we review some definitions and results related to group scheme actions.

**Definition 1.1.** Let $G$ be a group scheme over a field $k$ with identity section $e : \text{Spec} \ k \to G$ and multiplication $m : G \times_k G \to G$.

An action of $G$ on a scheme $X$ is a morphism $\mu : G \times X \to X$ such that:

1. The composite

$$X \cong \text{Spec} \ k \times_k X \xrightarrow{e \times 1_X} G \times_k X \xrightarrow{\mu} X$$

is the identity map.

2. The diagram below commutes:

$$\begin{array}{ccc}
G \times_k G \times_k X & \xrightarrow{m \times 1_X} & G \times_k X \\
\downarrow{1_G \times \mu} & & \downarrow{\mu} \\
G \times_k X & \xrightarrow{\mu} & X
\end{array}$$

In the future, we will identify elements $g \in G$ with the morphism $\mu_g : X \to X$ defined by $\mu_g(x) = \mu(g, x)$ and (by abuse of notation) refer to this morphism simply as $g$.

If $G$ is a finite group (= a constant étale finite group scheme) over $k$ acting on a quasi-projective scheme $X$ (also over $k$), there exists a quasi-projective variety $Y$ together with a finite, surjective $G$-invariant morphism $f : X \to Y$ universal for $G$-invariant morphisms $X \to Z$. The scheme $Y$ is called the quotient of $X$ by $G$, and is typically denoted $Y = X/G$.

**Definition 1.2.** We say a scheme $X$ is pseudo-smooth if it is the quotient of a smooth scheme by the action of a finite group.
1.1.1. **Notation and terminology: review of correspondences.** In this section we define the category of rational correspondences and rational Chow motives for pseudo-smooth projective varieties.

Let $k$ be a field and $\mathcal{V}_k$ the category of schemes pseudo-smooth and projective over $k$. If $X, Y$ are objects of $\mathcal{V}_k$ and $X$ has pure dimension $d$, we define the group of degree $r$ correspondences from $X$ to $Y$ by $\text{Corr}^r(X, Y) = \text{CH}^{d+r}(X \times_k Y) \otimes \mathbb{Q}$, the group of co-dimension $d+r$ (rational) cycles on $X \times_k Y$ modulo rational equivalence. In general, let $X_1, \ldots, X_n$ be the irreducible components of $X$; we then define $\text{Corr}^r(X, Y) = \bigoplus_{i=1}^n \text{Corr}^r_i(X_i, Y)$. When $\alpha \in \text{Corr}^r(X, Y)$ and $\beta \in \text{Corr}^s(Y, Z)$, we define their composition $\beta \circ \alpha \in \text{Corr}^{r+s}(X, Z)$ by the formula

$$
\beta \circ \alpha = (p_{i3})_*(p^*_{12} \alpha \circ p^*_{23} \beta);
$$

here $p_{ij}$ represents projection of $X \times_k Y \times_k Z$ on the $i$th and $j$th factors.

One then constructs a new category $\mathcal{M}_k(\mathbb{Q})$, the category of (rational) Chow motives of pseudo-smooth projective varieties. The objects of $\mathcal{M}_k(\mathbb{Q})$ are pairs $(X, \pi)$, where $X$ is an object of $\mathcal{V}_k$ of dimension $d$ and $\pi \in \text{Corr}^0(X, X)$ is a projector; that is, an element satisfying $\pi \circ \pi = \pi$. For any two Chow motives $(X, \pi)$ and $(Y, \rho)$, one then defines

$$
\text{Hom}_{\mathcal{M}_k}((X, \pi), (Y, \rho)) = \bigoplus_j \rho \circ \text{Corr}^0(Y, X) \circ \pi.
$$

If $\Delta_X$ is the diagonal of $X \times_k X$ and $[\Delta_X]$ its class in $\text{CH}^*(X \times_k X) \otimes \mathbb{Q}$, a straightforward computation shows that $\Delta_X$ is a projector, and furthermore that $\Delta_X \circ \alpha = \alpha = \alpha \circ \Delta_X$ for any pseudo-smooth projective scheme $Y$ and $\alpha \in \text{Corr}^r(X, Y)$. Thus, there is a functor $h : \mathcal{V}_k^{opp} \rightarrow \mathcal{M}_k(\mathbb{Q})$ defined on objects by $h(X) = (X, \Delta_X)$ and on morphisms by $h(X \rightarrow Y) = \Gamma_f$, where $\Gamma_f \in \text{Hom}_{\mathcal{M}_k(\mathbb{Q})}(h(Y), h(X))$ is the class of the graph of $f$. Furthermore, letting $\coprod$ denote disjoint union (of schemes), one may define the sum $\oplus$ and product $\otimes$ of motives thus:

$$
(X, p) \oplus (Y, q) = (X \coprod Y, p \coprod q)
$$

$$
(X, p) \otimes (Y, q) = (X \times_k Y, p \times q)
$$

We denote by $\mathbb{I}$ the “trivial” motive $h(\text{Spec } k)$, a neutral element for $\otimes$, and by $\mathbb{L}$ the “Lefschetz motive” $(\mathbb{P}_k^1, \mathbb{P}_k^1 \times_k \{x\})$; here $x \in \mathbb{P}_k^1$ is any rational point. Finally, if $\alpha \in \text{Corr}^r(X, Y)$ is any correspondence, we define its “transpose” $^t \alpha = s^* (\alpha) \in \text{Corr}^r(Y, X)$, where $s : X \times_k Y \rightarrow Y \times_k X$ is the exchange of factors. For further discussion of motives, we refer the reader to [22]. Also see [6] Example (8.3.12) and Example (16.1.12) for discussion that shows one can in fact define a category of Chow motives for pseudo-smooth schemes as we have done. In fact it is possible to consider the above theory for all smooth Deligne-Mumford stacks over $k$; some of our results extend to this situation readily.
1.2. **Abelian Varieties.** In this section we establish notation and cite a rigidity property for abelian varieties necessary in the sequel. A comprehensive treatment of abelian varieties may be found in [17] or [15].

Let \( k \) be a field and \( A \) an abelian variety over \( k \). Following [15], we denote by \( m : A \times_k A \to A \) the morphism representing composition on \( (\text{the group scheme}) \ A \) and use additive notation for this (commutative) operation. For any \( a \in A(k) \), we denote by \( \tau_a : A \to A \) (translation by \( a \)) the map defined by \( \tau_a(x) = x + a \).

A morphism \( f : A \to B \) between abelian varieties is called a **homomorphism** if for every \( a, a' \in A \), \( f(a + a') = f(a) + f(a') \). When \( n \in \mathbb{Z} \) we define \( n : A \to A \) by \( n(a) = na \) and set \( A[n] = \text{Ker} (A \xrightarrow{n} A) \), the (group scheme of) \( n \)-torsion points on \( A \). For clarity of notation, we write \( \sigma \) instead of \( -1 \).

The following important result is a consequence of a general rigidity principle; see [15], Corollary 2.2 for details:

**Proposition 1.3.** Let \( h : A \to B \) be a morphism of abelian varieties. Then there exists a homomorphism \( h_0 : A \to B \) and an element \( a \in A(k) \) such that \( h = \tau_a \circ h_0 \).

We remark that \( h_0 \) and \( a \) are in fact unique. Indeed, one must have \( a = -h(0) \); uniqueness of \( h_0 \) then follows immediately.

Let \( \hat{A} \) be the dual abelian variety; we will denote by \( \mathcal{L} \) the Poincaré bundle and \( \ell \) its class in \( CH^1_k(A \times_k \hat{A}) \).

We conclude this section by recalling the definition of strong Küneth and Chow Küneth decompositions.

**Definition 1.4.** Let \( X \) be any scheme of pure dimension \( d \) over a field \( k \).

We say that \( X \) possesses a **strong Küneth decomposition** if there exist elements \( a_{i,j}, b_{i,j} \in CH^i_q(X) \) such that

\[
[\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}
\]

Now suppose \( X \) is pseudo-smooth (over \( k \)). We say that \( X \) has a **Chow-Küneth decomposition** if there exist elements \( \pi_0, \ldots, \pi_{2d} \in CH^d_q(X \times_k X) \) such that:

- \( [\Delta_X] = \sum_{i=0}^{2d} \pi_i \)
- For every \( i \), \( \pi_i \circ \pi_i = \pi_i \) and for all \( j \neq i \), \( \pi_i \circ \pi_j = 0 \). (Thus, \( \pi_0, \ldots, \pi_{2d} \) form a system of mutually orthogonal projectors).
- Let \( H \) be a Weil cohomology theory \( H^* \) (cf. [12]) and, for any \( k \)-scheme \( Y \), let \( cl_Y : CH^i_q(Y) \to H^i(Y) \) denote the cycle map. We require that \( cl_{X \times_k X}(\pi) = \Delta(i) \), where \( \Delta(i) \) is the co-dimension \( i \) Küneth component of the class of \( \Delta_X \) in \( H^*(X \times_k X) \). (We will show later that any Weil cohomology theory admits an extension to the category of pseudo-smooth schemes.)
Observe that if \( X \) is projective, \( X \) having a Chow-Künneth decomposition is equivalent to asserting that \( h(X) \cong \bigoplus_{i=0}^{2d} h^i(X) \) where \( h^i(X) \) is the motive \( (X, \pi_i) \).

2. The strong Künneth decomposition for finite quotients

Now suppose \( X \) is a pseudo-smooth, projective, equidimensional scheme over a field \( k \) and \( G \) a finite group of automorphisms of \( X \). As in [17], we may form the quotient variety \( Y = X/G \) and ask whether an explicit strong Künneth decomposition for \( X \) may be used to construct a strong Künneth decomposition for \( Y \). We answer this question in the affirmative below.

First we consider an elementary calculation showing that strong Künneth decompositions are preserved under finite maps.

**Proposition 2.1.** Let \( X \) and \( Y \) be pseudo-smooth proper varieties and \( f : X \rightarrow Y \) a finite surjective map. If \( X \) has a strong Künneth decomposition, then \( Y \) also has a strong Künneth decomposition.

**Proof.**

Let \( d = \dim X, \) \( m = \deg f \). The hypothesis that \( X \) has a strong Künneth decomposition allows us to write

\[
[\Delta_X] = \sum_i \sum_j a_{i,j} \times b_{d-i,j}
\]

where as before \( a_{i,j}b_{i,j} \in CH^*_Q(X) \). Furthermore, \((f \times f)_*[\Delta_X] = m[\Delta_Y] \), so it suffices to prove that \((f \times f)_*(a_{i,j} \times b_{d-i,j}) = f_*(a_{i,j}) \times f_*(b_{d-i,j})\). This is accomplished by the next lemma, whose proof is immediate.

**Lemma 2.2.** Let \( f : X \rightarrow Y \) be a morphism of pseudo-smooth varieties

1. If \( f \) is proper, then for all \( \alpha, \beta \in CH^*_Q(X) \), \( f_*(\alpha \times \beta) = f_*(\alpha) \times f_*(\beta) \).
2. For all \( \gamma, \delta \in CH^*_Q(Y) \), \( f^*(\gamma \times \delta) = f^*(\gamma) \times f^*(\delta) \).

We note the following as a special case:

**Corollary 2.3.** Let \( X \) be a pseudo-smooth quasi-projective variety, \( G \) a finite group of automorphisms of \( X \). If \( X \) possesses a strong Künneth decomposition, so does \( Y = X/G \).

The utility of the previous statements becomes evident from the following easy result:

**Proposition 2.4.** Let \( X \) be a pseudo-smooth projective variety possessing a strong Künneth decomposition. Then \( X \) has a Chow-Künneth decomposition.

**Proof.**

Suppose \([\Delta_X] = \sum_{i=0}^{d} \sum_{j} a_{i,j} \times b_{d-i,j} \), where \( a_{i,j}b_{i} \in CH^*_Q(X) \). For \( 0 \leq r \leq d \), set \( \pi_r = \sum_j a_{r,j} \times b_{d-r,j} \) and for \( d+1 \leq r \leq 2d \), set \( \pi_r = 0 \). Then \([\Delta_X] = \sum_{r=0}^{2d} \pi_r \). We decorate \( p \) with subscripts and superscripts to denote the various projectors from and to subfactors of \( X \times_k X \times_k X \); for example, \( p^1_{13} : X \times_k X \times_k X \rightarrow X \times_k X \) sends \((x, y, z)\) to \((x, z)\), etc. Finally, we let \( \sigma : X \rightarrow \text{Spec } k \) and \( \tau : X \times_k X \rightarrow \text{Spec } k \) denote the respective structure maps. We claim that \( \pi_r \circ \pi_s = 0 \) when \( r \neq s \) and \( \pi_r \circ \pi_r = \pi_r \) for all \( r, 0 \leq r \leq 2d \). The first equality is a
consequence of the following more general fact proved in Lemma 2.5 (below). To conclude the proof of Proposition 2.4, we calculate:

\[ \pi_r \circ \pi_r = (\Delta_X - \sum_{s \neq r} \pi_s) \circ \pi_r = \Delta_X \circ \pi_r = \pi_r \]

**Lemma 2.5.** With notation as above, suppose \( a_r \in CH^r(X), b_{d-r} \in CH^{d-r}(X), a_s \in CH^s(X), b_s \in CH^{d-s}(X), \) and set \( \gamma_r = a_r \times b_{d-r}, \gamma_s = a_s \times b_{d-s}. \) If \( r \neq s, \) then \( \gamma_s \circ \gamma_r = 0. \)

**Proof.**

\[
\gamma_s \circ \gamma_r = p_{13}^{123} \cdot (p_{12}^{123} \cdot \gamma_r \cdot p_{23}^{123} \cdot \gamma_s) \\
= \sum_j p_{13}^{123} \cdot (p_{12}^{123} \cdot (p_1 a_r \cdot p_2^{123} b_{d-r}) \cdot p_{23}^{123} \cdot (p_2^{23} a_s \cdot p_3^{23} b_{d-s})) \\
= p_{13}^{123} \cdot (p_{12}^{123} \cdot (p_1^{13} a_r \cdot p_3^{13} b_{d-s}) \cdot p_2^{123} (a_s \cdot b_{d-r})) \\
= p_{13}^{13} a_r \cdot p_3^{13} b_{d-s} \cdot p_{13}^{123} p_2^{123} (a_s \cdot b_{d-r}) \\
= p_{13}^{13} a_r \cdot p_3^{13} b_{d-s} \cdot \tau^* \sigma_s (a_s \cdot b_{d-r})
\]

Note that \( \sigma_s (a_s \cdot b_{d-r}) \in CH^{d-r}(X), \) so if \( r \neq s, \) then \( \sigma_s (a_s \cdot b_{d-r}) = 0, \) and hence \( \gamma_s \circ \gamma_r = 0. \) This concludes the proof of Lemma 2.5.

As an application, we compute the strong Künneth decomposition for the \( n \)th symmetric product of projective space \( \mathbf{P}_k^m. \) Let \( \ell \in CH^1(Q_k^m) \) be the class of a generic hyperplane in \( \mathbf{P}_k^m. \) It is well-known (cf. [14], p. 455) that \( \mathbf{P}_k^m \) has a strong Künneth decomposition:

\[ \Delta_{\mathbf{P}_k^m} = \sum_{i=0}^{m} \ell^i \times \ell^{m-i} \]

Let \( X = (\mathbf{P}_k^m)^n. \) By the Künneth formula for motives, we have

\[ \Delta_X = \sum_{0 \leq i_1, \ldots, i_n \leq m} f_{i_1, \ldots, i_n} \]

where \( f_{i_1, \ldots, i_n} = \ell^{i_1} \times \cdots \times \ell^{i_n} \times \ell^{m-i_1} \times \cdots \ell^{m-i_n} \in CH_{Q_k}^m(X \times_k X). \)

Now consider the action of the symmetric group on \( n \) letters (denoted \( S_n \)) on \( X = (\mathbf{P}_k^m)^n \) by interchanging of factors. Let \( Y = X/S_n \) and \( q : X \longrightarrow Y \) the quotient map. Note also that for any \( \sigma \in S_n, \) \((q \times q)^* f_{i_1, \ldots, i_n} = (q \times q)^* f_{\sigma(i_1), \ldots, \sigma(i_n)}. \)

Applying \((q \times q)^* \) to the strong Künneth decomposition for \( \Delta_X \) given above, and noting that \( \deg q = n! \), we obtain

\[ (n!)\Delta_Y = \sum_{0 \leq i_1, \ldots, i_n \leq m} (q \times q)^* f_{i_1, \ldots, i_n} \]

\[ = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq m} \sum_{\sigma \in S_n} (q \times q)^* f_{\sigma(i_1), \ldots, \sigma(i_n)} \]

\[ = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq m} n! (q \times q)^* f_{i_1, \ldots, i_n} \]
Now let $\bar{t}^i = q_*(t^i)$. Then

\begin{equation}
\Delta_Y = \sum_{0 \leq i_1 \leq \ldots \leq i_n \leq m} (q \times q)_* f_{i_1, \ldots, i_n}
= \sum_{0 \leq i_1 \leq \ldots \leq i_n \leq m} \bar{t}^{i} \times \ldots \times \bar{t}^{i_n} \times \bar{t}^{m-i_1} \times \ldots \times \bar{t}^{m-i_n}
\end{equation}

giving a strong Künneth decomposition for $Y$.

**Corollary 2.6.** Let $Y$ denote the $n$-th symmetric product of $\mathbb{P}^n_k$. Then

$$CH^*(Y, Q, r) \cong CH^*(Y, Q, 0) \otimes CH^*(Spec \ k, Q, r)$$

where $CH^*(Z, Q, r) = \pi_r(z^*(Z, \cdot) \otimes Q)$ and $z^*(Z, \cdot)$ denotes the higher cycle complex of the scheme $Z$.

**Proof.** This follows readily from the above strong Künneth decomposition for the class $\Delta_Y$ and Theorem 4.1.

### 3. CHOW-KÜNNETH DECOMPOSITION FOR QUOTIENTS OF ABELIAN VARIETIES

Our goal in this section is to exhibit an explicit Chow-Künneth decomposition for the quotient of an abelian variety $A$ by the action of a finite group $G$, assuming only that $g(0)$ is a torsion point for each $g \in G$. As before, the quotient $A/G$ may be singular. We rely on the following result, originally due to Shermenev [23], but later proved in a somewhat more functorial setting by Deninger and Murre ([3], Theorem 3.1); in this latter source the result is proved more generally for abelian schemes over a smooth quasi-projective base:

**Theorem 3.1.** Let $A$ be an abelian variety of dimension $d$ over a field $k$. Then there exists a Chow-Künneth decomposition for $A$:

$$\Delta_A = \sum_{i=0}^{2d} \pi_i$$

Since we need to make explicit use of the projectors $\pi_i$, we will presently review their construction. First, consider $A \times_k \hat{A}$ as an abelian $A$-scheme via projection on the first factor; with respect to this structure, the dual abelian scheme is $A \times_k \hat{A}$. Consider then the Fourier transform (cf. [3], 2.12, [13], 1.3):

$$F_{CH} : CH^*_k(A \times_k A) \to CH^*_k(A \times_k \hat{A})$$

defined by $F_{CH}(\alpha) = p_{13,*}(p_{12,*} \alpha \cdot F)$, where

$$F = \sum_{i=0}^{\infty} \frac{1 \times \bar{t}^{i}}{i!} \in CH^*_k(A \times_k A \times_k \hat{A})$$

and the various $p_{ij}$ represent projections from $A \times_k A \times_k \hat{A}$ on the $i$th and $j$th factor. Note that the sum defining $F$ is actually finite.
Dualizing this construction, we may define
\[
\hat{F}_{CH} : CH^*_{\mathbb{Q}}(A \times_k \hat{A}) \to CH^*_{\mathbb{Q}}(A \times_k A)
\]
by \(\hat{F}_{CH}(\gamma) = q_{13*}(q_{12}^* \gamma \cdot \hat{F})\), where
\[
\hat{F} = \sum_{i=0}^{\infty} \frac{1 \times t^i}{i!} \in CH^*_{\mathbb{Q}}(A \times_k \hat{A} \times_k A)
\]
and \(q_{ij}\) represent the various projections from \(A \times_k \hat{A} \times_k A\). By switching the last two factors and changing notation appropriately, we see that in fact
\[
\hat{F}_{CH}(\gamma) = p_{12*}(p_{13}^* \gamma \cdot F).
\]

An argument involving the theorem of the square (cf. \([3]\), Cor. 2.22, also \([1]\), Prop. 3) then shows that \(\hat{F}_{CH}(F_{CH}(\alpha)) = (-1)^d \sigma^* \alpha\) for all \(\alpha \in CH^*(A \times_k A)\), and similarly for the other composition.

Observe that \(\Delta_A \in CH^d(A \times_k A)\), and write \(F_{CH}(\Delta_A) = \sum_{i=0}^{2d} \beta_i\), where \(\beta_i \in CH^i_{\mathbb{Q}}(A \times_k \hat{A})\). It is a fact ((\([3]\), p. 214-216) that \((1 \times n)^* \beta_i = n^i \beta_i\). Now define
\[
(3.0.3) \quad \pi_i = (-1)^d \sigma^* \hat{F}_{CH}(\beta_i)
\]

The main result to be proved is:

**Theorem 3.2.** Let \(A\) be an abelian variety of dimension \(d\) over a field \(k\) and \(G\) a finite group acting on \(A\) such that \(g(0) \in A(k)\) is a torsion point for each \(g \in G\). Let \(f : A \to A/G\) be the quotient map. Suppose \(\Delta_A = \sum_{i=0}^{2d} \pi_i\) is a Chow-Künneth decomposition for \(A\) and let
\[
\eta_i = \frac{1}{|G|}(f \times f)^* \pi_i.
\]
Then
\[
\Delta_{A/G} = \sum_{i=0}^{2d} \eta_i
\]
is a Chow-Künneth decomposition for \(A/G\).

**Remark.**

The hypothesis that \(g(0)\) be a torsion point of \(A\) is not always satisfied. For example, if \(a \in A(k)\) is any point of infinite order, then the automorphism \(g : x \mapsto -x + a\) defines an action of \(\mathbb{Z}/2\mathbb{Z}\) on \(A\) for which \(g(0) = a\) is not a torsion point. However, if \(k\) is an algebraic extension of a finite field, then it is clear that this hypothesis is always satisfied.

Our method of proof is based on that of \([3]\), Theorem 3.1; however, there are further technicalities which complicate it somewhat. The content of the proof is, of course, to show that the elements \(\frac{1}{|G|}(f \times f)^* \pi_i, 0 \leq i \leq 2d\), are mutually orthogonal projectors. Unfortunately, \(A/G\)
is in general not an abelian variety, so we cannot exploit any special properties of this variety. However, the map $f^*$ establishes an isomorphism ([6], Example 1.7.6):

$$CH^*_Q(A/G) \longrightarrow CH^*_Q(A)^G$$

with inverse $\frac{1}{|G|} f_*$. Thus, we will work in the group $CH^*_Q(A)^G$, constructing mutually orthogonal $G \times G$-invariant elements which may be descended to elements of $CH^*_Q(A/G)$ by the following device:

**Lemma 3.3.** Suppose $X$ is a pseudo-smooth projective variety of dimension $d$ and $G$ a group of automorphisms of $X$. Let $f : X \longrightarrow Y = X/G$ be the quotient map and suppose

$$\sum_{g, h \in G} (g \times h)^* \Delta_X = \sum_{i=0}^{2d} \rho_i$$

where $\rho_i \circ \rho_j = 0$ if $i \neq j$ and the $\rho_i$ are $G \times G$-invariant, i.e. for any $g, h \in G$, $(g \times h)^* \rho_i = \rho_i$. Then

$$\Delta_Y = \sum_{i=0}^{2d} \frac{1}{|G|^3} (f \times f)_* \rho_i$$

is a Chow-Küneth decomposition for $Y$.

**Proof.**

We have

$$(f \times f)_* (f \times f)^* = |G|^2, \sum_{g, h \in G} (g \times h)^* = (f \times f)^*(f \times f)_*$$

and $(f \times f)_* \Delta_X = |G| \Delta_Y$,

and therefore:

$$|G|^2 (f \times f)_* \Delta_X = (f \times f)_* \sum_{i} \rho_i$$

Hence

$$\Delta_Y = \frac{1}{|G|^3} \sum_{i} (f \times f)_* \rho_i$$

It remains to show that $(f \times f)_* \rho_i$ are mutually orthogonal. As in Proposition 2.4, we add subscripts and superscripts to $p$ (respectively, $q$) to denote the various projections between products of $X$ (respectively, $Y$), and for convenience of notation set $r = (f \times f \times f) : X \times_k X \times_k X \longrightarrow Y \times_k Y \times_k Y$. Now,

$$(3.0.4) \quad (f \times f)_* \rho_i \circ (f \times f)_* \rho_j = q_{13}^{123} \cdot (q_{12}^{123*} (f \times f)_* \rho_i) \cdot q_{23}^{123*} (f \times f)_* \rho_j)$$

Since the degree of $r$ is $|G|^3 = r_* r^*$, the last expression equals:

$$(3.0.5) \quad \frac{1}{|G|^3 q_{13}^{123} r_* q_{12}^{123*} (f \times f)_* \rho_i \cdot q_{23}^{123*} (f \times f)_* \rho_j)$$
Finally, because $q_{12}^{123} \circ r = (f \times f) \circ p_{12}^{123}$, the above simplifies to

$$\frac{1}{|G|^3} q_{13}^{123} \cdot (r \cdot p_{12}^{123} (f \times f)^* (f \times f) \cdot \rho_i \cdot q_{23}^{123} (f \times f) \cdot \rho_j)$$

Because the $\rho_i$ are $G \times G$-invariant, we have $(f \times f)^* (f \times f)_*$ is multiplication by $|G|^2$, so the expression equals:

$$\frac{1}{|G|^3} q_{13}^{123} \cdot (r \cdot p_{12}^{123} \cdot |G|^2 \cdot \rho_i \cdot q_{23}^{123} (f \times f) \cdot \rho_j)$$

Finally, applying the projection formula, the formula $q_{12}^{123} \circ r = (f \times f) \circ p_{12}^{123}$ and $(G \times G)$-invariance of the $\rho_i$, one may identify the last expression with:

$$\frac{1}{|G|} q_{13}^{123} \cdot r \cdot (p_{12}^{123} \cdot \rho_i \cdot r^* q_{23}^{123} (f \times f) \cdot \rho_j) = \frac{1}{|G|} p_{13}^{123} \cdot (p_{12}^{123} \cdot \rho_i \cdot p_{23}^{123} (f \times f)^* (f \times f) \cdot \rho_j)$$

$$= |G| p_{13}^{123} \cdot (p_{12}^{123} \cdot \rho_i \cdot p_{23}^{123} \rho_j)$$

$$= |G| (\rho_i \circ \rho_j) = 0$$

In [3], the crucial step in the proof of the Chow-Künneth decomposition for abelian varieties is the following computation, which may be proved using the seesaw theorem ([15], Corollary 5.2):

**Proposition 3.4.** ([3], 2.15)

For any integer $n$,

$$(1 \times n)^* \ell = n \ell$$

The analogous strategy in our context would seem to be to study the action of $(1 \times n)^*$ on $(g \times h)^* \ell$; however, there is a priori no action of $G$ on $\hat{A}$. (If $G$ acts on $A$ “by isogenies”; that is, if all of the maps $g : A \longrightarrow A$ are in fact homomorphisms of $A$, then duality gives a natural action of $G$ on $\hat{A}$, but we are not assuming this). Instead, we rely on the fact ([15], p.119) that the Poincaré bundle on $\hat{A} \times_k A$ is the transpose of the Poincaré bundle on $A \times_k \hat{A}$. Hence:

$$\Theta_{A \times_k \hat{A}} = \Theta_{\hat{A} \times_k A}$$

and we prove the following:

**Proposition 3.5.** There is an infinite subset $E \subset \mathbb{N}$ such that for all $n \in E$,

$$(n \times 1)^* (g \times 1)^* \ell = n (g \times 1)^* \ell.$$

**Proof.** For each $g \in G$, write $g = a_g \circ g_0$ as in Proposition 1.3. Let $m_g$ be the order of $a_g = -g(0)$; this is guaranteed to be finite by our hypothesis. Next, let $m = \prod_{g \in G} m_g$, and

$$E = \{ n \in \mathbb{N} : n \equiv 1 \pmod{m} \}$$

Note that if $n \in E$, $m_g$ divides $n - 1$ (for any $g$), so $na_g = a_g$. 

Now, if \( n \in E \), we have

\[
(3.0.7) \quad (n \times 1)^*(g \times 1)^* \ell = (n \times 1)^*(g_0 \times 1)^*(\tau_{a_g} \times 1)^* \ell
\]

Since \( g_0 \) is a homomorphism, \( n \circ g_0 = g_0 \circ n \); therefore the last expression equals

\[
(g_0 \times 1)^*(n \times 1)^*(\tau_{a_g} \times 1)^* \ell
\]

Since \( a_g = na_g \), this equals

\[
(g_0 \times 1)^*(\tau_{na_g} \times 1)^*(n \times 1)^* \ell = (g_0 \times 1)^*(\tau_{a_g} \times 1)^*(n \times 1)^* \ell
\]

By (3.0.6) the last term equals,

\[
n(g_0 \times 1)^*(\tau_{a_g} \times 1)^* \ell = n(g \times 1)^* \ell
\]

The next step in the proof of Theorem 3.2 is to construct the elements \( \rho_i \) appearing in Lemma 3.3; for each \( i \), we simply set

\[
\rho_i = \sum_{g, h \in G} (g \times h)^* \pi_i
\]

where \( \pi_i \) are the Chow-Künneth components of \( \Delta_A \) from Theorem 3.1. It is clear from the formula that the \( \rho_i \) are \( G \times G \)-invariant and that \( \sum_{i=0}^{2d} \rho_i = \sum_{g, h \in G} (g, h)^* \Delta_A \); so it remains to prove that they are mutually orthogonal. In preparation for this, we study the action of \((1 \times n)^*\) on \( \rho_i \):

**Proposition 3.6.** For \( n \in E \), \((1 \times n)^*(g \times h)^* \pi_i = n^i \pi_i\). Hence, \((1 \times n)^* \rho_i = n^i \rho_i\).

**Proof.**

Observe that \((1 \times n)^*(g \times h)^* \pi_i = (1 \times n)^*(g \times 1)^*(1 \times h)^* \pi_i = (g \times 1)^*(1 \times n)^*(1 \times h)^* \pi_i\), so it suffices to consider the case \( g = 1 \).

We recall the construction of \( \pi_i \) from (3.0.3):

\[
(3.0.8) \quad (1 \times n)^*(1 \times h)^* \pi_i = (-1)^g \sigma^*(1 \times n)^*(1 \times h)^* \hat{F}_{CH}(\beta_i)
\]

From the definition of \( \hat{F}_{CH} \) this identifies with:

\[
(3.0.9) \quad = (-1)^g \sigma^*(1 \times n)^*(1 \times h)^* p_{12*}(p_{13}^* \beta_i \cdot \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!})
\]

Next, in view of the Cartesian square:

\[
\begin{array}{ccc}
A \times_k A & \times_k A \\
\xrightarrow{p_{12}} & \\
1 \times h \times 1 & \xrightarrow{1 \times h} & \\
A \times_k A & \times_k A
\end{array}
\]
the above expression becomes
\[ (-1)^g \sigma^* (1 \times n)^* p_{12,*} (1 \times h \times 1)^* (p_{13}^* \beta_i \cdot \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!}) \]

Now using another Cartesian square:
\[ A \times_k A \times_k A \xrightarrow{p_{12}} A \times_k A \]
\[ A \times_k A \times_k A \xrightarrow{p_{12}} A \times_k A \]
this equals
\[ (-1)^g \sigma^* p_{12,*} (1 \times n \times 1)^* (1 \times h \times 1)^* (p_{13}^* \beta_i \cdot \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!}) \]

Since \( p_{1,3}^* \) leaves the second factor unchanged, this expression identifies with:
\[ (-1)^g \sigma^* p_{12,*} \left( p_{13}^* \beta_i \cdot (1 \times n \times 1)^* (1 \times h \times 1)^* \sum_{i=0}^{\infty} \frac{1}{i!} (1 \times (n \times 1)^* (h \times 1)^* \ell^i) \right) \]

By Proposition 3.5 the last term is given by
\[(3.0.10) \quad n^i (-1)^g \sigma^* p_{12,*} \left( p_{13}^* \beta_i \cdot \sum_{i=0}^{\infty} \frac{1}{i!} (1 \times (h \times 1)^* \ell^i) \right) \]
By applying the same steps above in essentially the opposite order one obtains the identification of the last expression with:
\[ n^i (-1)^g \sigma^* p_{12,*} \left( p_{13}^* \beta_i \cdot (1 \times h \times 1)^* \sum_{i=0}^{\infty} \frac{1}{i!} (1 \times \ell^i) \right) \]
\[ = n^i (-1)^g \sigma^* p_{12,*} (1 \times h \times 1)^* (p_{13}^* \beta_i \cdot \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!}) \]
\[ = n^i (1 \times h)^* p_{12,*} (p_{13}^* \beta_i \cdot \sum_{i=0}^{\infty} \frac{(1 \times \ell^i)}{i!}) \]
\[ = n^i (1 \times h)^* \pi_i \]

To prove orthogonality of the \( \rho_i \), we need a version of Liebermann’s trick (cf. [3], Proof of Theorem 3.1); first we prove the following simple lemma:

**Lemma 3.7.** For every \( g, h \in G \), \( \rho_j \circ (g \times h)^* \Delta_A = \rho_j \).
Proof.
Certainly the lemma is true if \( g = h = 1 \). In the general case,

\[
(3.0.11) \quad \rho_j \circ (g \times h)^* \Delta_A = p_{13_1}(p_{12}^*(g \times h)^* \Delta_A \cdot p_{23}^* \rho_j) = p_{13_1}((g \times h \times 1)^* p_{12}^* \Delta_A \cdot p_{23}^* \rho_j)
\]

\[
= p_{13_1}(g \times h \times 1)^* (p_{12}^* \Delta_A \cdot (g^{-1} \times h^{-1} \times 1)^* p_{23}^* \rho_j)
\]

\[
= (g \times 1)^* p_{13_1}(p_{12}^* \Delta_A \cdot p_{23}^* (h^{-1} \times 1)^* \rho_j)
\]

Since \( \rho_j \) is \( G \times G \)-invariant the last term equals

\[
(g \times 1)^* p_{13_1}(p_{12}^* \Delta_A \cdot p_{23}^* \rho_j) = (g \times 1)^* (\rho_j \circ \Delta_A) = (g \times 1)^* (\rho_j) = \rho_j
\]

Proposition 3.8. (Liebermann’s trick) For every \( i, j, i \neq j, \rho_i \circ \rho_j = 0 \).

Proof.
Suppose \( n \in E \). By Proposition 3.6,

\[
n^i \rho_j = (1 \times n)^* \rho_j
\]

\[
= (1 \times n)^* (\rho_j \circ \Delta_A)
\]

By Lemma 3.7, the last term equals

\[
\frac{1}{|G|^2}(1 \times n)^* (\rho_j \circ \sum_{g,h} (g \times h)^* \Delta_A) = \frac{1}{|G|^2}(1 \times n)^* (\rho_j \circ \sum_{i=0}^{2g} \rho_i)
\]

\[
= \frac{1}{|G|^2} \sum_{i=0}^{2g} (1 \times n)^* p_{13_1}(p_{12}^* \rho_j \cdot p_{23}^* \rho_i)
\]

\[
= \frac{1}{|G|^2} \sum_{i=0}^{2g} p_{13_1}(1 \times 1 \times n)^* (p_{12}^* \rho_j \cdot p_{23}^* \rho_i)
\]

\[
= \frac{1}{|G|^2} \sum_{i=0}^{2g} p_{13_1}(p_{12}^* \rho_j \cdot p_{23}^*(1 \times n)^* \rho_i)
\]

\[
= \frac{1}{|G|^2} \sum_{i=0}^{2g} n^i(\rho_j \circ \rho_i)
\]

Hence

\[
n^j((\rho_j \circ \rho_j) - |G|^2 \rho_j) + \sum_{i \neq j} n^i(\rho_i \circ \rho_j) = 0
\]

for all \( n \in E \). Since \( E \) is infinite, this forces \( \rho_i \circ \rho_j = 0 \) for all \( i \neq j \), and also \( \rho_j \circ \rho_j = |G|^2 \rho_j \).

This final step in the proof of Theorem 3.2 is to show that the images of the \( \eta_i \) under the cycle map \( cl_{A/G \times_k A/G} : CH^*(A/G \times_k A/G) \rightarrow H^*(X/G \times_k X/G) \) to any Weil cohomology
theory are in fact the K"unneth components of the class of the diagonal. This follows easily from the analogous fact for the variety $A$ and commutativity of the following diagram:

$$CH_{\mathbb{Q}}(A \times_k A) \xrightarrow{c_{A \times_k A}} H^*(A \times_k A)$$

\[
\begin{array}{cc}
CH^*(A/G \times_k A/G) & \xrightarrow{c_{A/G \times_k A/G}} H^*(A/G \times_k A/G) \\
(f \times_k f)_* & (f \times_k f)_*
\end{array}
\]

Here we will show that any Weil cohomology theory, $H^*$, extends to pseudo-smooth schemes and show that the above square commutes. First observe that if $G$ is a finite group acting on a smooth scheme $X$, each $g \in G$ acts on $X$ as an automorphism: therefore, the action of $G$ on $X$ induces an action on the given Weil cohomology theory applied to $X$, i.e. on $H^*(X)$. Since $H^*(X)$ are all vector spaces over a field of characteristic 0, one obtains a decomposition of $H^*(X)$ into irreducible representations of $G$. One defines $H^*(X/G)$ to be $H^*(X)^G$. Corresponding assertions also hold for the rational K"unneth groups.

Observe that if $f : X \to X/G$ is the quotient map, one may identify $f_* : CH_{\mathbb{Q}}(X) \to CH_{\mathbb{Q}}^*(X/G)$ $(f_* : H^*(X) \to H^*(X/G))$ with the projection $CH_{\mathbb{Q}}(X) \to CH_{\mathbb{Q}}^*(X/G)$ (the projection $H^*(X) \to H^*(X)^G$, respectively). Since the cycle map commutes with group action, one can now see that it commutes with $f_*$: we obtain the commutativity of the square above.

This concludes the proof of Theorem 3.2.

Among the formulas proved by K"unnemann is the so-called Poincaré duality for abelian varieties ([13], Theorem 3.1.1 (iii)); in our notation, this reads $\pi_{2d-i} = \pi_i$ for each $i$. This fact immediately implies the analogue for quotients:

**Corollary 3.9.** *(Poincaré duality for quotients)* The Chow-K"unneth decomposition for $A/G$ of Theorem 3.2 satisfies Poincaré duality; that is, for any $i$, $\eta_{2d-i} = \eta_{i}$.

**3.1. Examples.**

1. **Symmetric products of abelian varieties** Let $X$ denote an abelian variety. We let $X^n/\Sigma_n$ denote the $n$-fold symmetric power of $X$. Observe that for every $\sigma \in \Sigma_n$, $\sigma(0, \cdots, 0) = (0, \cdots, 0)$. Therefore the hypotheses of Theorem 3.2 are satisfied irrespective of the base field $k$. Therefore, we obtain a Chow K"unneth decomposition for $X^n/\Sigma_n$. (Observe that the action of $\Sigma_n$ is not in general free so that the quotient $X^n/\Sigma_n$ is only pseudo-smooth and not smooth.)

2. **Example of Mehta and Srinivas** (See [16] .) Let $X$ be an elliptic curve (or more generally any abelian variety) over $k$, with $\text{char}(k) \neq 2$. Let $t$ denote a point of order 2 on $X$. Let the group $\mathbb{Z}/2\mathbb{Z}$ act on $X \times X$ by : $(x, y) \mapsto (x + t, -y)$. Let $Y$ denote the quotient variety. Now one may see easily that the action is free so that $Y$ is smooth. Nevertheless, in positive characteristics, $Y$ need not be an abelian variety as is shown in [16]. Theorem 3.2 provides a Chow K"unneth decomposition for $Y$. 
4. Appendix: The strong relative Künneth decomposition of cohomology

In this appendix, we consider, with a view to applications elsewhere, a relative form of the strong Künneth decomposition in arbitrary cohomology theories satisfying certain mild conditions. It should be remarked that in this section, it suffices to assume the cohomology theory is, at least in principle, part of a twisted duality theory in the sense of Bloch-Ogus. (See [2].) It should be added that the arguments are modifications of the ones in [21] where he considered the case of K-theory.

Accordingly we will denote $H^*(X, \Gamma(r))$ by $H^*(X, (r))$. Throughout this section we will make the following additional hypotheses on our cohomology theories. (Observe these hypotheses are not identical to the ones in [2], but are implied by them.)

(H.1): for every flat map $f : X \to Y$, there is an induced map $f^* : H^*(Y, (r)) \to H^*(X, (r))$ and this is natural in $f$.

(H.2): for every proper smooth map $f : X \to Y$ of relative dimension $d$, there is a push-forward $f_* : H^i(X; j) \to H^{i-2d}(Y; j - d)$ so that if $g : Y \to Z$ is another proper smooth map of relative dimension $d'$, one obtains $g_* \circ f_* = (g \circ f)_*$. In this case the obvious projection formula $f_*(x \circ f^*(y)) = f_*(x) \circ y, x \in H^s(X, (r)), y \in H^s(Y, (r))$ holds.

(H.3): for each smooth scheme $X$ and closed smooth sub-scheme $Y$ of pure codimension $c$, there exists a canonical class $[Y] \in H^{2c}(X; c)$. Moreover the last class lifts to a canonical class $[Y] \in H^{2c}_X(X; c)$. (The latter has the obvious meaning in the setting Bloch-Ogus twisted duality theories. In case the cohomology theory is defined as hyper-cohomology with respect to a complex, we let $\mathbb{H}_Y(X; c) = \text{the canonical homotopy fiber of the obvious map } \mathbb{H}(X; c) \to \mathbb{H}(X - Y; c)$; now $H^{2c}_X(X, c) = H^{2c}(\mathbb{H}_Y(X; \Gamma(c)))$.) The cycle classes are required to pull-back under flat pull-back and push-forward under proper push-forwards.

(H.4): if $X$ is a smooth scheme, there exists the structure of a graded commutative ring on $H^r(X; \cdot) = \bigoplus_{r,s} H^r(X; s)$. i.e. $\circ : H^r(X; s) \otimes H^r(X; s') \to H^{r+r'}(X; s + s')$. In addition to this, there exists an external product $H^r(X; s) \otimes H^r(X; s') \to H^{r+r'}(X \times X; s + s')$ so that the internal product is obtained from the latter by pull-back with the diagonal.

For the purposes of this section it is also convenient to consider only cohomology theories that are singly graded or non-weighted. Given a bigraded cohomology theory $H^*(X; (r))$, we will re-index it as follows: we let

$$(4.0.1) \quad h^r(X; 2r - s) = H^s(X; (r)) \quad \text{and} \quad h^r(X; n) = \bigoplus_{r} h^r(X; n)$$

We view $\{h^r(X; n) | n\}$ as a singly graded cohomology theory. Observe that if $f : X \to Y$ is a proper smooth map of relative dimension $d$, the induced map $f_*$ sends $h^r(X; n)$ to $h^{r}(Y; n)$. Similarly if $f : X \to Y$ is a flat map, the induced map $f^*$ sends $h^*(Y; n)$ to $h^*(X; n)$. 

Theorem 4.1. Let \( f : X \to Y \) denote a proper smooth map of smooth schemes of relative dimension \( d \) and let \([\Delta] \in H^{2d}(X \times X; \mathcal{O})\) denote the class of the diagonal. Assume that \([\Delta] = \sum a_{i,j} a_{d-i,j} \times b_{d-i,j}, \) with each \( a_{i,j} \in H^{2i}(X; \mathcal{O}), b_{d-i,j} \in H^{2d-2i}(X; \mathcal{O}) \). Then for every fixed integer \( n \) one obtains the isomorphism:

\[
h^*(X; n) \cong h^*(X; 0) \otimes h^*(Y; n)
\]

**Proof.** We will first prove that the classes \( \{ a_{i,j} \mid i \} \) generate \( h^*(X; n) \) as a module over \( h^*(Y; \cdot) \) i.e. the obvious map from the right hand side to the left hand side of 4.0.2 (which we will denote by \( \rho \)) is surjective.

Let \( p_i : X \times Y \to X \) denote the projection to the \( i \)-th factor. For each \( x \in h^*(X; n) \) we will first observe the equality:

\[
x = p_{1*}(\Delta \circ p_2(x))
\]

(To see this observe that \([\Delta] = \Delta_* (1) \in H^1(X; \mathcal{O}). \) Therefore, \( \Delta \circ p_2(x) = \Delta_* (\Delta^* p_2^1(x)) \) and hence \( p_{1*}(\Delta \circ p_2^2(x)) = p_{1*} \Delta^* (\Delta^* p_2^2(x)) = (p_1 \circ \Delta)_* ((p_2 \circ \Delta)^* (x)) = x. \)

Now we substitute \([\Delta] = \sum a_{i,j} \circ p_{2d-i,j} \) into the above formula to obtain:

\[
x = p_{1*}(\Sigma a_{i,j} \circ p_{2d-i,j} (x))
\]

This proves the assertion that the classes \( \{ a_{i,j} \mid i \} \) generate \( h^*(X; \cdot) \) i.e. the map \( \rho \) is surjective.

The rest of the proof is to show that the map \( \rho \) is **injective**. The key is the following diagram:

\[
\begin{CD}
h^*(X; n) @>{\rho}>> h^*(X; 0) \otimes h^*(Y; n) \\
@VV{\mu}V @VV{\alpha}V \\
\text{Hom}_{h^*(Y; 0)}(h^*(X; 0), h^*(Y; n)) \end{CD}
\]

where the map \( \alpha (\mu(x), x \in h^*(X, 0)) \) is defined by \( \alpha (x \otimes y) = \text{the map } x' \mapsto f_* (x' \circ x) \circ y \) (the map \( x' \mapsto f_* (x' \circ x), \) respectively). The commutativity of the above diagram is an immediate consequence of the projection formula: observe \( \rho (x \otimes y) = x \circ f^* (y). \) Therefore, to show the map \( \rho \) is injective, it suffices to show the map \( \alpha \) is injective. For this we define a map \( \beta \) to be a splitting for \( \alpha \) as follows: if \( \phi \in \text{Hom}_{h^*(Y; 0)}(h^*(X, 0), h^*(Y; n)), \) we let \( \beta (\phi) = \Sigma a_{i,j} \otimes (\phi(b_{d-i,j})). \) Observe that \( \beta (\alpha (x \otimes y)) = \beta (\text{the map } x' \mapsto f_* (x' \circ x) \otimes y) = (\Sigma a_{i,j} \otimes f_* (b_{d-i,j} \circ x)) \otimes y. \)

Now observe that \( f_* (b_{d-i,j} \circ x) \in h^*(Y; 0) \) so that we may write the last term as \( = (\Sigma a_{i,j} \circ f^* f_* (b_{d-i,j} \circ x)) \otimes y. \) By ( 4.0.1), the last term \( = x \otimes y. \) This proves that \( \alpha \) is injective and hence that so is \( \rho. \)

We end this section by considering some explicit examples of what the last theorem implies for various cohomology theories.
Examples 4.2. 1. K-theory. In this case the theorem takes on the form:

\begin{equation}
K^i(X) \cong K^0(X) \otimes_{K^0(Y)} K^i(Y)
\end{equation}

See [21] for a proof of this. The last part of the proof of the above theorem is clearly an adaptation of this argument.

2. The theorem takes on the following simple form in the case of any of the following bigraded cohomology theories: absolute, motivic or Deligne cohomology.

\begin{equation}
H^n(X; t) = \left( \bigoplus_a H^{2a}(X; a) \right) \otimes \left( \bigoplus_a H^{n-2a}(Y; t-a) \right)
\end{equation}

Remarks 4.3. 1. Taking \( n = 0 \), the argument in the last part of the proof of the theorem shows that one obtains a non-degenerate pairing \( < \ , \ > : h^*(X, 0) \otimes h^*(Y, 0) \rightarrow h^*(Y, 0) \)

by \( \alpha \otimes \beta \rightarrow f_*(\alpha \otimes \beta) \).

2. As an example of the usefulness of the last theorem or the formula in 2, we consider the following result. Let \( H \) denote motivic cohomology. Recall that the Beilinson-Soulé vanishing conjecture for the motivic cohomology of a scheme is the following statement: \( H^s(X, r; \mathbb{Q}) = 0 \) if \( s < 0 \) or if \( (s = 0 \text{ and } r \neq 0) \) while \( H^0(X, 0; \mathbb{Q}) = \mathbb{Q} \). Now we leave it as an easy exercise to prove from 2 the following proposition.

Proposition 4.4. If \( X \rightarrow Y \) satisfies the hypotheses of Theorem 4.1 and \( X \) and \( Y \) are both connected, then the Beilinson-Soulé vanishing conjecture holds for \( X \) if it holds for \( Y \).

Corollary 4.5. The following class of varieties over a number field \( k \) satisfy the Beilinson-Soulé vanishing conjecture.

- All toric varieties over \( k \)
- All spherical varieties over \( k \). (A variety \( X \) is spherical if there exists a reductive group \( G \) defined over \( k \) acting on \( X \) so that there exists a Borel subgroup scheme \( S \) having a dense orbit. The orbits are products of tori and affine spaces (over \( k \)).)
- Any variety over \( k \) on which a connected solvable group scheme defined over \( k \) acts with finitely many orbits. (For example projective spaces and flag varieties over \( k \)).
- Any variety over \( k \) that has a stratification into strata each of which is the product of a torus with an affine space.

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